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Study on exponential synchronisation between the time-delay spatiotemporal network and the target system

YING LI, YUQING XU, LING LÜ*, GANG LI and CHENGREN LI*

School of Physics and Electronic Technology, Liaoning Normal University, Dalian 116029, China
*Corresponding authors. E-mail: luling1960@aliyun.com; lnnulicr@aliyun.com

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Abstract. In this paper, the problem of exponential synchronisation between time-delay spatiotemporal network and target system is studied. Based on the Lyapunov stability principle, the Lyapunov–Krasovskii generic function with exponential form is designed, and the form of the synchronisation controller is determined, so that the exponential synchronisation is realised between the time-delay spatiotemporal network and the target system. In simulation, Fisher–Kolmogorov system is further selected as the network node to verify the rationality of synchronisation scheme. This technology fully considers the effect of time delay on network synchronisation performance, making the synchronisation scheme more practical. In addition, exponential synchronisation technology can effectively adjust the rate of network synchronisation.

Keywords. Exponential synchronisation; delay; network; Lyapunov–Krasovskii functional.

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1. Introduction

Synchronisation is an interesting phenomenon in complex networks. In recent years, network synchronisation technology has solved key problems in production, such as information transmission, confidential communication and disease control [1–3], thus attracting great interest of researchers [4–8]. As there are many factors affecting synchronisation between complex networks, such as different number of nodes, different network topologies and complex interaction relationship between nodes, the synchronisation study of complex networks is much more complex than that of a single system [9–14].

As is known to all, time delay often appears in the actual network system, which often leads to the performance degradation and instability of the network. Therefore, it is necessary to consider the influence of time-delay factors in the study of network synchronisation [15–18]. In recent years, some synchronisation problems of time-delay networks have been reported. Xu *et al* [19] presented the synchronisation basis for complex nonlinear networks with time delays by designing state feedback controllers. Guo *et al* [20] proposed an adaptive control strategy using the local new law, and gave synchronisation criterion between the time-delay network and the target system. By designing feedback controllers, Liu *et al* [21] studied the synchronisation between a class of neural networks with time delay and target systems. By constructing appropriate Lyapunov–Krasovskii generic functions, Hu *et al* [22] gave synchronisation conditions of neural networks with time delay. Wu *et al* [23] applied fractional Razumikhin theorem and completed the synchronisation study of time-delay coupled generalised fractional complex networks by constructing simple Lyapunov–Krasovskii generic functions.

With increasing network related research, different types of synchronisation, such as cluster synchronisation, complete synchronisation, phase synchronisation and exponential synchronisation [24–26] have been reported successively. Exponential synchronisation is widely studied by researchers because it can control the synchronisation rate of complex networks by adjusting the parameters in the synchronisation process. Typical works include: By designing Lyapunov–Krasovskii universal function and linear inequality principle, Zhang *et al* [27] realised exponential synchronisation of discrete dynamical networks with time-delay random

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disturbance. Gao *et al* [28] obtained sufficient conditions for feedback time-delay network synchronisation using Lyapunov–Krasovskii generic function and graph theory methods.

So far, most of the network nodes involved in a large number of studies on network synchronisation have only evolved with time. In fact, there are not only network nodes that evolve with time, but also network nodes that evolve with space. Therefore, the synchronisation study of spatiotemporal network has more practical significance. At present, there are few researches on spatiotemporal network synchronisation. So it is very necessary to study it. Typical works include: He and Li [29] proposed a hybrid adaptive synchronisation strategy to achieve complete synchronisation between spatiotemporal neural networks. By using Lyapunov–Krasovskii generic function method and inequality theorem, Xu *et al* [30] proposed spatiotemporal synchronisation of coupled reaction-diffusion neural networks with switched topology and time delay.

Based on the above analysis, the exponential synchronisation problem between the time-delay spatiotemporal network and the target system is studied in this paper. Based on the Lyapunov stability principle, the Lyapunov–Krasovskii generic function with exponential form is designed, and the form of the synchronisation controller is determined, so that the exponential synchronisation is realised between the time-delay spatiotemporal network and the target system. Fisher–Kolmogorov system is further selected as the network node of simulation to verify the rationality of synchronisation scheme. This technology fully considers the effect of time delay on network synchronisation performance and makes the synchronisation scheme more practical. In addition, exponential synchronisation technology can effectively adjust the rate of network synchronisation.

