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Thermodynamics in the emergent Universe model

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Abstract. The emergent Universe scenario is obtained in flat Universe with equation of state (EoS) ($P = B\rho - A\rho^{1/2}$) (where A and B are constants) as per Mukherjee *et al.* The EoS of this emergent Universe (EU) model can describe the current accelerated expansion and the initial singularity of the Universe. Following the standard thermodynamical criteria, stability of the EU models has been discussed. It is noted from thermal stability and positivity of adiabatic sound speed that B satisfies the values $B = \frac{1}{3}, 1$ in the EU model. So, the emergent models with $B = 0, -\frac{1}{3}$ are not supported with the stability issue. Further, the third law of thermodynamics is obeyed in this case for $B < -1$ (or with $A = 0$, but it is outside the EU), i.e., any of the four discrete value of B $(= 0, -\frac{1}{3}, \frac{1}{3}, 1)$ does not support this third law. The specific heat at constant volume c_v obeys the relation $c_v \geq 0$ for $T \geq 0$ in the EU models. Two characteristic volume scales, critical volume V_c and flip volume V_f are obtained from zero pressure and zero deceleration condition in this model. Physically, these should follow the relation $V_f > V_c$, which are actually followed in the EU model for $B = \frac{1}{3}, 1$.

Keywords. Hubble parameter; dark energy; dark matter; structure formation.

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1. Introduction

It is widely accepted that the big-bang model is one of the best model of the Universe. According to this model, Universe was created billions of years ago from singularity. Although physics does not provide any suitable theoretical support to this singularity, the discovery of cosmic microwave background radiation [1,2] favours this big-bang model. The scar of singularity problem remained with this model. Recent discovery of present acceleration of the Universe from the supernovae redshift studies [3–6] has burdened the big-bang model with another problem.

Universe, consisting of normal matter, should now decelerate due to the action of gravity after its creation from big-bang. But supernovae studies show the evidence of acceleration at present, instead. This present acceleration may be a reality either (i) by modifying the curvature sector with higher order terms or (ii) by modifying the matter sector of the Einstein–Hilbert action with new type of energy (dark energy (DE)). Scalar tensor cosmology [7], brane-world models [8] and Horava–Lifshitz gravity [9] are few examples of modified theories of gravity in the first approach with some success. Second approach attracted many candidates of dark energy, of which the cosmological constant model is the most successful one. Though this model works wonderfully well at large scale, it has some theoretical problems, like cosmic coincidence and cosmological constant problem [10].

In addition to cosmological constant, dynamical DE models are also utilised to interpret the cosmic acceleration. The quintessence [11], phantom field [12], K-essence [13] and Chaplygin gas models [14] are some dynamical DE models. Generalised Chaplygin gas (GCG) [15,16], modified Chaplygin gas (MCG) [17], variable modified Chaplygin gas (VMCG) [18], extended Chaplygin gas (ECG) [19,20] etc. are the recent advanced versions of Chaplygin gas devoid of drawbacks of the earliest version.

As the cosmological constant model fits the observational [21] data better, a suitable DE model should be close to this model. Most of the models above, solve the late acceleration problem, but unable to solve the initial singularity problem. As a result, there are very few works in which both problems are addressed simultaneously and successfully. One such work is represented in [22] with a non-linear equation of state (EoS) that

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helps in overcoming both the initial singularity and the present acceleration issue. Such equation of state (EoS) gives rise to the emergent Universe (EU) model [22] in flat Universe scenario which is given by

$$P = B\rho - A\rho^{1/2}, \tag{1}$$

where A and B are constants with $A > 0$ and P and ρ are pressure and density respectively. This EoS (1) can be obtained from the EoS of modified Chaplygin gas EoS ($P = B\rho - \frac{A}{\rho^{\alpha}}$) for a particular value of the constant $\alpha = -\frac{1}{2}$. With discrete set of values of B ($= 0, -\frac{1}{3}, \frac{1}{3}, 1$) in the model, one obtains Universe with three different kinds of cosmic fluids [22] (dark energy is common to all). The EU model starts with initial static phase, followed by exponential inflation, standard reheating and classical thermal radiation dominated phase one by one [23]. EU models can be built in brane-world gravity [24,25], Brans–Dicke theory [26] and with various fluids and fields [23,27,28] without initial singularity. The model corresponds to (i) $B = 0$, consists of dark energy, exotic matter and dust, (ii) $B = -\frac{1}{3}$, dark energy, domain wall and cosmic string, (iii) $B = \frac{1}{3}$, dark energy, cosmic string and radiation and (iv) $B = 1$, dark energy, dust and stiff matter [22].

In the EU models, initial singularity is avoided with a scale factor of type $a(t) = a_i(\beta + \exp(\alpha t))^w$ (where β, α, w are constants) for all energy density $\rho > 0$ [22]. Although the strong energy condition $\rho + 3P \geq 0$ (i.e., $\omega \geq -\frac{1}{3}$) is violated in late time (when deceleration to acceleration transition occurs), the density ρ remains positive throughout (in this scenario). Original works of Mukherjee *et al* [22,29] avoided the issue of matching different regimes, prediction of correct tensor to scalar ratio and the scalar spectral index. Mukherjee *et al* [29] mentioned about the fields (in $B = 0$ model) responsible for early inflation but could not give direction, about how to exit from this inflation. But the deceleration-to-acceleration phase transitions of the Universe in these EU models are obtained from the values of A and B [30,31], which may also be understood from ω vs. V and q vs. V plots of this paper. In spite of its success in avoiding initial singularity, the present EU models found wanting to address those issues. The task of a thorough re-examination of this EU model (for its improvement) may be taken up as future work.

Recently, attention has been paid to address the thermodynamic stability of the fluids in different DE models. This is an effort to establish the models as physically realistic systems. As a result, there has been a revived interest among us to address the thermodynamic aspect of the system. Following this, Santos *et al* have studied the thermodynamical stability in generalised [32] and modified Chaplygin gas [33] model on the basis of standard prescription [34] where both (i) $(\partial P/\partial V)_S < 0$, $(\partial P/\partial V)_T < 0$ and (ii) $c_v > 0$ are satisfied simultaneously. Following the methods of [32–34] and [35,36], some physical parameters, such as pressure P, effective equation of state ω, deceleration parameter q etc. are investigated here in EU scenario on the basis of thermodynamics. Many thermal quantities are derived as functions of temperature or volume in this EU model for the analysis.

In this case of EU, thermal stability depends critically on B. From the adiabatic sound speed it is also noted that stability of the model depends on the same parameter.

The paper is presented as follows: In §2, relevant field equations are given. In §3, pressure P has been introduced for the model. In §4, EoS is introduced. In §5, deceleration parameter q is introduced. In §6, concept of adiabatic sound is introduced. In §7, concept of thermodynamic stability is discussed at length. In §8, concept of thermal EoS is introduced. In §9, pressure–volume relation is discussed. In §10, the conclusions are given.

