Pramana – J. Phys. (2021) 95:85
https://doi.org/10.1007/s12043-021-02103-2

Symmetries and integrability of the modified Camassa–Holm equation with an arbitrary parameter

A DURGA DEVI[1], K KRISHNAKUMAR[2],*, R SINUVASAN[3] and P G L LEACH[4]

[1]Department of Physics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India
[2]Department of Mathematics, Srinivasa Ramanujan Centre, SASTRA Deemed to be University, Kumbakonam 612 001, India
[3]Department of Mathematics, VIT-AP University, Amaravathi 522 237, India
[4]Department of Mathematics, Durban University of Technology, P.O. Box 1334, Durban 4000, Republic of South Africa
*Corresponding author. E-mail: krishapril09@gmail.com

MS received 5 May 2020; revised 12 August 2020; accepted 30 December 2021

Abstract. We study the symmetry and integrability of a modified Camassa–Holm equation (MCH), with an arbitrary parameter k, of the form

$$u_t + k(u - u_{xx})^2 u_x - u_{xxt} + (u^2 - {u_x}^2)(u_x - u_{xxx}) = 0.$$

The commutator table and adjoint representation of the symmetries are presented to construct one-dimensional optimal system. By using the one-dimensional optimal system, we reduce the order or number of independent variables of the above equation and also we obtain interesting novel solutions for the reduced ordinary differential equations. Finally, we apply the Painlevé test to the resultant nonlinear ordinary differential equation and it is observed that the equation is integrable.

Keywords. Modified Camassa–Holm equation; symmetry; Painlevé test; integrability.

PACS Nos 12.60.Jv; 12.10.Dm; 98.80.Cq; 11.30.Hv

1. Introduction

Linear and nonlinear partial differential equations have encompassed the landscape of many disciplines such as chemical, physical, biological sciences, engineering and mathematics due to their potential ability to provide adequate understanding of the underlying systems. There are many wave equations in engineering and physics such as the Korteweg–de-Vries (KdV) equation, nonlinear coupled KdV equations, modified Camassa–Holm (MCH) equation and so on [1–11] that are quite helpful and relevant. In general, the closed-form solutions of differential equations are very hard to achieve in most instances.

Numerous systematic techniques have been developed to find solutions of differential equations in recent years. An overview of the recent developments made in identifying/generating systems of Liénard-type nonlinear oscillators exhibiting isochronous properties is presented in [12,13]. Especially, Lie symmetry analysis [14–23], Painlevé analysis [24–29], Adomian decomposition method [30,31], variational iteration method, homotopy analysis method, homotopy perturbation method and variational homotopy perturbation method have been utilised successfully to solve and study the nature of integrability of partial differential equations [32]. Among these methods, symmetry analysis and Painlevé analysis have been used by researchers effectively due to their systematic procedure for finding the solution of the given differential equations [22,33,34].

Recently, researchers have been inspired to study the integrability and solutions of various Camassa–Holm (CH) types of equations with cubic or higher-order nonlinearities. Olver and Rosenau [35] and Fuchssteiner [8] derived the well-known modified Camassa-Holm

Published online: 19 May 2021

(MCH) equation

$$y_t + uy_x + 2u_x y = 0,\ y = u - u_{xx} \tag{1}$$

by employing the tri-Hamiltonian duality approach to the bi-Hamiltonian representation of the MKDV equation. The functions u and y represent the velocity of the fluid and its potential density [36]. The MCH equation is a water-wave equation and is a suitable approximation of the incompressible irrotational Euler system. Consequently, Qiao [36] discussed the nature of integrability and the structure of solutions of the MCH equation. In some literature, the MCH equation is otherwise called as FORQ equation [37,38].

The MCH equation is completely integrable [35]. It has a bi-Hamiltonian structure and also admits a Lax pair [39] and hence may be solved by the inverse scattering transform method. The MCH equation can be viewed as a cubic extension of the well-known CH equation which was proposed as a model to describe the unidirectional propagation of shallow water waves [40–42] and axially symmetric waves in hyperelastic rods [43].

The purpose of this work is to study the symmetry and integrability of MCH equation with an arbitrary parameter. The equation is given by

$$\begin{aligned} &u_t + k(u - u_{xx})^2 u_x - u_{xxt} \\ &\quad +(u^2 - {u_x}^2)(u_x - u_{xxx}) = 0, \end{aligned} \tag{2}$$

where the parameter $k \in R$ characterises the magnitude of the linear dispersion.