2. Network construction and mathematical preparation

Considering the time-delay spatiotemporal network composed of N nodes, the state equation of the network node is as follows:

$$\frac{\partial x_i(r,t)}{\partial t} = f(x_i(r,t)) + \varepsilon_i \sum_{j=1}^{N} b_{ij} x_j(r, t-\tau) + u_i(r,t), \tag{1}$$

where t is the time variable, r is the coordinate of spatial lattice point, $x_i(r,t) = (x_{i1}(r,t), x_{i2}(r,t), \ldots, x_{in}(r,t))^T \in R^n$ is the state variable of the network node and $f(x_i(r,t))$ is the state function of the network node. ε_i is the coupling strength of the network connection, N is the number of network nodes, τ is the time delay and $\tau \geq 0$. $u_i(r,t)$ is the network controller. $B = (b_{ij})_{N\times N}$ is the network coupling matrix, which represents the topology structure characteristics of the network. The coupling matrix element of the network is defined as follows: if there is a connection between the node i and the node $j (i \neq j)$, then $b_{ij} = 1$, otherwise $b_{ij} = 0$. The diagonal matrix elements of matrix B are defined as follows:

$$b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}, \quad i = 1, 2, \ldots, N. \tag{2}$$

Remark 1. The coupling matrix B of the network need not be symmetric and irreducible, which means that the topology of the network can be chosen arbitrarily.

Remark 2. The node number of the network has no effect on the synchronisation performance between the network and the target system.

Remark 3. The value range of the time delay is $\tau \in [0, n]\,(n > 0)$.

Equation (1) is taken as the response network, and the dynamical equation of the target system is

$$\frac{\partial S(r,t)}{\partial t} = f(S(r,t)). \tag{3}$$

The error function between the response network and the target system is defined as

$$e_i(r,t) = x_i(r,t) - S(r,t). \tag{4}$$

The derivative of eq. (4) with respect to time is obtained as

$$\begin{aligned}\frac{\partial e_i(r,t)}{\partial t} &= \frac{\partial x_i(r,t)}{\partial t} - \frac{\partial S(r,t)}{\partial t} \\ &= f(x_i(r,t)) + \varepsilon_i \sum_{j=1}^{N} b_{ij} e_j(r, t-\tau) \\ &\quad + u_i(r,t) - f(S(r,t)).\end{aligned} \tag{5}$$

Hypothesis 1 [31]. For any $x_i(r,t)S(r,t) \in R^n$, assume the existence of a normal number l_i, which satisfies the following inequality:

$$\| f(x_i(r,t)) - f(S(r,t))\| \leqslant l_i \, \|x_i(r,t) - S(r,t)\|, \tag{6}$$

where l_i is also called the Lipschitz constant.

Lemma 1 [31]. *For any vector X and Y, it satisfies the following inequality relation:*

$$2X^TY \leqslant X^TX + Y^TY. \tag{7}$$

DEFINITION 1

For the response network eq. (1) and the target system eq. (3), if at a certain time $t > 0$, there is a constant τ and it is not related to time, so that for any $i = 1, 2, \ldots, N$, $\lim_{t\to\infty} \|x_i(r,t) - S(r,t)\| = 0$, it can be said that there is a synchronisation between the time-delay spatiotemporal network and the target system.

3. Design of synchronisation scheme between the time-delay spatiotemporal network and the target system

Theorem 1. *To achieve exponential synchronisation between the time-delay spatiotemporal network and the target system, the following network controller shall be constructed:*

$$u_i(r,t) = -\left(l_i + \frac{1}{2}\varepsilon_i\lambda_i + \exp(\mu n) + \frac{1}{2}\mu\right)e_i(r,t), \tag{8}$$

where μ is the regulating parameter and λ_i is the eigenvalue of the network coupling matrix.

Proof. Construct the following Lyapunov–Krasovskii function with the exponential form

$$V(r,t) = \frac{1}{2}\exp(-\mu n)\sum_{i=1}^{N} e_i^T(r,t)e_i(r,t) + Q(r,t), \tag{9}$$

where

$$Q(r,t) = \sum_{i=1}^{N}\int_{t-\tau}^{t} \exp\left[-\mu(t-\theta)\right] \times e_i^T(r,\theta)e_i(r,\theta)\mathrm{d}\theta. \tag{10}$$