2. Relevant field equations for the emergent Universe models

The Friedmann–Robertson–Walker (FRW) metric ($c = 1$) for the spatially flat space–time is given by

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2], \tag{2}$$

where $a(t)$ is the scale factor of the Universe and r, θ, ϕ are the dimensionless co-moving coordinates. The equation of state (EoS) relating pressure P and density ρ for the EU is given by eq. (1). This density is related with the internal energy and volume as

$$\rho = \frac{U}{V}, \tag{3}$$

where U and V are the internal energy and volume filled by the fluid. The pressure is related with the internal energy [37] via the following equation:

$$\left(\frac{\partial U}{\partial V}\right)_S = -P. \tag{4}$$

From the equation of pressure and density, the EoS equation (1) transforms to

$$\left(\frac{\partial U}{\partial V}\right)_S = A\left(\frac{U}{V}\right)^{1/2} - B\frac{U}{V}. \tag{5}$$

Integrating the above equation, internal energy U is obtained as a function of volume V as

$$U = \left[\frac{AV^{1/2}}{B+1} + \frac{C}{V^{B/2}} \right]^2, \tag{6}$$

where C is the integration constant.

This internal energy can be expressed as

$$U = \left(\frac{A}{B+1} \right)^2 V \left[1 + \left(\frac{\epsilon}{V} \right)^{\frac{B+1}{2}} \right]^2, \tag{7}$$

where

$$\epsilon = \left[\frac{C(B+1)}{A} \right]^{\frac{2}{B+1}}.$$

So, the energy density may be expressed as

$$\rho = \left(\frac{A}{B+1} \right)^2 \left[1 + \left(\frac{\epsilon}{V} \right)^{\frac{B+1}{2}} \right]^2. \tag{8}$$

3. Pressure in the EU models

The expression of pressure P of the fluid of the EU model is obtained from eqs (1) and (8) as

$$P = -\left(\frac{A}{B+1} \right)^2 \left[1 + \left(\frac{\epsilon}{V} \right)^{\frac{B+1}{2}} \right] \left[1 - B \left(\frac{\epsilon}{V} \right)^{\frac{B+1}{2}} \right]. \tag{9}$$

For the model with (i) $B = 0$, it is

$$P = -\left(\frac{A}{1} \right)^2 \left[1 + \left(\frac{\epsilon}{V} \right)^{1/2} \right], \tag{10}$$

for (ii) $B = -\frac{1}{3}$, it is

$$P = -\left(\frac{3A}{2} \right)^2 \left[1 + \left(\frac{\epsilon}{V} \right)^{1/3} \right] \left[1 + \frac{1}{3} \left(\frac{\epsilon}{V} \right)^{1/3} \right], \tag{11}$$

for (iii) $B = \frac{1}{3}$, it is

$$P = -\left(\frac{3A}{4} \right)^2 \left[1 + \left(\frac{\epsilon}{V} \right)^{2/3} \right] \left[1 - \frac{1}{3} \left(\frac{\epsilon}{V} \right)^{2/3} \right] \tag{12}$$

and for (iv) $B = 1$, it is

$$P = -\left(\frac{A}{2} \right)^2 \left[1 + \left(\frac{\epsilon}{V} \right)^{1/1} \right] \left[1 - \left(\frac{\epsilon}{V} \right)^{1/1} \right]. \tag{13}$$

In general EU model the condition of zero pressure, $P = 0$, corresponds to the critical volume V_c which is given by

$$V_c = \left[\frac{C(B+1)B}{A} \right]^{\frac{2}{B+1}}. \tag{14}$$

When magnitude of volume V is below a certain volume V_c, i.e., $V < V_c$, pressure P is positive, which indicates a possible decelerated Universe. When $V = V_c$, pressure vanishes ($P = 0$). For volume greater than this critical volume, $V > V_c$, pressure becomes negative indicating exotic matter dominance and a possible accelerating Universe. From these considerations, V_c comes out as a new scale. As V_c and ϵ are of the same order of magnitude, $V \gg \epsilon$ represents a very large volume with a possible acceleration and for smaller volumes when $V \ll \epsilon$, the reverse decelerating motion is expected.

From eqs (8) and (9) one obtains

$$\rho + 3P = \left(\frac{A}{B+1} \right)^2 \left[1 + \left(\frac{\epsilon}{V} \right)^{\frac{B+1}{2}} \right] \times \left[(3B+1) \left(\frac{\epsilon}{V} \right)^{\frac{B+1}{2}} - 2 \right]. \tag{15}$$

The condition $\rho + 3P = 0$ is associated with another critical volume V_f.

4. Equation of state

Equation of state parameter ω is defined as $\omega = P/\rho$. In terms of parameters of the EU model, it is given as

$$\omega = B - \frac{B+1}{1 + (\epsilon/V)^{\frac{B+1}{2}}}. \tag{16}$$

For (i) volume $V \ll \epsilon$, i.e., $\frac{\epsilon}{V} = \infty$, and so $\omega = B$. In this region, the fluid acts like a barotropic one, i.e., $P = B\rho$, for very small volume of the Universe. (ii) For large volume $V \gg \epsilon$, $\frac{\epsilon}{V} = 0$, $\omega = -1$ and so $P = -\rho$. Then the EU model acts like a ΛCDM model in large volume.

Initially, when volume is sufficiently small, $\omega > 0$. As V increases to V_c, ω tends to zero. When

$$V = V_c = \left[\frac{C(B+1)B}{A} \right]^{\frac{2}{B+1}},$$

$\omega = 0$. Again V increases and ω becomes negative.

