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Gegenbauer spectral tau algorithm for solving fractional telegraph equation with convergence analysis

HODA F AHMED, M R A MOUBARAK and W A HASHEM*

Mathematics Department, Faculty of Science, Minia University, Minia, Egypt
*Corresponding author. E-mail: wabdelraheem@yahoo.com, waleed.hashim@mu.edu.eg

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Abstract. In this article, a novel shifted Gegenbauer operational matrix (SGOM) of fractional derivative in the Caputo sense is derived. Based on this operational matrix, an accurate and effective numerical algorithm is proposed. The SGOM of fractional derivative in conjunction with the tau method are used for solving the constant and variable coefficients space–time fractional telegraph equations (FTE) with various types of boundary conditions, namely, Neumann, Dirichlet and Robin conditions. The convergence analysis of the proposed method is established in $\mathcal{L}^2_{\omega_\alpha}$. Finally, miscellaneous test examples are given and compared with other methods to clarify the accuracy and efficiency of the presented algorithm.

Keywords. Telegraph equation; Caputo fractional derivative; shifted Gegenbauer tau method; operational matrices; convergence analysis.

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1. Introduction

The applications of fractional calculus in physics and engineering were not related to its appearance in mathematics. It was delayed for many years because of the widespread controversy about the multiple non-equivalent definitions of the fractional derivatives besides the absence of geometrical interpretation of the fractional derivatives due to their non-local properties [1,2]. However, it was proved in the last 10 years that the fractional calculus is a precious tool for modelling many phenomena and interdisciplinary applications. Fractional partial differential equations where the derivatives have more degrees of fluency are quite flexible and relevant for precisely modelling some phenomena in science and engineering more than the classical type; for instance, anomalous diffusion processes [3], seepage flow in porous media [4], the seismic analysis of mechanical models in the presence of viscoelastic dampers [5] and other scientific branches [6–8].

The telegraph model arose after noticing reflection of wave motion of electromagnetic type on the wire throughout the improvement of transmission line joining the voltage and current waves. The mathematical model of telegraph equation can be precisely represented by fractional partial differential equation where the dependent variable denotes the voltage and the independent variables denote position and time. Special choice of the coefficients and the fractional derivative domain of the fractional telegraph equation convert it into fractional non-homogeneous Klein–Gordon equation or fractional reaction dispersion equation or fractional diffusion-wave equation with damping or fractional wave equation.

Recently, a number of numerical approaches have been applied to solve the FTEs. Jiang and Lin [9] reproduced kernal theorem with Caputo fractional derivatives to find exact solution for the temporal FTE with Robin boundary condition. Zhao and Li [10] presented Galerkin finite element method in the Riemann–Liouville sense and finite difference method in the Caputo sense for solving the spatial and temporal FTE respectively. Mohebbi *et al* [11] proposed meshless method depending on the approximation of the solution by radial basis functions with the spectral collocation method to solve the one- and two-dimensional time FTE. Bhrawy *et al* [12] solved the two-sided time FTE using operational integration matrices based on shifted Chebyshev tau method with Riesz fractional derivative. Akram *et al* [13] used Caputo's fractional derivative

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combined with the extended cubic B-splines method as a basis for a finite difference scheme to solve time FTE.

Spectral methods such as Galerkin, collocation and tau methods are a class of the most elegant tools because of their ability to reach accurate numerical solutions with fewer degrees of freedom to the classical and fractional differential equations. We must know that the utilised basis functions are the main factor in obtaining the precise approximation characteristics of spectral methods. Using orthogonal polynomials as basis functions gives preferable results, such as, trigonometric functions for approximating the periodic problems, Gegenbauer, Jacobi, Chebyshev and Legendre functions for approximating non-periodic problems. The spectral numerical procedure first expresses approximate solution as an expansion of certain orthogonal polynomials, then the problem is transformed into an algebraic system of equations. The coefficients are selected in such a way to make the difference between the exact and approximate solutions as accurate as possible. In recent years, spectral methods, together with the operational matrices of fractional derivatives and integrals, were applied for finding numerical solutions of fractional differential equations of various kinds; for instance, see [14–17]. The present work focusses on the Gegenbauer polynomials (GPs) due to their fundamental properties. They are eigenfunctions of some differential operators. They achieve rapid rates of convergence for small range of spectral expansion terms. GPs are successfully applied to solve different kinds of fractional differential equations [18,19], fractional variational problems and optimal control problems [20]. In this respect, El-Kalaawy *et al* [21] derived fractional integral operational matrix of the fractional Gegenbauer functions, then applied it together with the Rayleigh–Ritz method for solving fractional variational and optimal control problems. Ahmed *et al* [22] presented a novel spectral collocation method together with the SGPs to provide accurate numerical solutions for the one- and two-dimensional time- and space-fractional coupled Burgers' equations.

The main objective of this paper is to present a new spectral algorithm depending on SGPs and operational matrix of fractional derivative in Caputo sense combined with the spectral tau method to discretise the second-order one-dimensional space–time fractional telegraph equation with space variable coefficients. Hence, a linear system of algebraic equations is generated which greatly simplifies the problem. Moreover, the convergence of the proposed method is analysed. The proposed method is an easily implementable tool for the numerical solution of the variable coefficients telegraph equation. It is free from round-off errors. Also, it has an exponential convergence rate in both spatial and temporal discretisation according to the numerical result. It is worth pointing out that, the spectral approximation based on Chebyshev functions and Legendre functions are special cases of the present approximation.

The structure of this article is arranged as follows: In §2, we present short notes on some fractional calculus concepts, Gegenbauer orthogonal polynomials and their properties in approximate functions of one and two independent variables. In §3, the operational matrix of fractional derivative of the shifted Gegenbauer polynomials is derived. In §4, a numerical algorithm based on shifted Gegenbauer tau method for solving constant and variable coefficient one-dimensional fractional telegraph equation with several kinds of boundary conditions is constructed. In §5, we investigate the convergence analysis of the proposed method. In §6, numerical examples are tested and compared with other methods to demonstrate the accuracy of the proposed algorithms. Conclusion is given in §7.

2. Preliminaries and basic definitions

In this section, we mention some of the preliminary definitions and properties of the fractional integration, fractional differentiation and shifted Gegenbauer polynomials which are required in the following sections.

2.1 *Fractional integration and differentiation*

DEFINITION 2.1

The integral of fractional order $v > 0$ according to Riemann–Liouville has the form

$$I^{v} f(x)=\frac{1}{\Gamma(v)}\int_0^x (x-\xi)^{v-1} f(\xi)\,\mathrm{d}\xi, \quad v>0, \quad x>0. \tag{1}$$

DEFINITION 2.2

The derivative of fractional order $v > 0$ according to Caputo has the form

$$\begin{aligned} D^{v} f(x) &= I^{m-v} D^{m} f(x) \\ &= \frac{1}{\Gamma(m-v)}\int_0^x (x-\xi)^{m-v-1}\frac{\mathrm{d}^m}{\mathrm{d}\xi^m} f(\xi)\,\mathrm{d}\xi, \\ & m-1 < v \le m, \quad x>0, \end{aligned} \tag{2}$$

where m is the ceiling function of v. The operator D^{v} satisfies the following equation:

$$D^{v} x^{\alpha} = \begin{cases} \dfrac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-v)} x^{\alpha-v} & \alpha \ge v, \\ 0 & \alpha < v. \end{cases} \tag{3}$$

DEFINITION 2.3

The two-parameter Mittag–Leffler function is defined as

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k+\beta)}, \quad \alpha,\beta \in \mathbb{R}^+, t \in \mathbb{C} \tag{4}$$

which is a generalisation of the one-parameter Mittag–Leffler function ($\beta = 1$) and the latter is a generalisation of the exponential function e^t.