Thus

$$\frac{\partial Q(r,t)}{\partial t} = -\mu Q(r,t) + \sum_{i=1}^{N} e_i^T(r,t)e_i(r,t) - \sum_{i=1}^{N}\exp(-\mu\tau)e_i^T(r,t-\tau)e_i(r,t-\tau). \tag{11}$$

In this way,

$$\begin{aligned}\frac{\partial V(r,t)}{\partial t} &= \exp(-\mu n)\sum_{i=1}^{N} e_i^T(r,t)[f(x_i(r,t)) \\ &\quad - f(S(r,t)) + \varepsilon_i\sum_{j=1}^{N} b_{ij}e_j(r,t-\tau) + u_i(r,t)] \\ &\quad -\mu Q(r,t) + \sum_{i=1}^{N} e_i^T(r,t)e_i(r,t) \\ &\quad - \sum_{i=1}^{N}\exp(-\mu\tau)e_i^T(r,t-\tau)e_i(r,t-\tau).\end{aligned} \tag{12}$$

According to Hypothesis 1,

$$\begin{aligned}\frac{\partial V(r,t)}{\partial t} &\le \exp(-\mu n)\sum_{i=1}^{N} e_i^T(r,t)[l_ie_i(r,t) \\ &\quad + \varepsilon_i\lambda_ie_j(r,t-\tau) + u_i(r,t)] \\ &\quad -\mu Q(r,t) + \sum_{i=1}^{N} e_i^T(r,t)e_i(r,t) \\ &\quad - \sum_{i=1}^{N}\exp(-\mu\tau)e_i^T(r,t-\tau)e_i(r,t-\tau) \\ &= \exp(-\mu n)\sum_{i=1}^{N} e_i^T(r,t)[l_ie_i(r,t) + \varepsilon_i\lambda_ie_j(r,t-\tau) \\ &\quad + \exp(\mu n)e_i(r,t) + u_i(r,t)] - \mu Q(r,t) \\ &\quad - \sum_{i=1}^{N}\exp(-\mu\tau)e_i^T(r,t-\tau)e_i(r,t-\tau).\end{aligned} \tag{13}$$

According to Lemma 1,

$$\begin{aligned}\frac{\partial V(r,t)}{\partial t} &\le \exp(-\mu n)\sum_{i=1}^{N} e_i^T(r,t)\Bigg[\Bigg(l_i + \frac{1}{2}\varepsilon_i\lambda_i \\ &\quad + \exp(\mu n)\Bigg)e_i(r,t) + u_i(r,t)\Bigg] \\ &\quad + \sum_{i=1}^{N}\left[\frac{1}{2}\varepsilon_i\lambda_i\exp(-\mu n) - \exp(-\mu\tau)\right] \\ &\quad \times e_i^T(r,t-\tau)e_i(r,t-\tau) - \mu Q(r,t).\end{aligned} \tag{14}$$

Equation (8) in Theorem 1, when substituted into eq. (14), gives

$$\begin{aligned}\frac{\partial V(r,t)}{\partial t} &\le -\frac{1}{2}\mu\exp(-\mu n)\sum_{i=1}^{N} e_i^T(r,t)e_i(r,t) \\ &\quad -\mu Q(r,t) \\ &\quad + \sum_{i=1}^{N}\left[\frac{1}{2}\varepsilon_i\lambda_i\exp(-\mu n) - \exp(-\mu\tau)\right] \\ &\quad \times e_i^T(r,t-\tau)e_i(r,t-\tau)\end{aligned}$$

$$= -\mu V(r,t) + \sum_{i=1}^{N}\left[\frac{1}{2}\varepsilon_i\lambda_i \exp(-\mu n) - \exp(-\mu\tau)\right] \times e_i^T(r,t-\tau)e_i(r,t-\tau) \quad (15)$$

If

$$\frac{1}{2}\varepsilon_i\lambda_i \exp(-\mu n) - \exp(-\mu\tau) \le 0 \quad (16)$$

then

$$\frac{\partial V(r,t)}{\partial t} \le 0. \quad (17)$$

Based on Lyapunov stability theory, the exponential synchronisation between the response network and the target system is achieved.

4. Simulation and discussion

In order to verify the above exponential synchronisation scheme, one-dimensional Fisher–Kolmogorov system, a real model describing the number of biological species, is selected as the node state equation of the response network and the target system for numerical simulation, so as to investigate the stability of the synchronisation between the time-delay spatiotemporal network and the target system.