5. Deceleration parameter in the models

The deceleration parameter q may be defined in terms of pressure and density as

$$q = \frac{1}{2} + \frac{3}{2}\frac{P}{\rho} \tag{17}$$

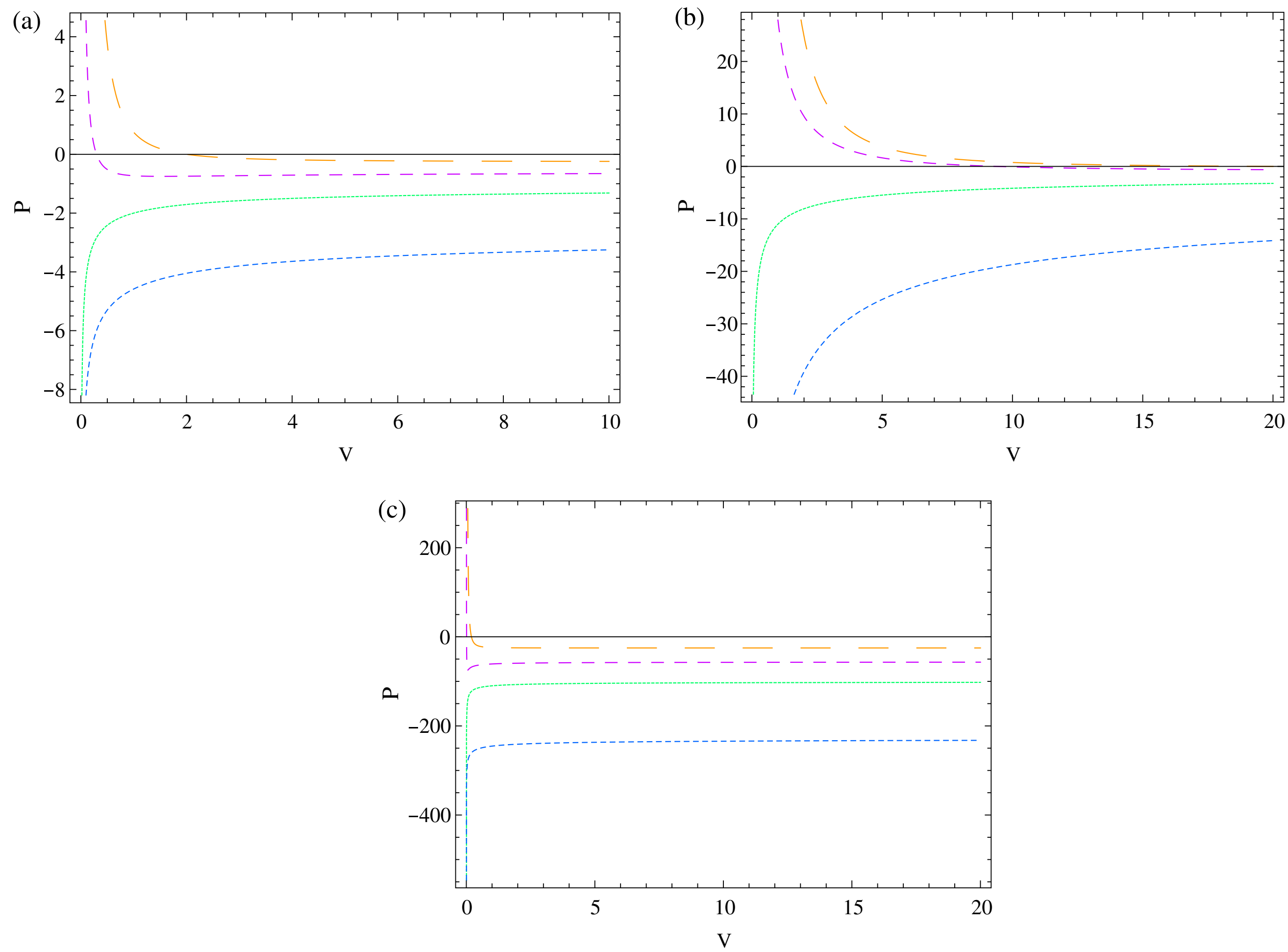

Figure 1. Plot of pressure P vs. volume V in various EU models for (**a**) $A = 1$, $C = 1$, (**b**) $A = 1$, $C = 10$ and (**c**) $A = 10$, $C = 1$. Here dotted line stands for $B = -\frac{1}{3}$, solid line for $B = 0$, close-dashed line for $B = \frac{1}{3}$ and wide-dashed line for $B = 1$ model.

which can be written in terms of the parameters of the model as

$$q = \frac{1}{2} + \frac{3}{2}\left[B - \frac{B+1}{1+(\epsilon/V)^{\frac{B+1}{2}}}\right]. \tag{18}$$

For (i) small volume $V \ll \epsilon$, $\frac{\epsilon}{V} = \infty$, $q = \frac{1}{2} + \frac{3}{2}B$. For small values of V, decelerating parameter q is positive for $B \geq 0$.

In the case of (ii) large volume $V \gg \epsilon$, $\frac{\epsilon}{V} = 0$, $q = -1$. For small volume, q is positive, and Universe decelerates. But for large volume, q is negative, Universe is in a possible accelerating phase. The change from positive to negative q occurs at volume V_f which may be expressed in terms of parameters of the model as

$$V_f = \left[\frac{C(B+1)(3B+1)}{2A}\right]^{\frac{2}{B+1}}. \tag{19}$$

Thus, for volumes less than a particular volume V_f, $V < V_f$, one expects decelerating phase and acceleration is a possibility above this volume, i.e., $V > V_f$. So, two scales of volume (V_c and V_f) are obtained from the condition of zero pressure and deceleration parameter, where possible deceleration–acceleration transition occurs. In FRW cosmology for flip to occur pressure should not only be negative, but also should be less than $\rho/3$ (i.e., $\rho + 3P < 0$). So, in this case one should get $V_c < V_f$, which is clear from eqs (14) and (19). It may also be proved that for this particular volume $V = V_f$, the condition between density and pressure, $\rho + 3P = 0$, is satisfied.

Now from eqs (14) and (19)