2.2 *Shifted Gegenbauer polynomials*

The Gegenbauer polynomials (GPs) with parameter $\alpha > -\frac{1}{2}$ and degree $i \in \mathbb{Z}^+$ are classical orthogonal polynomials on the finite interval $[-1, 1]$ which has the form $C_i^{(\alpha)}(x)$ and satisfy the following recurrence relation:

$$(i+1)C_{i+1}^{(\alpha)}(x) = 2(i+\alpha)xC_i^{(\alpha)}(x) - (i+2\alpha-1)C_{i-1}^{(\alpha)}(x), \quad i > 0, \tag{5}$$

where the first two terms are $C_0^{(\alpha)}(x) = 1$ and $C_1^{(\alpha)}(x) = 2\alpha x$. Moreover, GPs satisfy the following symmetry and inequality relations:

$$C_i^{(\alpha)}(x) = (-1)^i C_i^{(\alpha)}(-x), \quad i \geq 0,$$

$$|C_i^{(\alpha)}(x)| \leq C_i^{(\alpha)}(1), \quad i \geq 0, \quad \alpha > 0, \quad |x| \leq 1.$$

Doha [18] standardised the GPs, $C_i^{(\alpha)}(x)$ such that

$$C_i^{(\alpha)}(1) = 1, \quad i = 0, 1, 2, \ldots.$$

Under this standardisation, the GPs are defined as

$$C_i^{(\alpha)}(x) = \frac{i!\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma\left(i+\alpha+\frac{1}{2}\right)} P_i^{\left(\alpha-\frac{1}{2},\alpha-\frac{1}{2}\right)}(x), \quad i = 0, 1, 2, \ldots,$$

where $P_i^{(\alpha,\beta)}(x)$ is the Jacobi polynomial. The orthogonality relation of the GPs with respect to the weight function $\omega^\alpha(x) = (1-x^2)^{\alpha-\frac{1}{2}}$ is defined as

$$\langle C_i^{(\alpha)}(x), C_j^{(\alpha)}(x)\rangle = \int_{-1}^{1} C_i^{(\alpha)}(x)C_j^{(\alpha)}(x)\omega^\alpha(x)\,dx = h_i^{(\alpha)}\delta_{i,j},$$

where

$$h_i^{(\alpha)} = \frac{2^{2\alpha-1} i! \left(\Gamma\left(\alpha+\frac{1}{2}\right)\right)^2}{(i+\alpha)\Gamma(i+2\alpha)}.$$

It is worth mentioning that GPs can be identically the Legendre polynomial and Chebyshev polynomial. More specifically, we have

$$T_i(x) = C_i^{(0)}(x), \quad L_i(x) = C_i^{\left(\frac{1}{2}\right)}(x),$$

$$U_i(x) = (i+1)C_i^{(1)}(x), \quad i \geq 0,$$

where $T_i(x)$ is the Chebyshev polynomial of the first kind of degree i, $L_i(x)$ is the Legendre polynomial of degree i and $U_i(x)$ is the Chebyshev polynomial of the second kind of degree i.

To use GPs in the interval $[0, L]$, a change of variable $x \to \frac{2x}{L} - 1$ has been performed on GPs, $C_i^{(\alpha)}(x)$ turning it into the so-called shifted Gegenbauer polynomials (SGPs) $C_{L,i}^{(\alpha)}(x)$.

The SGPs have the orthogonality properties

$$\langle C_{L,i}^{(\alpha)}(x), C_{L,j}^{(\alpha)}(x)\rangle = \int_0^L C_{L,i}^{(\alpha)}(x)C_{L,j}^{(\alpha)}(x)\omega_L^\alpha(x)\,dx = h_{L,i}^{(\alpha)}\delta_{i,j}, \tag{6}$$

where

$$C_{L,i}^{(\alpha)}(x) = C_i^{(\alpha)}\left(\frac{2x}{L}-1\right),$$

$$\omega_L^\alpha(x) = \left(Lx-x^2\right)^{\alpha-\frac{1}{2}} \tag{7}$$

$$h_{L,i}^{(\alpha)} = \left(\frac{L}{2}\right)^{2\alpha} h_i^{(\alpha)}. \tag{8}$$

The analytic form of SGPs is given by

$$C_{L,i}^{(\alpha)}(x) = \sum_{k=0}^{i} \Theta_{k,i} x^k, \quad 0 < x < L, \tag{9}$$

where

$$\Theta_{k,i} = (-1)^{i-k} \frac{i!\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma(i+k+2\alpha)}{\Gamma\left(k+\alpha+\frac{1}{2}\right)\Gamma(i+2\alpha)(i-k)!k!L^k}.$$

- In terms of the first $(N+1)$ SGPs, we can approximate the square integrable function of the one independent variable $u(x)$, as follows:

$$u(x) \simeq \sum_{i=0}^{N} a_i C_{L,i}^{(\alpha)}(x) \simeq \mathbf{A}^T \phi_{L,N}(x), \tag{10}$$

where the coefficients a_i are calculated from

$$a_i = \frac{1}{h_{L,i}^{(\alpha)}} \int_0^L u(x)C_{L,i}^{(\alpha)}(x)\omega_L^{(\alpha)}(x)\,dx, \quad i = 0, 1, \ldots, N. \tag{11}$$

The last term of eq. (10) represents the matrix form of the approximated $u(x)$, with

$$\mathbf{A}^T \equiv [a_0, a_1, \ldots, a_N], \tag{12}$$

$$\phi_{L,N}(x) \equiv \left[C^{(\alpha)}_{L,0}(x), C^{(\alpha)}_{L,1}(x), \ldots, C^{(\alpha)}_{L,N}(x)\right]^T. \quad (13)$$

• The function of two independent variables $u(x,t)$ where $(x,t) \in (0, L] \times (0, \tau]$ is approximated by using double SGPs as

$$u_{N,M}(x,t) \simeq \sum_{i=0}^{M}\sum_{j=0}^{N} b_{i,j} C^{(\alpha)}_{\tau,i}(t) C^{(\alpha)}_{L,j}(x)$$
$$\simeq \phi^T_{\tau,M}(t)\mathbf{B}\phi_{L,N}(x), \quad (14)$$

where $\mathbf{B} = [b_{i,j}]$ with $i = 0, 1, \ldots, M$, $j = 0, 1, \ldots, N$ is an $(M+1)\times(N+1)$ matrix. The matrix coefficient $b_{i,j}$ can be determined from

$$b_{i,j} = \frac{1}{h^{(\alpha)}_{\tau,i} h^{(\alpha)}_{L,j}} \int_0^{\tau}\int_0^{L} [u(x,t) C^{(\alpha)}_{\tau,i}(t)$$
$$C^{(\alpha)}_{L,j}(x)\omega^{(\alpha)}_{\tau}(t)\omega^{(\alpha)}_{L}(x)\,\mathrm{d}x\,\mathrm{d}t],$$
$$i = 0, 1, \ldots, M, \quad j = 0, 1, \ldots, N. \quad (15)$$

3. SGOM of fractional differentiation

The main objective of this section is inferring the SGOM of fractional Caputo derivative. For this purpose, firstly, some lemmas are deduced to get the SGOM.

Lemma 3.1. *The Caputo fractional derivative of order $v > 0$ of SGPs has the form*

$$D^v C^{(\alpha)}_{L,j}(x) = \begin{cases} \sum_{k=\lceil v\rceil}^{j} \Theta_{k,j} \dfrac{\Gamma(k+1)}{\Gamma(k+1-v)} x^{k-v} & j \geq \lceil v\rceil, \\ 0 & j < \lceil v\rceil. \end{cases}$$

Proof. Using the linearity of the fractional derivative, we get

$$D^v C^{(\alpha)}_{L,j}(x) = \sum_{k=0}^{j} \Theta_{k,j} D^v x^k.$$

Now, by using eq. (3) the lemma is proved easily. □

Lemma 3.2. *For $v > 0$, the following relation is satisfied:*

$$\int_0^L D^v C^{(\alpha)}_{L,i}(x) C^{(\alpha)}_{L,j}(x)\omega^{\alpha}_L(x)\,\mathrm{d}x$$
$$= \sum_{k=\lceil v\rceil}^{i}\sum_{f=0}^{j} \Theta_{k,i}\Theta_{f,j} L^{f+k-v+2\alpha}$$
$$\times \frac{\Gamma(k+1)\Gamma\left(f+k-v+\alpha+\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)}{\Gamma(k+1-v)\Gamma(f+k-v+2\alpha+1)}.$$