The form of the Fisher–Kolmogorov system is as follows [32]:

$$\frac{\partial x(r,t)}{\partial t} = \eta x(r,t)(1-x(r,t)) + D\nabla^2 x(r,t), \quad (18)$$

where η, D are the parameters of the system, and $\eta = 0.5D = 5$. Periodic boundary conditions are selected for the system, and the initial values are randomly selected for simulation. The evolution image of the state variable of the Fisher–Kolmogorov system with empty space at any time is obtained, as shown in figure 1. It can be seen from figure 1 that the system has spatiotemporal chaotic behaviour.

In the simulation process, the target system selects a single Fisher–Kolmogorov system:

$$\frac{\partial S(r,t)}{\partial t} = \eta S(r,t)(1-S(r,t)) + D\nabla^2 S(r,t). \quad (19)$$

Based on eq. (18), the response network node equation is

$$\frac{\partial x_i(r,t)}{\partial t} = \eta x_i(r,t)(1-x_i(r,t)) + D\nabla^2 x_i(r,t) + \varepsilon_i \sum_{j=1}^{N} b_{ij}x_j(r,t-\tau) + u_i(r,t). \quad (20)$$

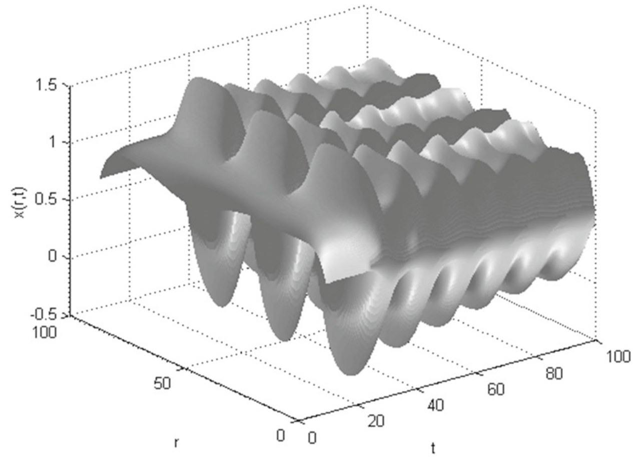

Figure 1. Spatiotemporal evolution of the state variable.

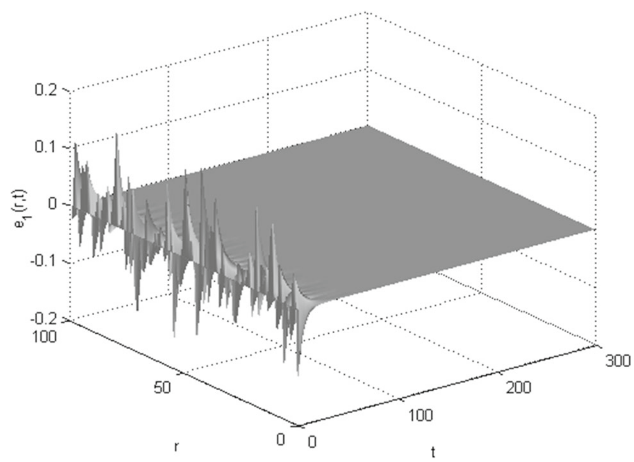

Figure 2. Spatiotemporal evolution of the error function ($e_1(r,t) = x_1(r,t) - S(r,t)$, $\mu = 0.1$).

The number of nodes in the response network is randomly selected as $N = 8$. Since the topology of the network can be arbitrary, the connection mode between 8 nodes of the response network is selected as follows:

$$B = \begin{bmatrix} -3 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -2 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -4 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & -3 \end{bmatrix}_{8\times 8}. \quad (21)$$

The parameters are taken as $n = 1, \tau = 0.05$ and $\varepsilon_i = 1$. The initial values of the target system and the response network state variables are all random values. During simulation, the evolution of the error at any time is shown in figures 2–9.

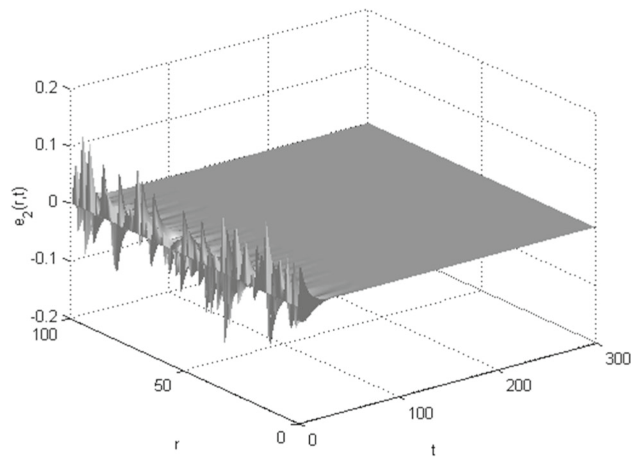

Figure 3. Spatiotemporal evolution of the error function ($e_2(r,t) = x_2(r,t) - S(r,t)$, $\mu = 0.1$).