$$\frac{V_f}{V_c} = \left(\frac{3B+1}{2B}\right)^{\frac{2}{B+1}} \tag{20}$$

which may be written as

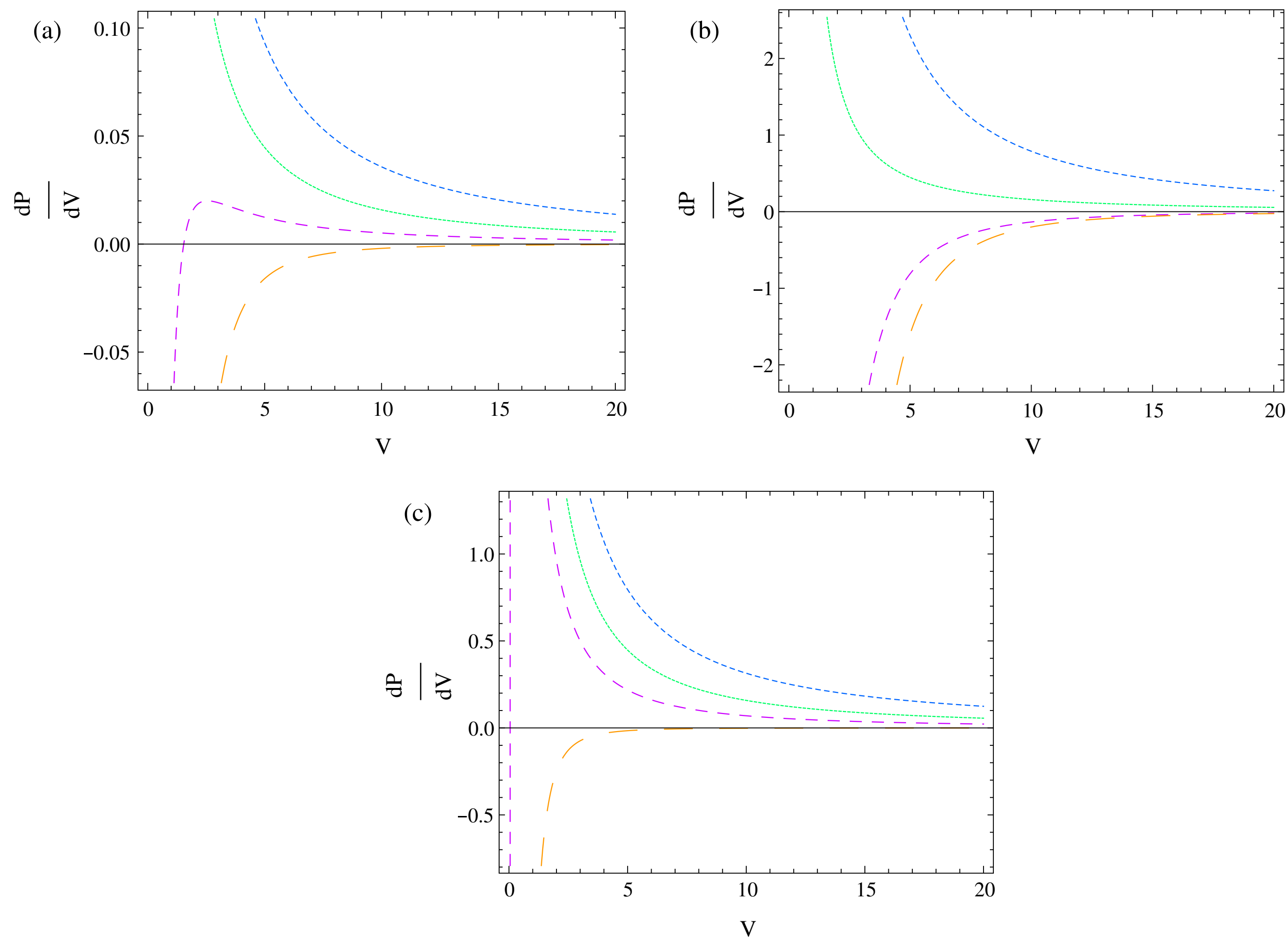

Figure 2. Plot of pressure gradient $\mathrm{d}P/\mathrm{d}V$ vs. volume V in various EU models for (**a**) $A = 1, C = 1$, (**b**) $A = 1, C = 10$ and (**c**) $A = 10, C = 1$. Here dotted line stands for $B = -\frac{1}{3}$, solid line for $B = 0$, close-dashed line for $B = \frac{1}{3}$ and wide-dashed line for $B = 1$ model.

$$\frac{V_f}{V_c} = \left(1 + \frac{B+1}{2B}\right)^{\frac{2}{B+1}}. \tag{21}$$

Theoretically, the condition $V_f > V_c$ will satisfy when

$$\frac{B+1}{2B} > 0 \quad \text{and} \quad \frac{2}{B+1} > 0$$

which leads to the set of conditions (i) $B > 0$, $B > -1$, (ii) $B < 0$, $B < -1$ and $B > -1$. Simply, $B > 0$ is essential for the validity of the condition $V_f > V_c$. Graphically, it can also be seen that the condition $V_f > V_c$ holds for $B > 0$. So, clearly models having $B = \frac{1}{3}, 1$ satisfies the relation ($B > 0$).

6. Adiabatic sound speed of the fluid

The outward force of radiation and inward attractive force of gravity in the primordial plasma produces sound waves (in the early Universe). Afterwards, matter and radiation gets separated as they grow differently with the expansion of the Universe. With the rapid expansion of the Universe, inhomogeneity in density develops which is magnified due to the action of gravity and consequently structures are formed. This adiabatic sound carries valuable information of the early Universe. The square of adiabatic sound speed is given as