Proof. According to the fractional derivative of the SGPs as in Lemma 3.1, with eqs (7) and (9) we can write

$$\int_0^L D^v C^{(\alpha)}_{L,i}(x) C^{(\alpha)}_{L,j}(x)\omega^{\alpha}_L(x)\,\mathrm{d}x$$
$$= \sum_{k=\lceil v\rceil}^{i}\sum_{f=0}^{j} \Theta_{k,i}\Theta_{f,j} \frac{\Gamma(k+1)}{\Gamma(k+1-v)}$$
$$\times \int_0^L x^{f+k-v+\alpha-\frac{1}{2}}(L-x)^{\alpha-\frac{1}{2}}\,\mathrm{d}x$$
$$= \sum_{k=\lceil v\rceil}^{i}\sum_{f=0}^{j} \Theta_{k,i}\Theta_{f,j} L^{f+k-v+2\alpha} \frac{\Gamma(k+1)}{\Gamma(k+1-v)}$$
$$\times \int_0^1 \left(\frac{x}{L}\right)^{f+k-v+\alpha-\frac{1}{2}} \left(1-\frac{x}{L}\right)^{\alpha-\frac{1}{2}} d\left(\frac{x}{L}\right).$$

The last integral part represents the beta function $B(f+k-v+\alpha+\frac{1}{2}, \alpha+\frac{1}{2})$, and using the relation between beta and gamma functions completes the proof. □

The fractional differentiation of the shifted Gegenbauer vector $\phi_{L,N}(x)$ of order $(v > 0)$ can be expressed by

$$D^v\phi_{L,N}(x) \simeq \mathbf{D}^{(v)}_L \phi_{L,N}(x), \quad (16)$$

where $\mathbf{D}^{(v)}_L$ denotes the operational matrix of differentiation of $\phi_{L,N}(x)$ which is the main objective to be computed in this section.

Theorem 3.3. *Let $\phi_{L,N}(x)$ be the shifted Gegenbauer vector and by applying the Caputo fractional derivative of order $(v > 0)$ we get*

$$D^v\phi_{L,N}(x) \simeq \mathbf{D}^{(v)}_L \phi_{L,N}(x),$$

where $\mathbf{D}_L^{(v)}$ is the $(N+1)$ square matrix called operational matrix of fractional differentiation of the form

$$\mathbf{D}_L^{(v)} = \begin{pmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \\ \Psi(\lceil v \rceil, 0, \alpha, v) & \Psi(\lceil v \rceil, 1, \alpha, v) & \dots & \Psi(\lceil v \rceil, N, \alpha, v) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi(i, 0, \alpha, v) & \Psi(i, 1, \alpha, v) & \dots & \Psi(i, N, \alpha, v) \\ \vdots & \vdots & \ddots & \vdots \\ \Psi(N, 0, \alpha, v) & \Psi(N, 1, \alpha, v) & \dots & \Psi(N, N, \alpha, v) \end{pmatrix}.$$

Here

$$\mathbf{D}_L^{(v)} = \begin{cases} \Psi(i, j, \alpha, v) & i \geq \lceil v \rceil, \\ 0 & i < \lceil v \rceil, \end{cases}$$

where

$$\Psi(i, j, \alpha, v) = \sum_{k=\lceil v \rceil}^{i} \Theta_{k,i} \frac{k! L^k}{\Gamma(k+1-v)} \times \sum_{f=0}^{j} \Theta_{f,j} \frac{2(j+\alpha)\Gamma(j+2\alpha)\Gamma(f+k-v+\alpha+\frac{1}{2})L^{f-v}}{j!\Gamma(\alpha+\frac{1}{2})\Gamma(f+k-v+2\alpha+1)}. \tag{17}$$

Proof. Applying eq. (16) and the orthogonality property in eq. (6),we can write

$$\mathbf{D}_L^{(v)} = \langle D^v \phi_{L,N}(x), \phi_{L,N}^T(x) \rangle \Upsilon^{-1}$$

for which $\langle D^v \phi_{L,N}(x), \phi_{L,N}^T(x) \rangle$ and Υ^{-1} are two $N+1$ square matrices defined by

$$\langle D^v \phi_{L,N}(x), \phi_{L,N}^T(x) \rangle = \left[\int_0^L D^v C_{L,i}^{(\alpha)}(x) C_{L,j}^{(\alpha)}(x) \omega_L^{\alpha}(x)\, \mathrm{d}x \right]_{i,j=0}^{N}$$

$$\Upsilon^{-1} = \left[\frac{\delta_{i,j}}{h_{L,i}^{(\alpha)}} \right]_{i,j=0}^{N},$$

where $\delta_{i,j}$ is the Kronecker delta function. Then, by using Lemma 3.2 we get the desired result. □

4. Space–time FTE

In this section, the shifted Gegenbauer operational matrix of fractional differentiation with the tau method are utilised to solve the space–time FTE with various boundary conditions.

4.1 *Operational matrices multiplication product*

We present the following operational matrices product for completely using the spectral tau method on the proposed 1D fractional telegraph equation in the subsequent subsection.

Consider the function $s(x)$ which is smooth and continuous. Therefore, $s(x)$ can be approximated by SGPs as

$$s_N(x) = \mathbf{S}^T \phi_{L,N}(x),$$

where $\mathbf{S}^T = [s_0, s_1, \dots, s_N]$ and the elements s_k; $k = 0, 1, \dots, N$ of the matrix $\mathbf{S}^T$ are calculated from

$$s_k = \frac{1}{h_{L,k}^{(\alpha)}} \int_0^L s(x) C_{L,k}^{(\alpha)}(x) \omega_L^{(\alpha)}(x)\, \mathrm{d}x, \quad k = 0, 1, \dots, N. \tag{18}$$

Now, we are looking for a matrix $\mathbf{W}$ such that

$$\phi_{L,N}(x) \phi_{L,N}^T(x) \mathbf{S} = \mathbf{W}^T \phi_{L,N}(x) \tag{19}$$

which can be written in the form

$$\sum_{k=0}^{N} s_k C_{L,k}^{(\alpha)}(x) C_{L,j}^{(\alpha)}(x) = \sum_{k=0}^{N} w_{kj} C_{L,k}^{(\alpha)}(x), \quad j = 0, 1, \dots, N.$$

Multiplying both sides of the previous relation by $C_{L,i}^{(\alpha)}(x)\omega_L^{\alpha}(x)$, $i = 0, 1, \ldots, N$ and integrating on the interval $[0, L]$, we get

$$\sum_{k=0}^{N} s_k \int_0^L C_{L,k}^{(\alpha)}(x) C_{L,j}^{(\alpha)}(x) C_{L,i}^{(\alpha)}(x) \omega_L^{\alpha}(x)\,\mathrm{d}x = \sum_{k=0}^{N} w_{ij} \int_0^L C_{L,k}^{(\alpha)}(x) C_{L,i}^{(\alpha)}(x) w_L^{\alpha}(x)\,\mathrm{d}x.$$

Now, by applying the orthogonality property on the right-hand side, we get

$$\mathbf{W} = \left[\frac{1}{h_{L,i}^{(\alpha)}} \sum_{k=0}^{N} s_k \int_0^L C_{L,k}^{(\alpha)}(x) C_{L,j}^{(\alpha)}(x) C_{L,i}^{(\alpha)}(x) w_L^{\alpha}(x)\,\mathrm{d}x \right]_{i,j=0}^{N} \tag{20}$$

which defines the desired $(N+1)$ square matrix $\mathbf{W}$.

4.2 *The proposed discretisation methodology*

In this subsection, we consider the general 1D space–time FTE of the following form:

$$\frac{\partial^{\eta} u(x,t)}{\partial t^{\eta}} + p(x) \frac{\partial^{v} u(x,t)}{\partial t^{v}} + q(x) u(x,t) = r(x) \frac{\partial^{\theta} u(x,t)}{\partial x^{\theta}} + f(x,t), \quad 0 < x < L,\ 0 < t \le \tau \tag{21}$$

with initial conditions

$$u(x,0) = h_0(x), \quad \frac{\partial u(x,0)}{\partial t} = h_1(x), \quad 0 \le x \le L \tag{22}$$

and Dirichlet boundary conditions

$$u(0,t) = g_0(t), \quad u(L,t) = g_1(t), \quad 0 \le t \le \tau \tag{23}$$

where $1 < \eta \le 2$, $0 < v \le 1$, $1 < \theta \le 2$, $p(x) > 0$, $q(x) \ge 0$, $r(x) > 0$.