We split this article into the following five sections. In §2 the general procedure of Lie's theory is given. In §3 we discuss Lie point symmetries and reductions of the order of eq. (2). Indeed, we reduce the order of ordinary or partial differential equations by using the symmetry. Hence, Lie symmetries play major roles in the discussion. In §4, we study the Painlevé analysis for eq. (2) and we are able to prove that eq. (2) is integrable. Section 5 gives Conclusion.

2. Lie's theory

Suppose that the given equation is of the form

$$\chi(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, u_{xxx}, \ldots) = 0, \tag{3}$$

where t, x are independent variables and u is a dependent variable. For each of the variables, the infinitesimal point transformations are given as follows:

$$\begin{aligned} \tilde{t}(t, x,\ \epsilon) &= t + \epsilon\xi^1(t, x) + \circ(\epsilon^2) = t + \epsilon X t + \circ(\epsilon^2), \\ \tilde{x}(t, x,\ \epsilon) &= x + \epsilon\xi^2(t, x) + \circ(\epsilon^2) = x + \epsilon X x + \circ(\epsilon^2), \\ \tilde{u}(t, x,\ \epsilon) &= u + \epsilon\eta^1(t, x) + \circ(\epsilon^2) = u + \epsilon X u + \circ(\epsilon^2), \end{aligned}$$

where X is called the 'infinitesimal generator' which is denoted by

$$X = \xi^1(t, x)\partial_t + \xi^2(t, x)\partial_x + \eta^1(t, x)\partial_u.$$

Based on the theory, the invariant condition for eq. (3) is given by

$$\chi(t, x, u) = \chi(\tilde{t}, \tilde{x}, \tilde{u}).$$

It is well known that one can reduce the order of differential equations as well as the number of independent variables by using the infinitesimal generator which is known as a symmetry of eq. (3).

Therefore, if X is a Lie point symmetry for equation $\chi \equiv 0$, then the following condition is true:

$$X^{[n]}\chi = \lambda\chi|_{\chi=0},$$

where λ is an arbitrary function and $X^{[n]}$ is the nth extension of X in the jet space.

3. The symmetries of the modified Camassa–Holm equation

Qiao discussed eq. (2) which is integrable only when $k = 2$ [36]. He also suggested to examine the integrability of the above equation for all possible values of k. Therefore, we propose a more general form of eq. (2) by taking k as $k(t)$ which is a function of t. Hence, the MCH equation with arbitrary function, $k(t)$, as discussed in [36,44], can be represented as

$$\begin{aligned} &u_t + k(t)(u - u_{xx})^2 u_x - u_{xxt} \\ &\quad +(u^2 - {u_x}^2)(u_x - u_{xxx}) = 0. \end{aligned} \tag{4}$$

When $k(t)$ is an arbitrary function of t, eq. (4) admits a symmetry ∂_x. The corresponding characteristic equation for ∂_x is given by

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}.$$

The solution of the auxiliary equation gives similarity variables as $\alpha = t$ and $u = f(t)$. This leads to the trivial solution $u(x, t) = C$ to eq. (4). When $k(t) = k$, an arbitrary parameter, the symmetries of eq. (2) are

$$\mathbf{X}_1 = \partial_t \tag{5}$$

$$\mathbf{X}_2 = \partial_x \tag{6}$$

$$\mathbf{X}_3 = 2t\partial_t - u\partial_u. \tag{7}$$

Throughout this work the Mathematica add-on Sym [45–47] is used to compute the symmetries. As Lie symmetries can be used for the classification of differential equations, eq. (2) admits the Lie algebra $\{2A_1 \oplus_s A_1\}$ which is identified from the Morozov–Mubarakzyanov

Table 1. Commutator table.

$[\mathbf{X}_I, \mathbf{X}_J]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$
$\mathbf{X}_1$	0	0	$2\mathbf{X}_1$
$\mathbf{X}_2$	0	0	0
$\mathbf{X}_3$	$-2\mathbf{X}_1$	0	0

Table 2. Adjoint representation.

$\mathrm{Ad}[e^{\epsilon \mathbf{X}_I}]\mathbf{X}_J$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$
$\mathbf{X}_1$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3 - 2\epsilon\mathbf{X}_1$
$\mathbf{X}_2$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$
$\mathbf{X}_3$	$e^{2\epsilon}\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$

classification scheme [48–51] based on the Lie brackets of the Lie symmetries of eq. (2) as given in table 1.