$$c_s^2 = \left(\frac{\mathrm{d}P}{\mathrm{d}\rho}\right)_S \tag{22}$$

which in terms of EoS parameter ω can be expressed as

$$c_s^2 = \frac{\omega + B}{2}. \tag{23}$$

In terms of parameters of the model, sound speed can be expressed as

$$c_s^2 = B - \frac{B+1}{2[1 + (\epsilon/V)^{\frac{B+1}{2}}]}. \tag{24}$$

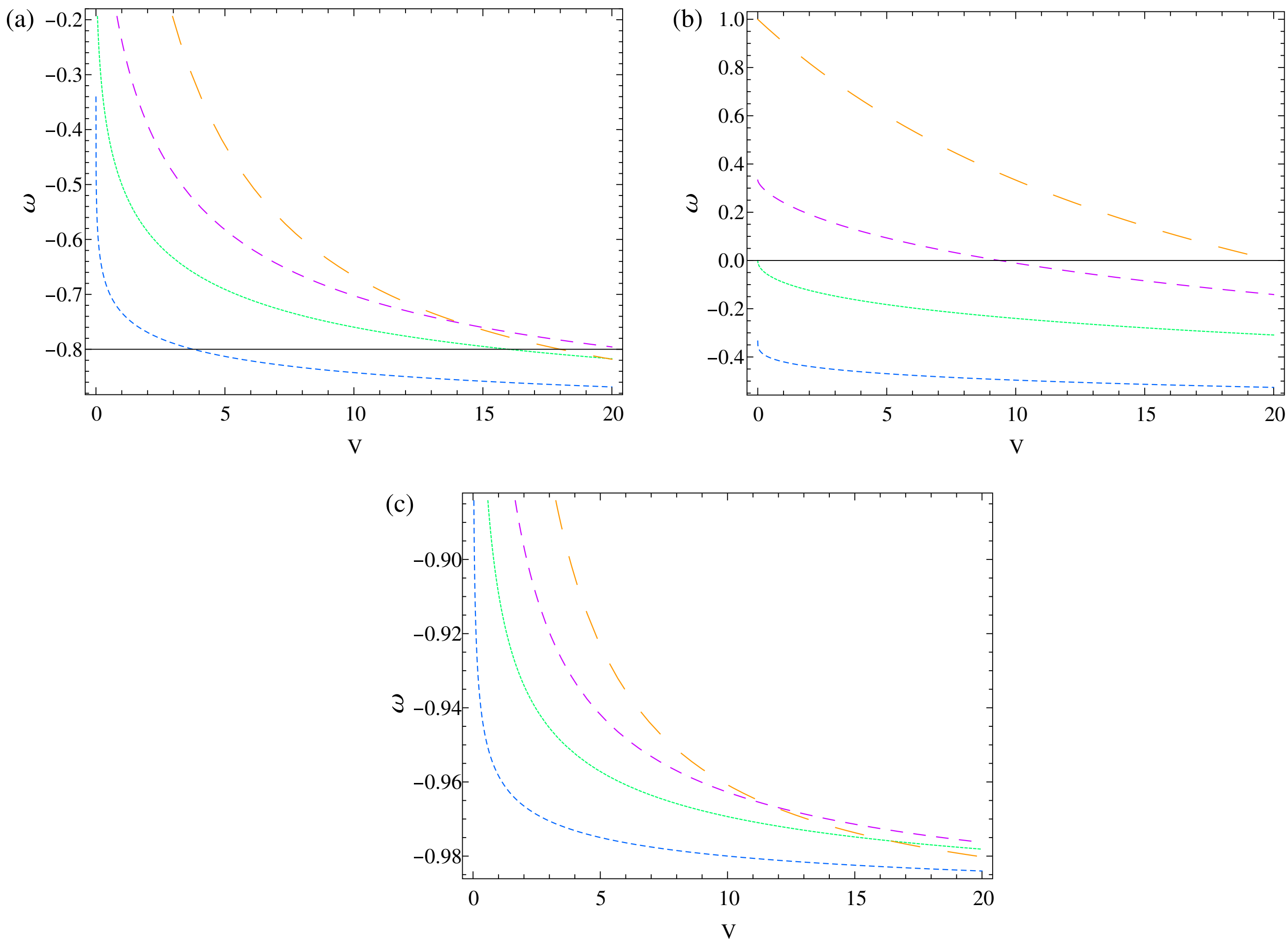

Figure 3. Plot of EoS (ω) vs. volume V in various EU models for (**a**) $A = 1$, $C = 1$, (**b**) $A = 1$, $C = 10$ and (**c**) $A = 10$, $C = 1$. Here dotted line stands for $B = -\frac{1}{3}$, solid line for $B = 0$, close-dashed line for $B = \frac{1}{3}$ and wide-dashed line for $B = 1$ model.

For (i) small volume, $V \ll \epsilon$, $c_s^2 = B$. So, $c_s^2 > 0$ for positive values of B and (ii) for large volume, $V \gg \epsilon$, $c_s^2 = \frac{B}{2} - \frac{1}{2}$. So, $c_s^2 = 0$ for $B = 1$ and c_s^2 is negative for other values of B. So, the signs of structure formation ($c_s^2 > 0$) is noticed in the early Universe for models having positive values of B. In late time, signs of structure formation are not noticed even for $B = 1$ model.

7. Thermodynamic stability of the fluid

The partial derivative of P with V is expressed as

$$\left(\frac{\partial P}{\partial V}\right)_S = -\left(\frac{A}{B+1}\right)^2 \frac{(B+1)}{2V}\left(\frac{\epsilon}{V}\right)^{\frac{B+1}{2}} \times\left[B - 1 + 2B\left(\frac{\epsilon}{V}\right)^{\frac{B+1}{2}}\right], \quad (25)$$

which, in terms of pressure is given as

$$\left(\frac{\partial P}{\partial V}\right)_S = \frac{P(N/V)\left[B - 1 + 2B\,(\epsilon/V)^N\right]}{[1 + (\epsilon/V)^{-N}][1 - B\,(\epsilon/V)^N]}, \quad (26)$$