The known and unknown functions $f(x,t)$, $p(x)$, $q(x)$, $r(x)$ and $u(x,t)$ are approximated by finite terms of SGPs and written in matrix forms as follows:

$$\begin{aligned} u_{N,M}(x,t) &= \phi_{\tau,M}^{T}(t) \mathbf{B} \phi_{L,N}(x), \\ f_{N,M}(x,t) &= \phi_{\tau,M}^{T}(t) \mathbf{F} \phi_{L,N}(x), \\ p_N(x) &= \mathbf{P}^T \phi_{L,N}(x), \\ q_N(x) &= \mathbf{Q}^T \phi_{L,N}(x), \\ r_N(x) &= \mathbf{R}^T \phi_{L,N}(x), \end{aligned} \tag{24}$$

where $\mathbf{B}$ is the unknown $(M+1) \times (N+1)$ coefficient matrix to be determined, $\mathbf{F}$ is known as $(M+1) \times (N+1)$ coefficient matrix whose elements $f_{i,j}$ can be determined as in eq. (15), $\mathbf{P}^T$, $\mathbf{Q}^T$ and $\mathbf{R}^T$ are $(N+1)$ row coefficient matrices which can be calculated as in eq. (11).

Now by employing eqs (16) and (24) in eq. (21), we get

$$\begin{aligned} &\phi_{\tau,M}^{T}(t) (\mathbf{D}_{\tau}^{(\eta)})^T \mathbf{B} \phi_{L,N}(x) \\ &\quad + (\mathbf{P}^T \phi_{L,N}(x)) (\phi_{\tau,M}^{T}(t) (\mathbf{D}_{\tau}^{(v)})^T \mathbf{B} \phi_{L,N}(x)) \\ &\quad + \big(\mathbf{Q}^T \phi_{L,N}(x)\big) \big(\phi_{\tau,M}^{T}(t) \mathbf{B} \phi_{L,N}(x)\big) \\ &= \big(\mathbf{R}^T \phi_{L,N}(x)\big) \big(\phi_{\tau,M}^{T}(t) \mathbf{B} \mathbf{D}_{L}^{(\theta)} \phi_{L,N}(x)\big) \\ &\quad + \phi_{\tau,M}^{T}(t) \mathbf{F} \phi_{L,N}(x) \end{aligned}$$

which can be written as

$$\begin{aligned} &\phi_{\tau,M}^{T}(t) (\mathbf{D}_{\tau}^{(\eta)})^T \mathbf{B} \phi_{L,N}(x) \\ &\quad + \phi_{\tau,M}^{T}(t) (\mathbf{D}_{\tau}^{(v)})^T \mathbf{B} (\phi_{L,N}(x) \phi_{L,N}^{T}(x) \mathbf{P}) \\ &\quad + \phi_{\tau,M}^{T}(t) \mathbf{B} (\phi_{L,N}(x) \phi_{L,N}^{T}(x) \mathbf{Q}) \\ &= \phi_{\tau,M}^{T}(t) \mathbf{B} \mathbf{D}_{L}^{(\theta)} \big(\phi_{L,N}(x) \phi_{L,N}^{T}(x) \mathbf{R}\big) \\ &\quad + \phi_{\tau,M}^{T}(t) \mathbf{F} \phi_{L,N}(x). \end{aligned}$$

Applying eq. (19) yields

$$\begin{aligned} &\phi_{\tau,M}^{T}(t) (\mathbf{D}_{\tau}^{(\eta)})^T \mathbf{B} \phi_{L,N}(x) \\ &\quad + \phi_{\tau,M}^{T}(t) (\mathbf{D}_{\tau}^{(v)})^T \mathbf{B} \mathbf{W}_1^T \phi_{L,N}(x) \\ &\quad + \phi_{\tau,M}^{T}(t) \mathbf{B} \mathbf{W}_2^T \phi_{L,N}(x) \\ &= \phi_{\tau,M}^{T}(t) \mathbf{B} \mathbf{D}_{L}^{(\theta)} \mathbf{W}_3^T \phi_{L,N}(x) + \phi_{\tau,M}^{T}(t) \mathbf{F} \phi_{L,N}(x), \end{aligned}$$

where $\mathbf{W}_1$, $\mathbf{W}_2$ and $\mathbf{W}_3$ are calculated from eq. (20). Then the residual has the form

$$\begin{aligned} \textbf{Residual} &= \phi_{\tau,M}^{T}(t) ((\mathbf{D}_{\tau}^{(\eta)})^T \mathbf{B} + (\mathbf{D}_{\tau}^{(v)})^T \mathbf{B} \mathbf{W}_1^T \\ &\quad + \mathbf{B} \mathbf{W}_2^T - \mathbf{B} \mathbf{D}_{L}^{(\theta)} \mathbf{W}_3^T - \mathbf{F}) \phi_{L,N}(x) \\ &= \phi_{\tau,M}^{T}(t) \mathbf{H} \phi_{L,N}(x) \end{aligned} \tag{25}$$

with

$$\mathbf{H} = (\mathbf{D}_{\tau}^{(\eta)})^T \mathbf{B} + (\mathbf{D}_{\tau}^{(\upsilon)})^T \mathbf{B} \mathbf{W}_1^T + \mathbf{B} \mathbf{W}_2^T - \mathbf{B} \mathbf{D}_L^{(\theta)} \mathbf{W}_3^T - \mathbf{F}, \tag{26}$$

where $\mathbf{H}$ is an $(M+1) \times (N+1)$ matrix whose elements are linear algebraic equations of the unknown coefficients; a_{ij} with $i = 0, \ldots, M$, $j = 0, \ldots, N$.

By applying eqs (14) and (16), the initial conditions; eq. (22) takes the form

$$\begin{aligned} &\phi_{\tau,M}^T(0) \mathbf{B} \phi_{L,N}(x) = h_0(x), \\ &\phi_{\tau,M}^T(0) \big(\mathbf{D}_{\tau}^{(1)}\big)^T \mathbf{B} \phi_{L,N}(x) = h_1(x). \end{aligned} \tag{27}$$

Employing eq. (14), the Dirichlet boundary condition eq. (23), can be expressed as

$$\phi_{\tau,M}^T(t) \mathbf{B} \phi_{L,N}(0) = g_0(t), \quad \phi_{\tau,M}^T(t) \mathbf{B} \phi_{L,N}(L) = g_1(t). \tag{28}$$

4.2.1 *Robin boundary condition.* If the boundary conditions of the FTE is defined as a linear combination of the dependent variable value and its normal derivative which have the form

$$\begin{aligned} &\mu_1 u(0,t) + \mu_2 u_x(0,t) = g_0(t), \\ &\sigma_1 u(L,t) + \sigma_2 u_x(L,t) = g_1(t). \end{aligned} \tag{29}$$

Then in terms of eqs (14) and (16), we get

$$\begin{aligned} &\phi_{\tau,M}^T(t)(\mu_1 \mathbf{B} + \mu_2 \mathbf{B} \mathbf{D}_L^{(1)}) \phi_{L,N}(0) = g_0(t), \\ &\phi_{\tau,M}^T(t)(\sigma_1 \mathbf{B} + \sigma_2 \mathbf{B} \mathbf{D}_L^{(1)}) \phi_{L,N}(L) = g_1(t). \end{aligned} \tag{30}$$

4.2.2 *Neumann boundary condition.* If the boundary conditions of the FTE is defined as a value of dependent variable normal derivatives, which have the form

$$u_x(0,t) = g_0(t), \quad u_x(L,t) = g_1(t) \tag{31}$$

then in terms of eq. (16), we get

$$\begin{aligned} &\phi_{\tau,M}^T(t) \mathbf{B} \mathbf{D}_L^{(1)} \phi_{L,N}(0) = g_0(t), \\ &\phi_{\tau,M}^T(t) \mathbf{B} \mathbf{D}_L^{(1)} \phi_{L,N}(L) = g_1(t). \end{aligned} \tag{32}$$

The outline of the SG tau algorithm for solving the 1D-FTE represented by eqs (21)–(23) is described in the sequel.