Olver [15] discussed the adjoint representation and optimal system which gives unique possible combination of symmetries to perform the reductions of a differential equation. Later, Zhao and Han [52,53] used a simple way to find the optimal system. Hence, the adjoint representations of the symmetries, based on the formula $\mathrm{Ad}\left[e^{\epsilon \mathbf{X}_I}\right]\mathbf{X}_J = X_J - \epsilon[X_I, X_J] + \frac{\epsilon^2}{2}[X_I, [X_I, X_J]] - \cdots$, are given in table 2.

According to their idea, the commutator table 1 and adjoint representation table 2 provide the optimal system of eq. (2) as follows: Let the generic symmetry vector field be taken as $X = l_1X_1 + l_2X_2 + l_3X_3$.

Case 1. $l_3 = 0$

Subcase 1.1. If $l_1 \neq 0$, then the adjoint representation gives that X_1 and X_2 are linearly independent symmetries and the generic symmetry vector field becomes the linear combination of X_1 and X_2. That is, $X_1 + cX_2$.

Subcase 1.2. If $l_1 = 0$, then the generic symmetry vector field is X_2.

Subcase 1.3. If $l_2 = 0$, then the generic symmetry vector field is X_1.

Case 2. $l_3 \neq 0$

Subcase 2.1. If $l_2 = 0$, then the adjoint representation gives that $X_1 = X_3/2\epsilon$. Therefore, the generic symmetry vector field becomes X_3.

Consequently the one-dimensional optimal systems are

$$X_1,\ X_2,\ X_3 \text{ and } X_1 + CX_2.$$

3.1 *Reductions based on optimal system*

3.1.1 *Reduction with* X_1. This gives the static solution of eq. (2) through the invariant function which is obtained from symmetry X_1. The corresponding characteristic equation for X_1 is given by

$$\frac{\mathrm{d}t}{1} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{0}.$$

The solution of the characteristic equation gives the similarity variables as

$$\xi = x \text{ and } u = U(x).$$

Thus, the invariant solution of eq. (2) is $u(x,t) = U(x)$. This leads to an ordinary differential equation

$$(U''' - U')(U^2 - U'^2) - kU'(U'' - U)^2 = 0. \tag{8}$$

The above equation can be rewritten as

$$\frac{(U''' - U')}{(U'' - U)} + \frac{k}{2}\frac{2U'(U - U'')}{(U^2 - U'^2)} = 0, \tag{9}$$

where $'$ represents differentiation with respect to x. Integrating eq. (9) once, we get

$$(U'' - U)(U^2 - U'^2)^{k/2} + I_1 = 0, \tag{10}$$

where I_1 is a constant of integration.

As eq. (10) is an autonomous equation, it has the trivial symmetry, ∂_x. The corresponding canonical variable is

$$U' = V(U). \tag{11}$$

This gives $U'' = VV'$. Therefore eq. (10) becomes

$$(VV' - U)(U^2 - V^2)^{k/2} + I_1 = 0, \tag{12}$$

where $'$ denotes the derivative with respect to U and also eq. (12) can be written as

$$(U^2 - V^2)^{k/2}d(U^2 - V^2) - 2I_1 = 0. \tag{13}$$

The solution of eq. (13) is given by

$$(U^2 - V^2)^{(k/2)+1} - 2I_1U - I_2 = 0. \tag{14}$$

This implies

$$V = \pm\sqrt{U^2 - (2I_1U + I_2)^{2/(k+2)}}. \tag{15}$$

From eq. (11) the above equation can be written as

$$\frac{\mathrm{d}U}{\mathrm{d}x} = \pm\sqrt{U^2 - (2I_1U + I_2)^{2/(k+2)}}. \tag{16}$$

This yields

$$\int \left(\frac{\mathrm{d}U}{\sqrt{U^2 - (2I_1U + I_2)^{2/(k+2)}}}\right) = \pm x + I_3. \tag{17}$$

3.1.2 *Reduction with* X_2. It gives trivial solution $u(x,t) = c$ because the canonical transformation based on X_2 is $u(x,t) = c$.