where

$$N = \frac{B+1}{2}.$$

(i) At early times for small volume $V \ll \epsilon$,

$$\left(\frac{\partial P}{\partial V}\right)_S \approx -\frac{P(B+1)}{V}.$$

In early time, pressure can be approximated as $P = B\rho$ and so

$$\left(\frac{\partial P}{\partial V}\right)_S \approx -\frac{B\rho(B+1)}{V}.$$

(ii) For large volume $V \gg \epsilon$,

$$\left(\frac{\partial P}{\partial V}\right)_S \approx 0.$$

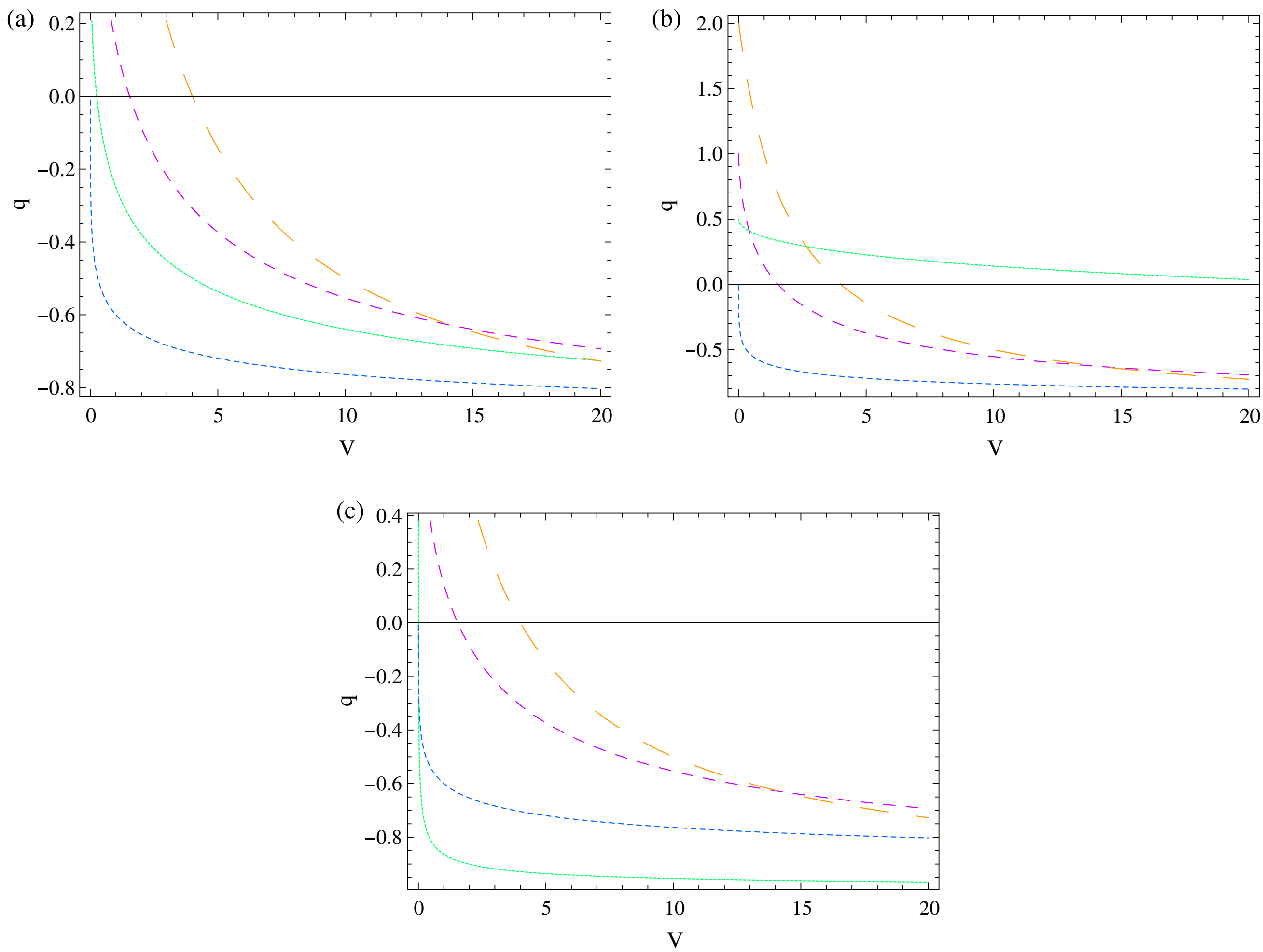

Figure 4. Plot of deceleration parameter q vs. volume V in various EU models for (**a**) $A = 1, C = 1$, (**b**) $A = 1, C = 10$ and (**c**) $A = 10$, $C = 1$. Here dotted line stands for $B = -\frac{1}{3}$, solid line for $B = 0$, close-dashed line for $B = \frac{1}{3}$ and wide-dashed line for $B = 1$ model.

The temperature of the fluid is given by the equation $T = (\partial U/\partial S)_V$ [37], which can be written as

$$T = \left(\frac{\partial U}{\partial C}\right)_V \left(\frac{\partial C}{\partial S}\right)_V .$$

So, temperature T becomes

$$T = 2V^{\frac{1-B}{2}} \left[\frac{A}{B+1} + \frac{C}{V^{\frac{B+1}{2}}}\right] \left(\frac{\partial C}{\partial S}\right)_V . \tag{27}$$

If C is assumed to be a universal constant, then $(\mathrm{d}C/\mathrm{d}S) = 0$, and temperature will be zero for any value of its volume and pressure. Therefore, to discuss the thermodynamic stability of the fluid of the EU whose temperature varies during expansion, it is necessary to assume that the derivative $(\mathrm{d}C/\mathrm{d}S) \neq 0$. There is no *a priori* knowledge of the functional dependence of C. From physical considerations, however, one can understand that this function gives positive temperature and cooling along an adiabatic expansion and so $(\mathrm{d}C/\mathrm{d}S) > 0$ is chosen [32,35,36].

From the dimensional analysis

$$[U] = \left(\frac{[C]}{[V^{B/2}]}\right)^2 . \tag{28}$$

Since

$$[U] = [T][S], \tag{29}$$

the constant parameter C is given by

$$[C] = [U^{1/2}][V^{B/2}] = [T^{1/2}][S^{1/2}][V^{B/2}]. \tag{30}$$

Alternately, in terms of τ and υ it is given as

$$C = (\tau \upsilon^B)^{1/2} S^{1/2}. \tag{31}$$

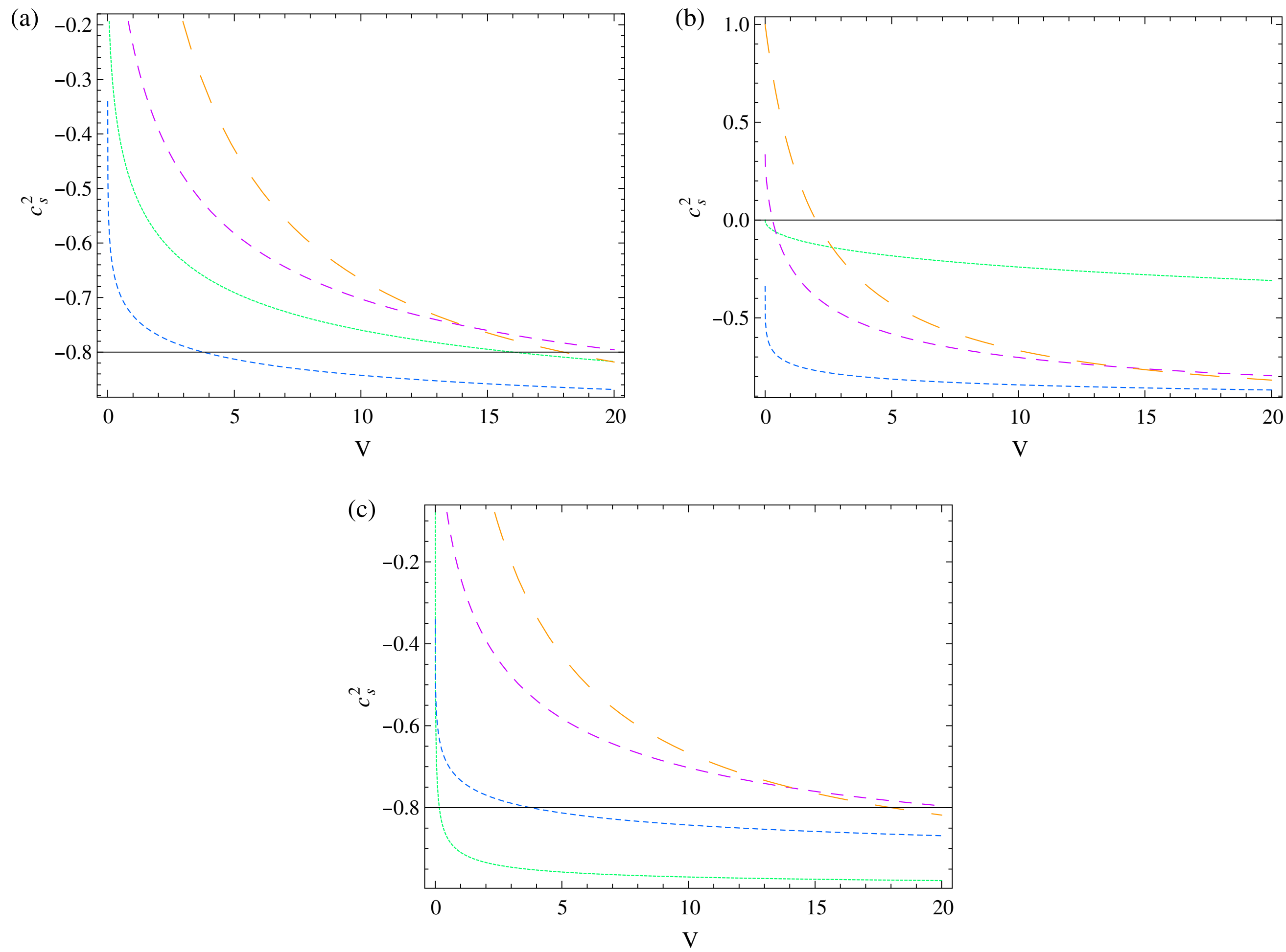

Figure 5. Plot of the square of adiabatic sound speed c_s^2 vs. volume V in various EU models for (**a**) $A = 1, C = 1$, (**b**) $A = 1$, $C = 10$ and (**c**) $A = 10$, $C = 1$. Here dotted line stands for $B = -\frac{1}{3}$, solid line for $B = 0$, close-dashed line for $B = \frac{1}{3}$ and wide-dashed line for $B = 1$ model.