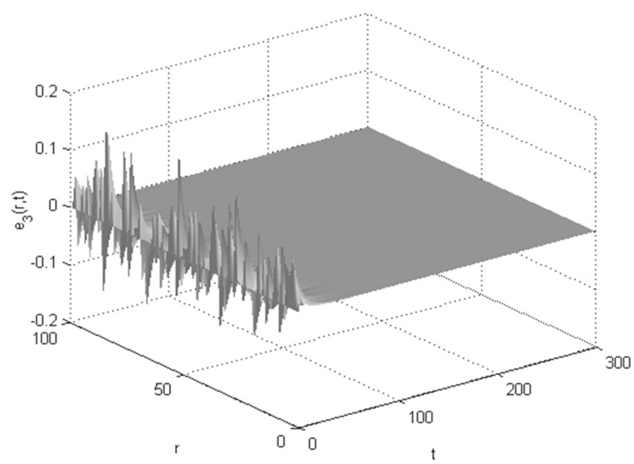

Figure 4. Spatiotemporal evolution of the error function ($e_3(r,t) = x_3(r,t) - S(r,t)$, $\mu = 0.1$).

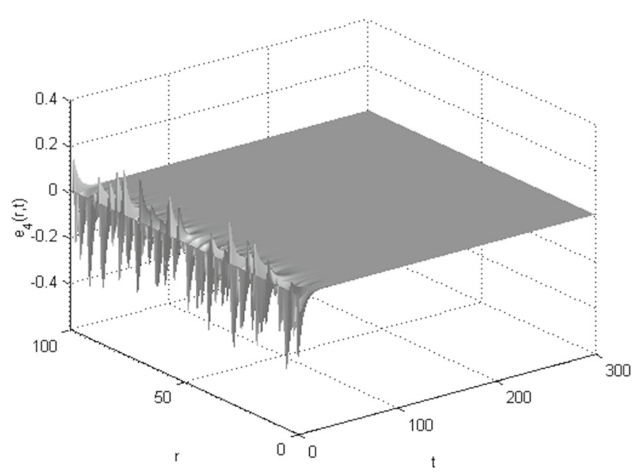

Figure 5. Spatiotemporal evolution of the error function ($e_4(r,t) = x_4(r,t) - S(r,t)$, $\mu = 0.1$).

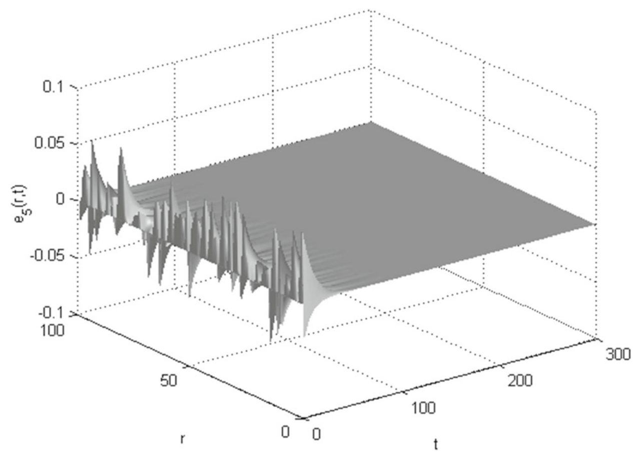

Figure 6. Spatiotemporal evolution of the error function ($e_5(r,t) = x_5(r,t) - S(r,t)$, $\mu = 0.1$).