3.1.3 *Reduction with* X_3. It gives a scaling solution through the invariant solution $u(x,t) = t^{-1/2}U(x)$ which is given by X_3. With the use of $u(x,t) = t^{-1/2}U(x)$, eq. (2) can be reduced to the equation

$$2(U''' - U')(U^2 - U'^2) \\ -(U'' - U)(1 - 2kU'(U'' - U)) = 0. \tag{18}$$

This equation can be rewritten as

$$\frac{(U''' - U')}{(U'' - U)} + \frac{k}{2}\frac{2U'(U - U'')}{(U^2 - U'^2)} - \frac{1}{2(U^2 - U'^2)} = 0, \tag{19}$$

where $'$ represents the differentiation with respect to x. Integrating eq. (19) once, we get

$$(U''-U)(U^2-U'^2)^{k/2}+I_1 \exp\left[\frac{1}{2}\int \frac{\mathrm{d}x}{U^2 - U'^2}\right] = 0, \tag{20}$$

where I_1 is a constant of integration.

As eq. (20) is an autonomous equation, it has the trivial symmetry ∂_x. The corresponding canonical variable is given by

$$U' = V(U). \tag{21}$$

$$U'' = VV'. \tag{22}$$

Therefore, (20) becomes

$$(U - VV')(U^2 - V^2)^{k/2} \\ -I_1 \exp\left[\frac{1}{2}\int \frac{\mathrm{d}U}{V(U^2 - V^2)}\right] = 0, \tag{23}$$

where $'$ denotes the derivative with respect to U and eq. (23) can be written as

$$(U^2 - V^2)^{k/2}\mathrm{d}(U^2 - V^2) \\ -2I_1 \exp\left[\frac{1}{2}\int \frac{\mathrm{d}U}{V(U^2 - V^2)}\right] = 0. \tag{24}$$

Integrating eq. (24) we have

$$(U^2 - V^2)^{(k/2)+1} \\ -2I_1 \int \exp\left[\frac{1}{2}\int \frac{\mathrm{d}U}{V(U^2 - V^2)}\right]\mathrm{d}U - I_2 = 0. \tag{25}$$

This provides

$$V^2 = U^2 \\ -\left(2I_1 \int \exp\left[\frac{1}{2}\int \frac{\mathrm{d}U}{V(U^2-V^2)}\right]\mathrm{d}U + I_2\right)^{2/(k+2)}. \tag{26}$$

By using eq. (21) one can find U by solving the following equation:

$$\left(\frac{\mathrm{d}U}{\mathrm{d}x}\right)^2 = U^2 - \left(2I_1 \int \frac{\mathrm{d}U}{\mathrm{d}x} \times \exp\left[\frac{1}{2}\int \frac{\mathrm{d}x}{(U^2 - (\mathrm{d}U/\mathrm{d}x)^2)}\right]\mathrm{d}x + I_2\right)^{2/(k+2)} \tag{27}$$

Now we discuss the travelling-wave solution of eq. (2) by taking the linear combination of $\mathbf{X}_1$ and $\mathbf{X}_2$. We take the linear combination of these two symmetries as $\mathbf{X}_4 = \partial_t + C\partial_x$. The corresponding canonical variables are $r = x - Ct$ and $u(x,t) = Q(r)$. Based on these canonical variables, eq. (2) can be reduced to

$$(C - Q^2 + Q'^2)(Q''' - Q') \\ +kQ'(Q'' - Q)^2 = 0, \tag{28}$$

where Q is the function of the new independent variable r. The symmetry of eq. (28) is ∂_r for which the canonical variables are

$$Q(r) = Q \quad \text{and} \quad Q' = W(Q). \tag{29}$$

Therefore, Q'' and Q''' become

$$Q'' = WW' \quad \text{and} \quad Q''' = W(WW'' + W'^2). \tag{30}$$

Based on these canonical variables (29) and eq. (30), eq. (28) can be reduced to

$$(C - Q^2 + W^2)(WW'' + W' - 1) \\ +k(WW' - Q)^2 = 0, \tag{31}$$

where W is a function of Q. When we find the symmetries of eq. (31), we arrive at the following two possible cases:

Case 1

If $k \neq -2$, then the symmetries of eq. (31) are

$$\Gamma_1 = \partial_Q + \frac{Q}{W}\partial_W$$

$$\Gamma_2 = \frac{(C - Q^2 + W^2)^{-k/2}}{W}\partial_W$$

$$\Gamma_3 = \frac{Q(C - Q^2 + W^2)^{-k/2}}{W}\partial_W$$

$$\Gamma_4 = \frac{(C - Q^2 + W^2)}{(k+2)W}\partial_W$$

$$\Gamma_5 = 2Q\partial_Q + \frac{(C + (3+2k)Q^2 + W^2)}{(k+2)W}\partial_W$$

$$\Gamma_6 = Q^2\partial_Q + \frac{Q(C + (k+1)Q^2 + W^2)}{(k+2)W}\partial_W$$

$$\Gamma_7 = \frac{(C - Q^2 + W^2)^{(k+2)/2}}{k+2}\partial_Q \\ + \frac{Q(C - Q^2 + W^2)^{(k+2)/2}}{(k+2)W}\partial_W$$

$$\Gamma_8 = \frac{Q(C-Q^2+W^2)^{(k+2)/2}}{k+2}\partial_Q + \frac{(C-Q^2+W^2)^{(k+2)/2}(C+(k+1)Q^2+W^2)}{(k+2)^2 W}\partial_W.$$

Case 2

If $k = -2$, then the symmetries of eq. (31) are

$$\Gamma_1 = \partial_Q + \frac{Q}{W}\partial_W$$

$$\Gamma_2 = \frac{(C-Q^2+W^2)}{W}\partial_W$$

$$\Gamma_3 = \frac{Q(C-Q^2+W^2)}{W}\partial_W$$

$$\Gamma_4 = \log[C-Q^2+W^2]\partial_Q + \frac{Q\log[C-Q^2+W^2]}{W}\partial_W$$

$$\Gamma_5 = \frac{(C-Q^2+W^2)\log[C-Q^2+W^2]}{W}\partial_W$$

$$\Gamma_6 = 4Q\partial_Q + \frac{1}{W}(4Q^2+(C-Q^2+W^2) \times \log[C-Q^2+W^2])\partial_W$$

$$\Gamma_7 = 2Q^2\partial_Q + \frac{1}{W}(2Q^3+Q(C-Q^2+W^2) \times \log[C-Q^2+W^2])\partial_W$$

$$\Gamma_8 = 2Q\log[C-Q^2+W^2]\partial_Q + \frac{1}{W}(\log[C-Q^2+W^2](2Q^2 + (C-Q^2+W^2)\log[C-Q^2+W^2]))\partial_W.$$

As eq. (31) has eight Lie point symmetries, which is the maximal symmetry, for both $k \neq -2$ and $k = 2$ cases, based on Lie's theory, eq. (31) can be linearised as follows:

Equation (31) can be rewritten as

$$\frac{(WW''+W'-1)}{(WW'-Q)} + k\frac{(WW'-Q)}{(C-Q^2+W^2)} = 0. \tag{32}$$

After integration we obtain

$$(WW'-Q)(C-Q^2+W^2)^{k/2}I_1 = 0. \tag{33}$$

This leads to Bernoulli's equation of the form

$$W' = \frac{Q}{W}. \tag{34}$$

Equation (34) can be linearised to

$$Z' = 2Q, \tag{35}$$

by the use of the transformation $Z = W^2$ and $'$ denotes differentiation with respect to Q. The solution of eq. (35) is given by

$$Z = Q^2 + {I_2}^2. \tag{36}$$

By the use of backward substitution, we can obtain the solution of (2) as follows:

$$W^2 = Q^2 + {I_2}^2, \tag{37}$$

$$W = \pm\sqrt{Q^2 + {I_2}^2}. \tag{38}$$

Now by the use of (29) the above equation can be written as

$$Q' = \pm\sqrt{Q^2 + {I_2}^2}, \tag{39}$$

where $'$ represents differentiation with respect to r. The solution of eq. (39) is given by

$$Q = \pm I_2 \sinh[r + I_3]. \tag{40}$$

Hence, the travelling-wave solution of eq. (2) is given by

$$u(x,t) = \pm I_2 \sinh[x - Ct + I_3], \tag{41}$$

where I_2 and I_3 are constants of integration.