So,

$$\left(\frac{\partial C}{\partial S}\right)_V = (\tau \upsilon^B)^{1/2} \frac{1}{2} S^{-1/2}. \tag{32}$$

The temperature takes the form

$$T = V^{\frac{1-B}{2}} \rho^{1/2} (\tau \upsilon^B)^{1/2} S^{-1/2}. \tag{33}$$

This temperature can be simplified to

$$T = \frac{\tau \upsilon^B}{V^B} \left[1 + \left(\frac{V}{\epsilon}\right)^{\frac{B+1}{2}} \right]. \tag{34}$$

Using the value of C in the expression of energy density in eq. (33), entropy is given by

$$S = \left(\frac{A V^{\frac{B+1}{2}}}{B+1}\right)^2 \left(\frac{1}{\tau \upsilon^B}\right) \frac{1}{(1 - \frac{T V^B}{\tau \upsilon^B})^2}. \tag{35}$$

The expression for C may be obtained using eqs (35) and (31) as

$$C = \left(\frac{A V^{\frac{B+1}{2}}}{B+1}\right) \frac{1}{(1 - \frac{T V^B}{\tau \upsilon^B})}. \tag{36}$$

The entropy S can also be expressed as

$$S = \left(\frac{A}{B+1}\right)^2 \left(\frac{V^{1/2}}{T^{1/2}}\right)^2 \left(\frac{T V^B}{\tau \upsilon^B}\right) \frac{1}{(1 - \frac{T V^B}{\tau \upsilon^B})^2}. \tag{37}$$

For positive and finite entropy, one must consider $0 < T V^B < \tau \upsilon^B$, where they individually satisfies the limits $0 < T < \tau$ and $\upsilon < V < \infty$. Here τ represents the maximum temperature, whereas υ represents the minimum volume attainable. From eq. (37)

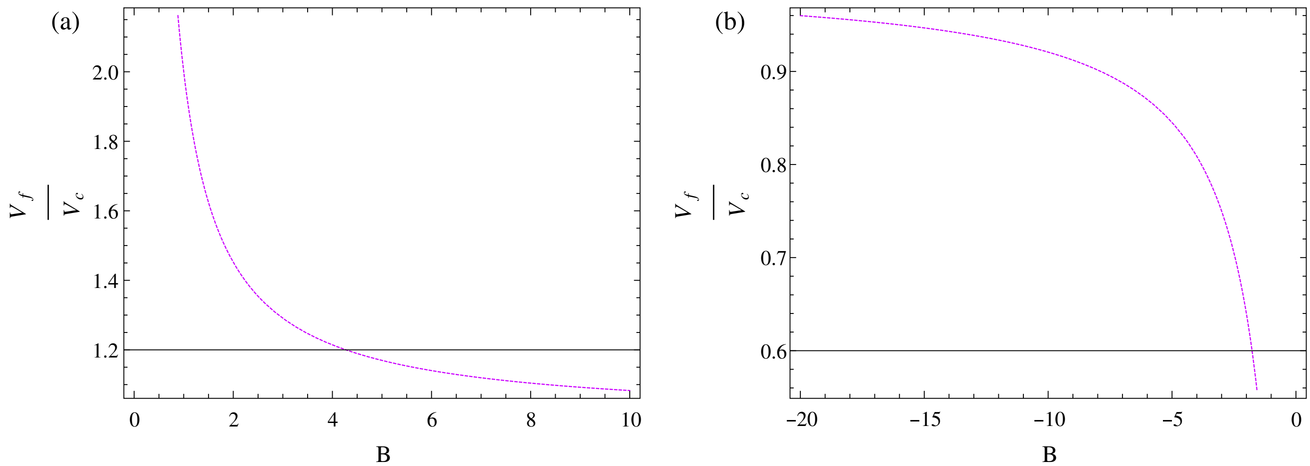

Figure 6. Plot of V_f/V_c with B. (**a**) For B between 0 and 10 and (**b**) for B between −20 and 0.

for $T \to 0$, entropy becomes

$$S = \left(\frac{A}{B+1}\right)^2 \left(\frac{V^{B+1}}{\tau v^B}\right). \tag{38}$$

In order to hold third law of thermodynamics, $S \to 0$ for $T \to 0$, two conditions arise in this model, which are $B < -1$ or $A = 0$. Clearly, third law satisfies for $B < -1$, as the other condition $A = 0$, is not permissible in the EU scenario.

The specific heat at constant volume [38] is defined as

$$c_v = T\left(\frac{\partial S}{\partial T}\right)_V \tag{39}$$

which becomes

$$c_v = 2\left(\frac{AV^{\frac{B+1}{2}}}{B+1}\right)^2 \frac{TV^B}{(\tau v^B)^2} \frac{1}{(1-\frac{TV^B}{\tau v^B})^3}. \tag{40}$$

It can be expressed as

$$c_v = 2\left(\frac{A}{B+1}\right)^2 \left(\frac{V^{1/2}}{T^{1/2}}\right)^2 \left(\frac{TV^B}{\tau v^B}\right)^2 \frac{1}{(1-\frac{TV^B}{\tau v^B})^3}. \tag{41}$$

For $0 < TV^B < \tau v^B$, $0 < T < \tau$ and $v < V < \infty$, c_v is positive. So, the entropy and specific heat are positive for the same conditions of the model. Here the specific heat at constant volume tends to vanish at low temperature, i.e., $c_v \to 0$, for $T \to 0$. Positivity of specific heat is strongly desirable in relativity. In a recent paper, Luongo and Quevedo [39] argued that in FRW Universe, a model, which is discussed in [36], a negative c_v and a vanishingly small specific heat at constant pressure c_P are compatible with observational data. It also overcomes the fine-tuning and the coincidence problems of the ΛCDM model.