Input: 1D-FTE; eqs (21)–(23)
$(p(x), q(x), r(x), h_0(x), h_1(x), g_0(t), g_1(t), f(x,t))$

Step 1. Choose $M, N, \alpha, \upsilon, \beta, L, \tau$.
Step 2. Compute $H_{ij} = 0; i = 0, \ldots, M-2,\quad j = 0, \ldots, N-2$ by using eq. (26).
Step 3. Collocate every initial condition of eq. (27) at $(N-1)$ roots of $C_{L,N-1}^{(\alpha)}(x)$.
Step 4. Collocate every boundary condition of eq. (23) at $(M+1)$ roots of $C_{\tau,M+1}^{(\alpha)}(t)$.
Step 5. Join the $(M+1) \times (N+1)$ linear algebraic equations resulting from Steps 2–4 and solve for b_{ij};$i = 0, \ldots, M,\quad j = 0, \ldots, N$.
Output. The approximate solution $u_{N,M}(x,t)$ is computed from eq. (14).

5. Convergence analysis

The aim of this section is to analyse the convergence of the shifted Gegenbauer tau method for the numerical solution of FTEs based on the SGPs by using the operator theory. The spectral rate of convergence for the proposed method is established in the $\mathcal{L}^2_{\omega_\alpha}$-norm. Here, we shall confine ourselves to the FTE of the following form:

$$\begin{cases} \dfrac{\partial^{\eta} u(x,t)}{\partial t^{\eta}} + p(x) \dfrac{\partial^{\upsilon} u(x,t)}{\partial t^{\upsilon}} + q(x) u(x,t) = r(x) \dfrac{\partial^{\theta} u(x,t)}{\partial x^{\theta}} + f(x,t), \\ u(x,0) = 0, \quad \dfrac{\partial u(x,0)}{\partial t} = 0, \quad 0 < x < 1, \\ u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \leq 1, \end{cases} \tag{33}$$

where $1 < \eta \leq 2, 0 < \alpha \leq 1, 1 < \theta \leq 2$. The more complicated boundary conditions for convergence will be presented in each individual case in the examples presented in the next section. The error function of the tau approximation is defined as $e_{N,M}(x,t) = u_{N,M}(x,t) - u(x,t)$, where $u_{N,M}(x,t)$ is the shifted Gegenbauer tau approximation solution of eq. (33) and $u(x,t)$ is the exact solution of eq. (33). Throughout this section, we will denote by C_i; $i = 0, 1, 2, \ldots$ the positive constants independent of N, M but will depend on m, n, υ, η and θ.

The set of SGPs forms a complete $\mathcal{L}^2_{\omega_\alpha}(\Lambda)$ orthogonal system, where $\Lambda = (0,1)$, ω_α is the shifted weight function and $\mathcal{L}^2_{\omega_\alpha}(\Lambda)$ functions $u : \Lambda \to \mathbb{R}$ with $\|u\|_{\mathcal{L}^2_{\omega_\alpha}(\Lambda)} < \infty$. We define

$$\|u\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Lambda)} = \langle u, u \rangle_{\mathcal{L}^2_{\omega_\alpha}} = \int_{\Lambda} u^2(x) \omega_\alpha(x) \, dx.$$

$\mathcal{H}^m_{\omega_\alpha}$ indicates the Sobolev space of all functions $u(x)$ on Λ such that $u(x)$ and all its weak derivatives up to order m are in $\mathcal{L}^2_{\omega_\alpha}(\Lambda)$. The norm and the semi-norm of $\mathcal{H}^m_{\omega_\alpha}$ are defined by

$$\|u\|^2_{\mathcal{H}^m_{\omega_\alpha}(\Lambda)} = \sum_{k=0}^{m} \left\| \frac{\partial^k}{\partial t^k} u(x) \right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Lambda)},$$

$$|u|^2_{\mathcal{H}^{m:M}_{\omega_\alpha}(\Lambda)} = \sum_{k=\min(m,N+1)}^{N} \left\| \frac{\partial^k}{\partial t^k} u(x) \right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Lambda)}.$$

Let $\mathcal{P}_N(\Lambda)$ be the space of all polynomials of degree up to N. Denote Π_N as the orthogonal projection operator $\Pi_N : \mathcal{L}^2_{\omega_\alpha}(\Lambda) \to \mathcal{P}_N(\Lambda)$ such that

$$\langle \Pi_N u - u, v \rangle = 0, \quad \forall v \in \mathcal{P}_N.$$

According to [23], the estimate of the truncation error of SGPs for $\lambda \geq 0$, $\mu \leq \lambda$ and $u \in \mathcal{H}^\lambda_{\omega_\alpha}(\Lambda)$:

$$\|\Pi_N u - u\|_{\mathcal{H}^\mu_{\omega_\alpha}(\Lambda)} \leq C N^{\sigma(\mu,\lambda)} \|u\|_{\mathcal{H}^\lambda_{\omega_\alpha}(\Lambda)}, \tag{34}$$

where

$$\sigma(\mu, \lambda) = \begin{cases} 2\mu - \lambda - \frac{1}{2}, & \mu > 1, \\ \frac{3}{2}\mu - \lambda, & 0 \leq \mu \leq 1, \\ \mu - \lambda, & \mu < 0. \end{cases}$$

Now, we consider a two-dimensional domain, say $\Omega = \Lambda_t \times \Lambda_x$ and consider

Theorem 5.1. *Consider the orthogonal projection $\Pi_{N,M} : \mathcal{L}^2_{\omega_\alpha}(\Omega) \to \mathcal{P}_{N,M}$, i.e., for any function $u \in \mathcal{L}^2_{\omega_\alpha}(\Omega)$*

$$\Pi_{N,M}(u(x,t)) = u_{N,M}(x,t).$$

Then, for all $a, b \geq 0$, we have

$$\left\|u - \Pi_{N,M} u\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \leq C_1 M^{-a} \|u\|_{\mathcal{H}^{a,0}_{\omega_\alpha}(\Omega)} + C_2 N^{-b} \|u\|_{\mathcal{H}^{0,b}_{\omega_\alpha}(\Omega)}$$

for all u in which the norms on the right-hand side are finite.

Proof. Let Π_N and Π_M be one-dimensional orthogonal projections. Then,

$$\Pi_{N,M} u = \Pi_N \circ (\Pi_M u).$$

Using inequality (34) leads to

$$\begin{aligned} &\left\|u - \Pi_{N,M} u\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \\ &\quad \leq \|u - \Pi_N u\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} + \|\Pi_N \circ (u - \Pi_M u)\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \\ &\quad \leq \|u - \Pi_N u\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} + C_2 \|(u - \Pi_M u)\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \\ &\quad \leq C_1 M^{-a} \|u\|_{\mathcal{H}^{a,0}_{\omega_\alpha}(\Omega)} + C_2 N^{-b} \|u\|_{\mathcal{H}^{0,b}_{\omega_\alpha}(\Omega)}. \end{aligned} \tag{35}$$

This completes the proof. □

$$\mathcal{P}_{N,M} = \operatorname{span}\left\{C^{(\alpha)}_{L,0}(x), C^{(\alpha)}_{L,1}(x), ..., C^{(\alpha)}_{L,N}(x), C^{(\alpha)}_{L,0}(t), C^{(\alpha)}_{L,1}(t), ..., C^{(\alpha)}_{L,M}(t)\right\}.$$

The Hilbert space $\mathcal{H}^{a,b}_{\omega_\alpha}(\Omega)$ of measurable functions $u : \Omega \to \mathbb{R}$ is defined as follows:

$$\mathcal{H}^{a,b}_{\omega_\alpha}(\Omega) = \mathcal{H}^b_{\omega_\alpha}(\Lambda_t; \mathcal{H}^a_{\omega_\alpha}(\Lambda_x)) = \left\{u \in \mathcal{L}^2_{\omega_\alpha}(\Omega) \middle| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \in \mathcal{L}^2_{\omega_\alpha}(\Omega), 0 \leq i \leq a, 0 \leq j \leq b\right\}$$

endowed with the norm

$$\|u\|^2_{\mathcal{H}^{a,b}_{\omega_\alpha}} = \sum_{i=0}^{a} \sum_{j=0}^{b} \left\| \frac{\partial^{i+j} u}{\partial x^i \partial t^j} \right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)},$$

where $\mathcal{H}^{a,0}_{\omega_\alpha} = \mathcal{L}^2_{\omega_\alpha}(\Lambda_t; \mathcal{H}^a_{\omega_\alpha}(\Lambda_x))$ and $\mathcal{H}^{0,b}_{\omega_\alpha} = \mathcal{H}^b_{\omega_\alpha}(\Lambda_t; \mathcal{L}^2_{\omega_\alpha}(\Lambda_x))$, endowed with the norms

$$\|u\|^2_{\mathcal{H}^{a,0}_{\omega_\alpha}} = \int_0^1 \|u(\cdot, t)\|^2_{\mathcal{H}^a_{\omega_\alpha}(\Lambda_x)} \, \mathrm{d}t,$$

$$\|u\|^2_{\mathcal{H}^{0,b}_{\omega_\alpha}} = \sum_{j=0}^{b} \left\| \frac{\partial^j u}{\partial t^j} \right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}.$$

Now, we state and prove the convergence theorem analysis of the shifted Gegenbauer tau method which is the main result of this section.