In what follows, we reduce the order of another form of eq. (2) by using an equivalence form of the symmetry, $\mathbf{X}_3$, as follows. Therefore, firstly consider another form of eq. (2),

$$v_t + k(t)v^2u_x + (u^2 - u_x^2)v_x = 0,$$
$$v = u - u_{xx}. \tag{42}$$

According to eq. (42) the equivalence form of $\mathbf{X}_3$ is represented as $\Gamma_3 = 2t\partial_t - u\partial_u - v\partial_v$. Also the associated transformations for the variables are:

$$u(x,t) = \frac{y(x)}{\sqrt{t}} \quad \text{and} \quad v(x,t) = \frac{z(x)}{\sqrt{t}}.$$

Therefore, eq. (42) becomes

$$2(y^2 - y'^2)z' + 2kz^2y' - z = 0$$
$$y'' - y + z = 0. \tag{43}$$

The symmetry of eq. (43) is ∂_x. The corresponding resulting system of ordinary differential equation is given by

$$2(y^2 - p^2)q + 2kz^2p - z = 0,$$
$$pp' - y + z = 0, \tag{44}$$

by using the canonical variables $y(x) = y$, $z(x) = z$, $y' = p(y)$ and $z' = q(z)$ which are obtained from the symmetry ∂_x. One can find the solution of eq. (42) by solving the system of equations (44).

4. Painlevé analysis

Paul Painlevé, a French mathematician, is responsible for the Painlevé test. By applying this test, one can determine the integrability of a given ordinary differential equation or a partial differential equation in terms of analytic functions [26,54–57].

Mark Feix and his collaborators have investigated the procedure of the Ablowitz–Ramani–Segur (ARS) algorithm. Also they found the underlying relation between ARS algorithm and Painlevé test [24–26]. Based on their outcomes, the solution can be identified in two possible ways: the right Painlevé series and the left Painlevé series. The right Painlevé series means the Laurent expansion is in an ascending form obtained from negative integral power and it constitutes the solution of the given differential equation within a punctured disc centred on the singularity [28,58]. On the other hand, the left Painlevé series means an expansion descending from the same negative integral power and constitutes the solution of the given differential equation without the disc [28,58].

The existence of non-generic negative and positive resonances was demonstrated by Andriopoulos and Leach [28]. They proved that the solution of the given differential equation in an annulus can be studied by standard singularity analysis.

In this paper, we can also identify that the given differential equation (2) is integrable, when the ordinary differential equation (28) which is obtained from (2) possesses Painlevé property. Also we can find the corresponding solution to the differential equation (28).

4.1 *Painlevé test*

For an ODE to be of Painlevé (P-) type, it is necessary that its solution possesses no movable branch points, either algebraic or logarithmic (as well as essential singularities). Consider an nth order ODE of the form

$$V^{(n)} = F(t, V, V', V'', \ldots, V^{(n-1)}). \tag{45}$$

Here F is analytic in t and rational in its other arguments and $'$ represents the derivative with respect to t. If we expand the solution of the above equation as a Laurent series in the neighbourhood of a movable singular point t_0, the algorithmic procedure called the ARS algorithm essentially consists of the following three steps [24,25]:

1. Determine the leading-order behaviour of the Laurent series in the neighbourhood of the movable singular point t_0 by taking $V = a_0(t - t_0)^\alpha$, where α is to be calculated.
2. Determine resonances, denoted by r, that is, the powers at which arbitrary constants of the solution of (45) can enter into the Laurent series expansion by using $V = a_0(t - t_0)^\alpha + b_0(t - t_0)^{\alpha+r}$, where r is to be calculated.
3. Verify that a sufficient number of arbitrary constants exist without the introduction of movable critical points.

At the end of the above three steps one is able to identify specific parametric choices at which the Painlevé property holds so that the general solution is meromorphic.

Now consider the generalised CH equation (2), by substituting $r = x - Ct$, changed to a third-order nonlinear autonomous ordinary differential equation

$$(C - Q^2 + Q'^2)(Q''' - Q') + kQ'(Q'' - Q)^2 = 0, \tag{46}$$

where Q is a function of r. If we use this form of eq. (46), it does not pass the Painlevé test. Therefore, we substitute $Q(r) = 1/v[r]$. Then the equation becomes

$$v^2(v^2 - Cv^4 - v'^2)v''' + vv'v''(6Cv^4 - 2(k+3)v^2 + 2(2k+3)v'^2 - kvv'') - v'(2(2k+3)v'^4 - Cv^6 - (4k+7)v^2v'^2 + v^4(1 + k + 6Cv'^2)) = 0. \tag{47}$$

According to the Painlevé test, the dominant terms of eq. (47) are given by

$$v^2(-Cv^4 - v'^2)v''' + vv'v''(6Cv^4 + 2(2k+3)v'^2 - kvv'') - 6Cv^4v'^3 - 2(2k+3)v'^5. \tag{48}$$

The behaviour of the leading order is $a_{-1}w^{-1}$, where $w = (x - x_0)$. Next, to find the resonances (s) we substitute $v(w) = a_{-1}w^{-1} + mw^{-1+s}$ into eq. (48) and equate the coefficients of m to zero. Then we have $s = -1,\ 0$ and 1.