8. Thermal equation of state and internal energy equation

The internal energy U in terms of the parameters of the model is given as

$$U = \left[\frac{AV^{1/2}}{B+1} + \frac{C}{V^{B/2}}\right]^2 \tag{42}$$

which becomes

$$U = V\left(\frac{A}{B+1}\right)^2 \left[1 - \frac{1}{(1-\frac{TV^B}{\tau v^B})}\right]^2. \tag{43}$$

The pressure is given by

$$P = \rho\left[B - \frac{B+1}{[1-\frac{1}{1-\frac{TV^B}{\tau v^B}}]}\right], \tag{44}$$

where density ρ is given as

$$\rho = \left(\frac{A}{B+1}\right)^2 \left[1 - \frac{1}{1-\frac{TV^B}{\tau v^B}}\right]^2. \tag{45}$$

So, EoS parameter is given by

$$\omega = \left[B - \frac{B+1}{[1-\frac{1}{1-\frac{TV^B}{\tau v^B}}]}\right]. \tag{46}$$

The thermal stability conditions to be satisfied by the pressure of the fluid in this EU model are $(\partial P/\partial V)_T <$

0 and $(\partial P/\partial V)_S < 0$. In the adiabatic scenario, these conditions are satisfied for $B = \frac{1}{3}, 1$ in the EU model.

The internal energy equation [40] is given as

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P. \tag{47}$$

This equality can be verified by the internal energy and pressure equations (43) and (44). The expression of U and P given by (43) and (44) are derived by assuming an ansatz about the constant C to be a function of entropy, i.e., $C = C(S)$ (31). Derivation of eq. (47) has shown that ansatz is viable in EU model (in line with [32,35,36]).

9. Pressure–volume relation of the fluid

For (i) $V \ll \epsilon$, $(\epsilon/V) = \infty$, and so $P \approx B\rho$. In this case, pressure and energy density are very high. Now using eq. (34) temperature can be written as

$$T = \frac{\tau v^B}{V^B}. \tag{48}$$

At the early stage, $V \to v$ (minimum volume) corresponds to $T \to \tau$ (maximum temperature). Using eq. (48) and $\rho = U/V = TS/V$, one gets

$$\rho = S\frac{\tau v^B}{V^{B+1}}. \tag{49}$$

So,

$$UV^B = \rho V^{B+1} = S\tau v^B. \tag{50}$$

Now at this stage, $P \approx B\rho$, i.e., $PV \approx BU$, which gives $PV^{B+1} = S\tau v^B$ (by using eq. (50)). When the entropy is constant (adiabatic process), this relation gives PV^{B+1} =constant. So, for this fluid $\gamma = (c_p/c_v) = B + 1$. The EoS can be written as $P = (\gamma - 1)\rho$, where $\gamma = 1 + B$. From the pressure–volume relation $PV^{4/3}$ = constant for $B = \frac{1}{3}$; PV^2 = constant for $B = 1$; $PV^{2/3}$ = constant for $B = -\frac{1}{3}$ and PV^1 = constant for $B = 0$.

For (ii) large volume $V \gg \epsilon$, $(\epsilon/V) = 0$. Using eq. (16) one gets

$$P = -\rho \tag{51}$$

(as $\omega = -1$). From eq. (8) density becomes

$$\rho \approx \left(\frac{A}{B+1}\right)^2. \tag{52}$$

From eqs (51) and (52) one gets

$$P \approx -\left(\frac{A}{B+1}\right)^2. \tag{53}$$

The pressure is negative and constant. This is the condition of dark energy.

10. Conclusion

It is generally accepted that the Universe is now accelerating. This discovery of late acceleration of the Universe led to the emergence of the DE models. A very general exotic fluid with EoS ($P = B\rho - A\rho^{1/2}$) is considered, whose cosmological implications and thermodynamical stability are studied in this paper.

The standard prescription of the thermodynamic stability criteria, (i) $(\partial P/\partial V)_S < 0$, $(\partial P/\partial V)_T < 0$ and (ii) $c_v > 0$ are applied on the EU model. In figures 1 and 2 pressure P and $(\partial P/\partial V)_S$ are plotted with volume V for different possible combinations of constants (A, C). It is found that at early epoch, positivity of pressure, i.e., $P > 0$ and negativity of $(\partial P/\partial V)_S$, are satisfied for the models with $B = \frac{1}{3}, 1$. In the plot of ω, q with volume V in figures 3 and 4, the desired evolutionary trajectory is noticed for $B = \frac{1}{3}, 1$ values. Positivity of squared adiabatic sound speed, which signifies stability of the models, is noticed for $B = \frac{1}{3}, 1$ in figure 5. From eq. (41), it is clear that c_v satisfies the relation, $c_v \geq 0$ for $T \geq 0$ for all values of B in the EU models except for $B = -1$.

The EoS of this model at early stage becomes $P = B\rho$, which is positive for $B = \frac{1}{3}, 1$ in the EU model. At later stage, it reduces to equation

$$P \approx -\left(\frac{A}{B+1}\right)^2,$$

a negative pressured dark energy. So, thermal stability and positivity of squared sound speed show that the EU models are stable for $B = \frac{1}{3}, 1$. $(P-V)$ relation at early times in an isentropic EU model also indicates that $B = \frac{1}{3}, 1$ values are the most suitable ones.

Cosmological dynamics are governed by two characteristic volumes, namely, critical volume V_c and flip volume V_f. From physical considerations one should get $V_f > V_c$, which is satisfied for positive values of B, i.e., for $(\frac{1}{3}, 1)$ (figure 6). In order for the deceleration–acceleration transition to occur, only a negative pressure is not the necessary and sufficient condition, else, magnitude of its pressure should also be less than $\rho/3$, which is satisfied for $B = \frac{1}{3}, 1$.

Third law of thermodynamics, $S \to 0$ for $T \to 0$, is satisfied in the EU model for $B < -1$ (other solution, $A = 0$, is contrary to EU EoS), i.e., any of the four discrete value of B $(= 0, -\frac{1}{3}, \frac{1}{3}, 1)$ does not satisfy this third law. It can be noted here that in this model, $T \to 0$ for $V \to \infty$ (eq. (48)).

In the thermal EoS, entropy and specific heat equation, temperature and volume satisfy the limits, $0 < T < \tau$, $\upsilon < V < \infty$. In early times, $T \to \tau$ for $V \to \upsilon$ and at late times, $T \to 0$ for $V \to \infty$. It very much suits an EU model, as its volume does not vanish at any time in past (it will attain some lowest finite volume), in fact, at its lowest volume υ, temperature attains its highest value τ.

In order to determine thermal equation of state $P = P(T, V)$ and temperature (§7) analytically, the ansatz $C = C(S)$ and $(\mathrm{d}C/\mathrm{d}S) > 0$ is assumed. The assumption $C = C(S)$ is an ansatz, whose viability has been tested via the derivation of internal energy equation (47). The reasoning behind the assumption is that, the field equations are connected with nature and nature is strongly connected with entropy (S) and as a result, the constant C may be related with S. Although V_c and V_f now became function of entropy (S), the relative value of V_f/V_c remained unchanged.

Finally, here cosmological and thermodynamical behaviour of the EU models are studied only on the basis of theoretical understandings. The results of these study mostly favours positive values of $B = \frac{1}{3}$, 1 in the EU model.

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