Theorem 5.2. *Let $u(x,t)$ be the exact solution of FTE eq. (33) and $\Pi_{N,M}(u(x,t)) = u_{N,M}(x,t)$ be the spectral Gegenbauer tau approximation eq. (14) to $u(x,t)$ defined by eq. (25). Assume that $u \in \mathcal{L}^2_{\omega_\alpha}(\Omega)$. Then, for sufficiently smooth functions $p(x)$, $q(x)$ and $r(x)$ in eq. (33) and for sufficiently large N and M we get*

$$\left\|u_{N,M}(x,t) - u(x,t)\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \to 0.$$

Proof. Let us define the error function $e_{N,M}(u(x,t)) = \Pi_{N,M}(u(x,t)) - u(x,t)$ where $u(x,t)$ is a continuous function on Ω. Following the proposed method, we get

$$\begin{aligned} q(x)u_{N,M}(x,t) &= \frac{r(x)}{\Gamma(m-\theta)}\Pi_{N,M}\left(\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right) \\ &\quad - \frac{1}{\Gamma(m-\eta)}\Pi_{N,M}\left(\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m u_{N,M}(x,\tau)}{\partial\tau^m}\,\mathrm{d}\tau\right) \\ &\quad - \frac{p(x)}{\Gamma(n-v)}\Pi_{N,M}\left(\int_0^t (t-\tau)^{n-v-1}\frac{\partial^n u_{N,M}(x,\tau)}{\partial\tau^n}\,\mathrm{d}\tau\right) \\ &\quad + \Pi_{N,M}\left(f(x,t)\right). \end{aligned} \tag{36}$$

Again, we can write eq. (33) as

$$\begin{aligned} q(x)u(x,t) &= \frac{r(x)}{\Gamma(m-\theta)}\left(\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right) \\ &\quad - \frac{1}{\Gamma(m-\eta)}\left(\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m u(x,\tau)}{\partial\tau^m}\,\mathrm{d}\tau\right) \\ &\quad - \frac{p(x)}{\Gamma(n-v)}\left(\int_0^t (t-\tau)^{n-v-1}\frac{\partial^n u(x,\tau)}{\partial\tau^n}\,\mathrm{d}\tau\right) \\ &\quad + (f(x,t)). \end{aligned} \tag{37}$$

Subtracting eq. (37) from eq. (36), we get

$$q(x)e_{N,M}(x,t) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7, \tag{38}$$

where

$$\begin{aligned} I_1 &= \frac{r(x)}{\Gamma(m-\theta)}e_{N,M}\left(\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right), \\ I_2 &= \frac{r(x)}{\Gamma(m-\theta)}\left(\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m e_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right), \\ I_3 &= \frac{1}{\Gamma(m-\eta)}e_{N,M}\left(\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m u_{N,M}(x,\tau)}{\partial\tau^m}\,\mathrm{d}\tau\right), \\ I_4 &= \frac{1}{\Gamma(m-\eta)}\left(\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m e_{N,M}(x,\tau)}{\partial\tau^m}\,\mathrm{d}\tau\right), \\ I_5 &= \frac{p(x)}{\Gamma(n-v)}e_{N,M}\left(\int_0^t (t-\tau)^{n-v-1}\frac{\partial^n u_{N,M}(x,\tau)}{\partial\tau^n}\,\mathrm{d}\tau\right) \\ I_6 &= \frac{p(x)}{\Gamma(n-v)}\left(\int_0^t (t-\tau)^{n-v-1}\frac{\partial^n e_{N,M}(x,\tau)}{\partial\tau^n}\,\mathrm{d}\tau\right) \\ I_7 &= e_{N,M}(f(x,t)). \end{aligned}$$

Let C_0 be a positive constant such that $\|q(x)\|_{\mathcal{L}^1_{\omega_\alpha}(\Lambda_x)} \geq C_0$. Then we can write

$$\left\|e_{N,M}(x,t)\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \leq \frac{1}{C_0}\sum_{j=1}^{7}\left\|I_j\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}. \tag{39}$$

Now, it is sufficient to show that the right-hand side of inequality (39) tends to zero for sufficiently large N and M. For this, by using inequality (35), we obtain

$$\begin{aligned} &\|I_1\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \\ &\leq C_1 M^{-1}\left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|_{\mathcal{H}^{1,0}_{\omega_\alpha}} \\ &+ C_2 N^{-1}\left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|_{\mathcal{H}^{0,1}_{\omega_\alpha}}. \end{aligned} \tag{40}$$

The first term on the right-hand side of the previous inequality can be written as follows:

$$\left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|^2_{\mathcal{H}^{1,0}_{\omega_\alpha}} = \left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} + \left\|\frac{\partial}{\partial x}\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}. \tag{41}$$

Due to eq. (2), we get the following relation:

$$\frac{\partial}{\partial x}\int_0^x (x-\xi)^{m-\theta-1}u(\xi)\,\mathrm{d}\xi = \int_0^x (x-\xi)^{m-\theta-1}\frac{\partial u(\xi)}{\partial\xi}\,\mathrm{d}\xi. \tag{42}$$

Also, using Young inequality [24], we obtain

$$\begin{aligned}
&\left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|^2_{\mathcal{H}^{1,0}_{\omega_\alpha}}\\
&= \left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} + \left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^{m+1} u_{N,M}(\xi,t)}{\partial\xi^{m+1}}\,\mathrm{d}\xi\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&\le \left\|(x-\xi)^{m-\theta-1}\right\|^2_{\mathcal{L}^1_{\omega_\alpha}(\Lambda_x)}\left\|\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} + \left\|(x-\xi)^{m-\theta-1}\right\|^2_{\mathcal{L}^1_{\omega_\alpha}(\Lambda_x)}\left\|\frac{\partial^{m+1} u_{N,M}(\xi,t)}{\partial\xi^{m+1}}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&\le C_3\left\|\frac{\partial^m u_{N,M}(x,t)}{\partial x^m}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} + C_3\left\|\frac{\partial^{m+1} u_{N,M}(x,t)}{\partial x^{m+1}}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&= C_3\left\|\frac{\partial^m u_{N,M}(x,t)}{\partial x^m}\right\|^2_{\mathcal{H}^{1,0}_{\omega_\alpha}}\\
&\le C_3\left\|u_{N,M}(x,t)\right\|_{\mathcal{H}^{m+1,0}_{\omega_\alpha}}\\
&= C_3\left\|e_{N,M}(x,t)-u(x,t)\right\|_{\mathcal{H}^{m+1,0}_{\omega_\alpha}}\\
&\le C_3\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{m+1,0}_{\omega_\alpha}} + \left\|u(x,t)\right\|_{\mathcal{H}^{m+1,0}_{\omega_\alpha}}\right)^2.
\end{aligned} \tag{43}$$

Also, for the second term on the right-hand side of inequality (40)

$$\begin{aligned}
&\left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m u_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|^2_{\mathcal{H}^{0,1}_{\omega_\alpha}}\\
&\le C_3\left\|\frac{\partial^m u_{N,M}(x,t)}{\partial x^m}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} + C_3\left\|\frac{\partial^{m+1} u_{N,M}(x,t)}{\partial x^m\partial t}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&= C_3\left\|u_{N,M}(x,t)\right\|^2_{\mathcal{H}^{m,1}_{\omega_\alpha}}\\
&\le C_3\left\|e_{N,M}(x,t)-u(x,t)\right\|^2_{\mathcal{H}^{m,1}_{\omega_\alpha}}\\
&\le C_3\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{m,1}_{\omega_\alpha}} + \left\|u(x,t)\right\|_{\mathcal{H}^{m,1}_{\omega_\alpha}}\right)^2.
\end{aligned} \tag{44}$$

With reference to inequalities (40), (43) and (44), we get

$$\begin{aligned}
&\|I_1\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&\le C_4M^{-1}\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{m+1,0}_{\omega_\alpha}} + \left\|u(x,t)\right\|_{\mathcal{H}^{m+1,0}_{\omega_\alpha}}\right)\\
&\quad + C_5N^{-1}\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{m,1}_{\omega_\alpha}} + \left\|u(x,t)\right\|_{\mathcal{H}^{m,1}_{\omega_\alpha}}\right).
\end{aligned} \tag{45}$$

From inequality (34), we can derive $\|I_1\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \to 0$ for sufficiently large N and M.