Now we substitute the right Painlevé expansion, $v(w) = a_{-1}\frac{1}{w} + a_0 + a_1 w + a_2 w^2 + \cdots$, into eq. (47). After the substitution, collect the coefficients of various powers of w. Then equate them to zero to calculate the arbitrariness of a_{-1}, a_0 and a_1. Finally, we have that a_0 and a_1 are both arbitrary constants and the remaining constants depend upon a_{-1}, a_0 and a_1. Therefore, we conclude that the reduced form of the MCH equation (47) passes the Painlevé test. As a consequence, eq. (2) is obviously an integrable partial differential equation.

5. Conclusion

We have examined the Lie point symmetries and Painlevé test for the MCH equation with an arbitrary parameter. By performing a symmetry analysis we have successfully found the Lie point symmetries of eq. (2) with the arbitrary parameter k and commutator table, adjoint representation table and optimal system were constructed. Based on the optimal system we have reduced the order of eq. (2) from third order to second order and then we have also obtained a set of solutions for the resultant second-order ordinary differential equation (31). Through the Painlevé test, we observed that the MCH equation (2) is an integrable partial differential equation even with an arbitrary parameter k.

Acknowledgements

ADD and KK thank the anonymous reviewers for their valuable comments and suggestions to improve the quality of this article. They also would like to thank Prof. Stylianos Dimas, Sáo José dos Campos/SP, Brasil, for providing us a new version of the SYM-Package. PGLL thanks the National Research Foundation of South Africa for its continued support.