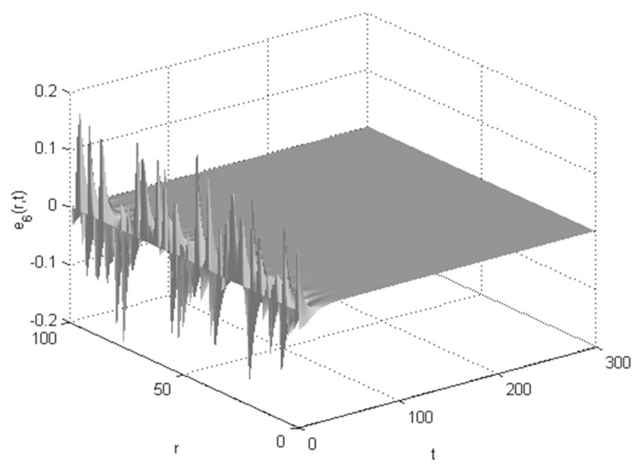

Figure 7. Spatiotemporal evolution of the error function ($e_6(r,t) = x_6(r,t) - S(r,t)$, $\mu = 0.1$).

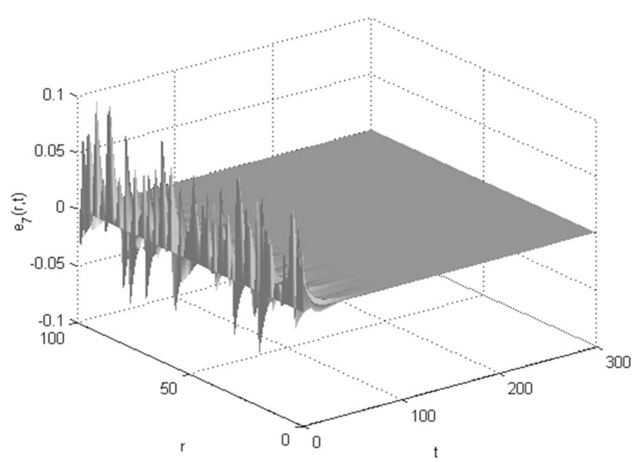

Figure 8. Spatiotemporal evolution of the error function ($e_7(r,t) = x_7(r,t) - S(r,t)$, $\mu = 0.1$).

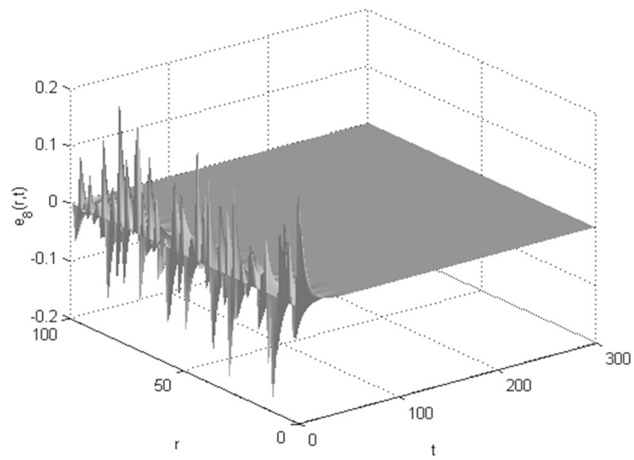

Figure 9. Spatiotemporal evolution of the error function ($e_8(r, t) = x_8(r, t) - S(r, t), \mu = 0.1$).

The error oscillation is obvious at the beginning due to different initial values. After a short period of time evolution, the oscillation weakens under the controller action of the network and gradually approaches zero, which means that exponential synchronisation between the response network and the target system is realised. At the same time, by changing the value of parameter μ when all other parameters remain the same, the error oscillation tends to zero at different speeds, indicating that exponential synchronisation can control the synchronisation rate of complex network by adjusting a parameter in the synchronisation process. We are not going to show the simulated image again. In the whole time series [0,300], the synchronisation process is always stable and not affected by the time delay, indicating that the designed exponential synchronisation scheme between the time-delay network and the target system is effective.

5. Conclusion

The problem of exponential synchronisation between time-delay spatiotemporal network and the target system is studied. The Lyapunov–Krasovskii function with exponential form is designed to effectively synchronise the time-delay spatiotemporal network. Fisher–Kolmogorov system is used as the node state equation to construct the response network for numerical simulation. The simulation results show that the error gradually approaches zero after a short oscillation, which means that the exponential synchronisation between the response network and the target system is realised. Especially when other parameters remain unchanged, the synchronisation rate can be effectively adjusted by changing the value of parameter μ. In addition, the exponential synchronisation scheme designed in this paper is applicable to network with arbitrary topology structure. In other words, the network synchronisation effect is not affected by the number of nodes, time delay and network topology structure, which not only reflects the good synchronisation performance of spatiotemporal network, but also reflects the universality and practicability of this synchronisation scheme.

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