Similar to I_1, using Young inequality and properties of Sobolev norm, we get

$$\begin{aligned}
&\|I_2\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&\le \left\|\int_0^x (x-\xi)^{m-\theta-1}\frac{\partial^m e_{N,M}(\xi,t)}{\partial\xi^m}\,\mathrm{d}\xi\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&\le \left\|(x-\xi)^{m-\theta-1}\right\|_{\mathcal{L}^1_{\omega_\alpha}(\Lambda_x)}\left\|\frac{\partial^m e_{N,M}(\xi,t)}{\partial\xi^m}\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&\le C_3\left\|\frac{\partial^m e_{N,M}(x,t)}{\partial x^m}\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&\le C_3\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{m,0}_{\omega_\alpha}(\Omega)}.
\end{aligned} \tag{46}$$

Using inequality (34), we conclude $\|I_2\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \to 0$ for sufficiently large N and M. For the third term, we may write

$$\begin{aligned}
&\|I_3\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\\
&\le C_6M^{-1}\left\|\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m u_{N,M}(x,\tau)}{\partial\tau^m}\mathrm{d}\tau\right\|_{\mathcal{H}^{1,0}_{\omega_\alpha}}\\
&+ C_7N^{-1}\left\|\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m u_{N,M}(x,\tau)}{\partial\tau^m}\mathrm{d}\tau\right\|_{\mathcal{H}^{0,1}_{\omega_\alpha}}.
\end{aligned} \tag{47}$$

In a similar manner with eq. (42) for the variable t, the first term on the right-hand side of inequality (47) is written as

$$\left\|\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m u_{N,M}(x,\tau)}{\partial \tau^m}\,d\tau\right\|^2_{\mathcal{H}^{1,0}_{\omega_\alpha}}$$
$$=\left\|\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m u_{N,M}(x,\tau)}{\partial \tau^m}\,d\tau\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$+\left\|\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^{m+1} u_{N,M}(x,\tau)}{\partial x\partial \tau^m}\,d\tau\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$\le C_8\left\|\frac{\partial^m u_{N,M}(x,t)}{\partial t^m}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$+C_8\left\|\frac{\partial^{m+1} u_{N,M}(x,t)}{\partial x\partial t^m}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$=C_8\left\|u_{N,M}(x,t)\right\|^2_{\mathcal{H}^{1,m}_{\omega_\alpha}}$$
$$\le C_8\left\|e_{N,M}(x,t)-u(x,t)\right\|^2_{\mathcal{H}^{1,m}_{\omega_\alpha}}$$
$$\le C_8\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{1,m}_{\omega_\alpha}}+\|u(x,t)\|_{\mathcal{H}^{1,m}_{\omega_\alpha}}\right)^2. \quad (48)$$

Moreover, the second term can be estimated as follows:

$$\left\|\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m u_{N,M}(x,\tau)}{\partial \tau^m}\,d\tau\right\|^2_{\mathcal{H}^{0,1}_{\omega_\alpha}}$$
$$\le C_8\left\|\frac{\partial^m u_{N,M}(x,t)}{\partial t^m}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$+C_8\left\|\frac{\partial^{m+1} u_{N,M}(x,t)}{\partial t^{m+1}}\right\|^2_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$=C_8\left\|u_{N,M}(x,t)\right\|^2_{\mathcal{H}^{0,m+1}_{\omega_\alpha}}$$
$$\le C_8\left\|e_{N,M}(x,t)-u(x,t)\right\|^2_{\mathcal{H}^{0,m+1}_{\omega_\alpha}}$$
$$\le C_8\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{0,m+1}_{\omega_\alpha}}+\|u(x,t)\|_{\mathcal{H}^{0,m+1}_{\omega_\alpha}}\right)^2. \quad (49)$$

Making use of inequalities (47)–(49), we get

$$\|I_3\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$\le C_9M^{-1}\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{1,m}_{\omega_\alpha}}+\|u(x,t)\|_{\mathcal{H}^{1,m}_{\omega_\alpha}}\right)$$
$$+C_{10}N^{-1}\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{0,m+1}_{\omega_\alpha}}+\|u(x,t)\|_{\mathcal{H}^{0,m+1}_{\omega_\alpha}}\right). \quad (50)$$

Using inequality (34), we derive $\|I_3\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \to 0$ for sufficiently large N and M. For the fourth term, we may write

$$\|I_4\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$\le\left\|\int_0^t (t-\tau)^{m-\eta-1}\frac{\partial^m e_{N,M}(x,\tau)}{\partial \tau^m}\,d\tau\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$\le C_8\left\|\frac{\partial^m e_{N,M}(x,t)}{\partial t^m}\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\le C_8\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{0,m}_{\omega_\alpha}} \quad (51)$$

Making use of inequality (34), we conclude $\|I_4\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \to 0$ for sufficiently large N and M.

For the fifth term, in a similar manner with I_3, we can conclude that

$$\|I_5\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$\le C_{11}M^{-1}\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{1,n}_{\omega_\alpha}}+\|u(x,t)\|_{\mathcal{H}^{1,n}_{\omega_\alpha}}\right)$$
$$+C_{12}N^{-1}\left(\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{0,n+1}_{\omega_\alpha}}+\|u(x,t)\|_{\mathcal{H}^{0,n+1}_{\omega_\alpha}}\right). \quad (52)$$

For the sixth term, in a similar manner with I_4, we can conclude that

$$\|I_6\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}\le C_{13}\left\|\frac{\partial^n e_{N,M}(x,t)}{\partial t^n}\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}$$
$$\le C_{13}\left\|e_{N,M}(x,t)\right\|_{\mathcal{H}^{0,n}_{\omega_\alpha}}. \quad (53)$$

From inequality (34), we can conclude $\|I_5\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \to 0$ and $\|I_6\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \to 0$ for sufficiently large N and M.

$$\|I_7\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}=\left\|e_{N,M}(f(x,t))\right\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)}. \quad (54)$$

From inequality (34), we can conclude $\|I_7\|_{\mathcal{L}^2_{\omega_\alpha}(\Omega)} \to 0$ for sufficiently large N and M.

The convergence result of the proposed space–time SG tau method is given by joining inequalities (39), (45), (46), (50)–(54). The obtained results show that the convergence of the approximate solution $u_{N,M}(x,t)$ as $N, M \to \infty$ depends on how many times $u(x,t)$ is differentiable with respect to x and t. □

6. Numerical applications

In this section, four examples are introduced for constant and variable coefficient space–time FTEs with boundary conditions, namely, Neumann, Dirichlet and Robin conditions. The absolute error (AE) is defined as

Table 1. MAEs at different time level with $N = M = 8$, $L = 1$, $v = 0.9$ and $\beta = 0.95$ for Example 1.