References

[1] D J Korteweg and G de Vries, *Phil. Mag.* **39**, 422 (1895)
[2] R Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971)
[3] R Hirota, *J. Math. Phys.* **14**, 805 (1973)
[4] R Hirota, *J. Math. Phys.* **14**, 810 (1973)
[5] R Hirota and J Satsuma, *J. Phys. Soc. Japan* **41**, 214 (1976)
[6] R Hirota, *Prog. Theor. Phys.* **52**, 1498 (1974)
[7] G Gui, Y Liu, P J Olver and C Qu, *Commun. Math. Phys.* **319**, 731 (2013)
[8] B Fuchssteiner, *Physica D* **95**, 229 (1996)
[9] A S Fokas, *Acta Appl. Math.* **39**, 295 (1995)
[10] B Xia, Z Qiao and J B Li, *Commun. Nonlin. Sci.* **63**, 292 (2018)
[11] Y Fu, G Gui, C Qu and Y Liu, *J. Diff. Eqn.* **255**, 1905 (2013)
[12] Ajey K Tiwari, A Durga Devi, R Gladwin Pradeep and V K Chandrasekar, *Pramana – J. Phys.* **85**, 789 (2015)
[13] A Durga Devi, R Gladwin Pradeep, V K Chandrasekar and M Lakshmanan, *J. Nonlin. Math. Phys.* **17**, 251 (2010); *J. Nonlin. Math. Phys.* **20**, 78 (2013)
[14] S Lie, *Math. Annal.* **8**, 328 (1874)
[15] P J Olver, *Applications of Lie groups to differential equations* (Springer, New York, 1986)
[16] P G L Leach, M R Feix and S Bouquet, *J. Math. Phys.* **29**, 2563 (1988)
[17] P G L Leach, K S Govinder and B Abraham-Shrauner, *J. Math. Anal. Appl.* **235**, 58 (1999)
[18] P G L Leach, K S Govinder and K Andriopoulos, *J. Appl. Math.* **2012**, Article ID 890171 (2012)
[19] K Andriopoulos, S Dimas, P G L Leach and D Tsoubelis, *J. Comput. Appl. Math.* **230**, 224 (2009)
[20] K M Tamizhmani, K Krishnakumar and P G L Leach, *Appl. Math. Comput.* **247**, 115 (2014)
[21] K M Tamizhmani, R Sinuvasan, K Krishnakumar and P G L Leach, *Afrika Mat.* **26**, 1343 (2014)
[22] M A Rodríguez, P Tempesta and P Winternitz, *J. Phys. Conf. Ser.* **175**, 01213 (2009)
[23] R Sinuvasan, K M Tamizhmani and P G L Leach, *Pramana – J. Phys.* **88**: 1 (2017)
[24] M J Ablowitz, A Ramani and H Segur, *J. Math. Phys.* **21**, 715 (1980)
[25] M J Ablowitz, A Ramani and H Segur, *J. Math. Phys.* **21**, 1006 (1980)
[26] A Ramani, B Grammaticos and T Bountis, *Phys. Rep.* **108**, 159 (1989)
[27] M R Feix, C Geronimi, L Cairó, P G L Leach, R L Lemmer and S É Bouquet, *J. Phys. A – Math. Theor.* **30**, 7437 (1997)
[28] K Andriopoulos and P G L Leach, *Phys. Lett. A* **359**, 199 (2006)
[29] R L Lemmer and P G L Leach, *J. Phys. A – Math. Theor.* **26**, 5017 (1993)
[30] G Adomian, *J. Math. Anal. Appl.* **135**, 501 (1988)
[31] A M Wazwaz, *Phys. Lett. A* **352**, 500 (2006)
[32] A Majeed Yousif, Bewar A Mahmood and Fadhil H Easif, *Am. J. Comput. Math.* **5**, 267 (2015)
[33] S Gravel and P Winternitz, *J. Math. Phys.* **43**, 5902 (2002)
[34] P Winternitz and I Yurdusen, *J. Phys. A – Math. Theor.* **42**, 385203 (2009)
[35] P J Olver and P Rosenau, *Phys. Rev. E* **53**, 1900 (1996)
[36] Z Qiao, *J. Math. Phys.* **47**, 112701 (2006)
[37] A Himonas and D Mantzavinos, *Nonlin. Anal.* **95**, 499 (2014)
[38] M Yang, Y Li and Y Zhao, *Appl. Anal.*, https://doi.org/10.1080/00036811.2017.1359565(2017)
[39] Z Qiao and X Li, *Theor. Math. Phys.* **267**, 584 (2011)
[40] R Camassa and D Holm, *Phys. Rev. Lett.* **71**, 1661 (1993)
[41] A Constantin and D Lannes, *Arch. Rat. Mech. Anal.* **192**, 165 (2009)
[42] B Fuchssteiner and A Fokas, *Physica D* **4**, 47 (1981/1982)
[43] A Constantin and W A Strauss, *Phys. Lett. A* **270**, 140 (2000)
[44] Z Gao, X Liu, X Liu and C Qu, *J. Diff. Eqns.* **266**, 7749 (2019)
[45] S Dimas and D Tsoubelis, *Group analysis of differential equations* edited by N H Ibragimov, C Sophocleous and P A Damianou (University of Cyprus, Nicosia, 2005) pp. 64-70, See also http://www.math.upatras.gr/~spawn
[46] S Dimas and D Tsoubelis, *8th International Mathematica Symposium* (Avignon, France, 2006)
[47] S Dimas, *Partial differential equations, algebraic computing and nonlinear systems*, Thesis (University of Patras, Patras, Greece, 2008)
[48] V V Morozov, *Izvestia Vysshikh Uchebn Zavendenií Matematika* **5**, 161 (1958)
[49] G M Mubarakzyanov, *Izvestia Vysshikh Uchebn Zavendenií Matematika* **32**, 114 (1963)
[50] G M Mubarakzyanov, *Izvestia Vysshikh Uchebn Zavendenií Matematika* **34**, 99 (1963)
[51] G M Mubarakzyanov, *Izvestia Vysshikh Uchebn Zavendenií Matematika* **35**, 104 (1963)
[52] Z Zhao and B Han, *Eur. Phys. J. Plus* **130**, 1 (2015)

[53] Z Zhao and B Han, *J. Math. Phys.* **58**, 1 (2017)
[54] P Painlevé, (Leçons de Stockholm, 1895) (Hermann, Paris, 1897). Reprinted, Œuvres de Paul Painlevé, vol. I, Éditions du CNRS, Paris. (1973)
[55] P Painlevé , *B. Soc. Math. Fr.* **28**, 201 (1900)
[56] P Painlevé, *Acta Math.* **25**, 1 (1902)
[57] M Lakshmanan and R Sahadevan, *Phys. Rep.* **224**, 1 (1993)
[58] K Krishnakumar, *A study of symmetries, reductions and solutions of certain classes of differential equations*, Thesis (Pondicherry Central University, Puducherry, India, 2016)