$\tau = 1$		$\tau = 2$		$\tau = 3$	
t	MAEs(x, t)	t	MAEs(x, t)	t	MAEs(x, t)
0.0	1.01×10^{-10}	0.0	5.54×10^{-10}	0.0	1.46×10^{-7}
0.1	1.27×10^{-10}	0.2	2.91×10^{-9}	0.3	4.86×10^{-8}
0.2	1.37×10^{-10}	0.4	2.37×10^{-8}	0.6	5.26×10^{-7}
0.3	1.36×10^{-10}	0.6	1.33×10^{-8}	0.9	3.79×10^{-7}
0.4	1.81×10^{-10}	0.8	3.23×10^{-8}	1.2	7.10×10^{-7}
0.5	8.21×10^{-11}	1.0	1.25×10^{-8}	1.5	4.14×10^{-7}
0.6	1.60×10^{-10}	1.2	3.72×10^{-8}	1.8	8.59×10^{-7}
0.7	8.16×10^{-11}	1.4	6.82×10^{-9}	2.1	3.00×10^{-7}
0.8	1.14×10^{-10}	1.6	2.99×10^{-8}	2.4	7.49×10^{-7}
0.9	7.50×10^{-11}	1.8	1.38×10^{-8}	2.7	3.86×10^{-7}
1.0	6.87×10^{-11}	2.0	3.49×10^{-9}	3.0	1.21×10^{-7}

Table 2. A comparison of MAEs at various values of N=M for Example 1.

N	$v = 0.2, \beta = 0.9$			$v = 0.9, \beta = 0.6$		
	MAE (x, t)	CO	CPU time	MAE (x, t)	CO	CPU time
2	6.64×10^{-2}	–	4.734	6.19×10^{-2}	–	4.859
4	1.01×10^{-4}	9.3639	11.859	3.99×10^{-4}	7.2770	11.938
6	2.79×10^{-7}	14.5233	28.469	1.22×10^{-6}	14.2807	29.547
8	2.22×10^{-10}	24.8110	59.594	2.62×10^{-9}	21.3586	59.922

$$\mathrm{AE}(x, t) = |u(x, t) - u_{N,M}(x, t)|, \tag{55}$$

where $u(x, t)$ is the exact solution and $u_{N,M}(x, t)$ is the approximate solution. Moreover, the maximum absolute errors (MAE) can be computed by

$$\mathrm{MAEs}(x, t) = \max_{(x,t)\in\Omega} \mathrm{AE}(x, t), \quad \Omega = [0, L] \times [0, \tau]. \tag{56}$$

The numerical order of convergence is calculated from

$$\text{Convergence order (CO)} = \frac{\log(\text{error}(M_1)/\text{error}(M_2))}{\log(M_2/M_1)}. \tag{57}$$

All computations are carried out by using Mathematica 10, on a DESKTOP-2L1UJVK with the processor: Intel(R) Core(TM) i3-6100 CPU, 3.70GHz and 8.00G RAM, with 64 bit operating system.

Example 1. Consider the following space–time FTE [25]:

$$\frac{\partial^{v+1}u(x, t)}{\partial t^{v+1}} + \frac{\partial^{v}u(x, t)}{\partial t^{v}} + u(x, t) = \frac{\partial^{2\beta}u(x, t)}{\partial x^{2\beta}} + f(x, t),$$

$$0 < x < 1, 0 < t \le \tau. \tag{58}$$

With the initial and Neumann boundary conditions:

$$u(x, 0) = \mathrm{e}^{x}, \quad \frac{\partial u(x, 0)}{\partial t} = -\mathrm{e}^{x}, \quad 0 < x < 1 \tag{59}$$

$$u_x(0, t) = \mathrm{e}^{-t}, \quad u_x(1, t) = \mathrm{e}^{1-t}, \quad 0 < t \le \tau. \tag{60}$$

With the exact solution $u(x, t) = \mathrm{e}^{x-t}$ and

$$f(x, t) = \left(\mathrm{e}^{x} - x^{2-\beta}E_{1,3-\beta}(x)\right)\mathrm{e}^{-t}.$$

In table 1, we depict the absolute error with different temporal end values $\tau = 1, 2, 3$ at various time levels for $N = M = 8$ when $v = 0.9$ and $\beta = 0.95$. Table 2 illustrates the MAEs, convergence order and computational time with various choices of $N = M$ at two different parameter values $v = 0.2, \beta = 0.9$ and $v = 0.9, \beta = 0.6$ which demonstrates that the errors are depreciated rapidly with small increase in N and M whatever the values of v and β. Figure 1 shows the error behaviour at time levels $t = 0.1, 0.5, 0.9$ when $v = 0.65$ and $\beta = 0.925$.

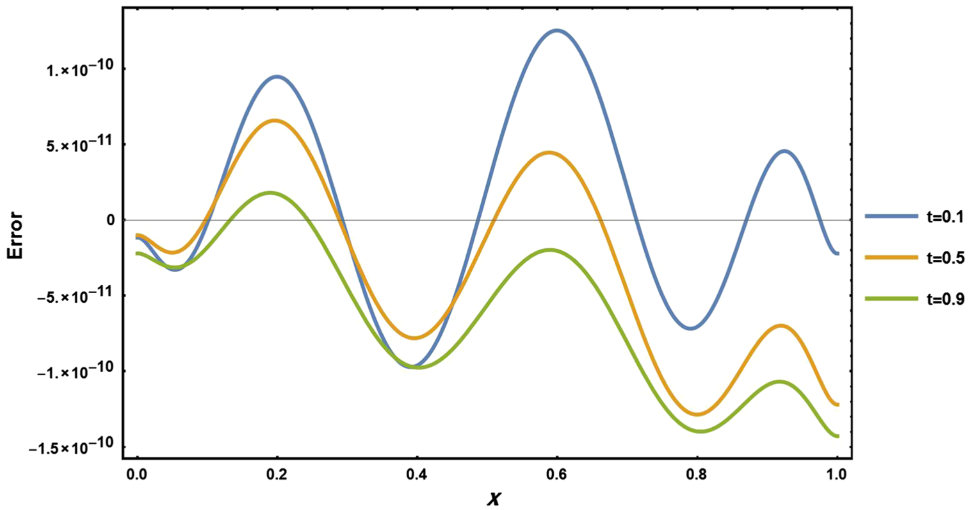

Figure 1. Errors at different time level with $\upsilon = 0.65$ and $\beta = 0.925$ for Example 1.

Example 2. Consider the following variable coefficients TFTE [12]:

$$\frac{\partial^{\upsilon} u(x,t)}{\partial t^{\upsilon}} + p(x)\frac{\partial^{\upsilon-1} u(x,t)}{\partial t^{\upsilon-1}} + q(x)u(x,t) = r(x)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < 1, 0 < t \leq 1. \tag{61}$$

With the initial and Robin boundary conditions:

$$u(x,0) = x\mathrm{e}^{-x}, \frac{\partial u(x,0)}{\partial t} = \mathrm{e}^{-x}, \quad 0 < x < 1, \tag{62}$$

$$u(0,t) + \frac{\partial u(0,t)}{\partial x} = 1, \quad u(1,t) - \frac{1}{2}\frac{\partial u(1,t)}{\partial x} = \frac{1}{2}(2+3t)\mathrm{e}^{-1}, \quad 0 < t \leq 1. \tag{63}$$

With the exact solution $u(x,t) = (x+t)\mathrm{e}^{-x}$

$$f(x,t) = \left(p(x)\frac{t^{1-\upsilon}}{\Gamma(2-\upsilon)} + q(x)(x+t) - r(x)(x+t-2) \right)\mathrm{e}^{-x}.$$

In table 3, we list the MAEs at different values of $p(x)$, $q(x)$ and $r(x)$. It is worth mentioning that the FTE covers various classes of fractional partial differential equation. Cases I and II represent FTE with constant and variable coefficients, Case III defines the time-fractional non-homogeneous Klien–Gordon equation, Case IV describes the time-fractional wave equation, Case V for $r(x) > 0$ defines variable coefficients time-fractional diffusion-wave equation with damping and finally, Case VI for $q(x) > 0$ and $r(x) > 0$ depicts variable coefficients time-fractional reaction dispersion equation.

Example 3. Consider the following variable coefficients time FTE [26]:

$$\frac{\partial^{2\upsilon} u(x,t)}{\partial t^{2\upsilon}} + 2x^2\frac{\partial^{\upsilon} u(x,t)}{\partial t^{\upsilon}} + x^2 u(x,t) = (1+x)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 < x \leq 1, 0 < t \leq 1. \tag{64}$$

With the initial and Dirichlet boundary conditions:

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 < x < 1, \tag{65}$$

$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \leq 1. \tag{66}$$

With the exact solution $u(x,t) = t^2(1-x)\sinh(x)$

$$f(x,t) = \left(\frac{\Gamma(3)}{\Gamma(3-2\upsilon)}t^{2-2\upsilon} + \frac{2\Gamma(3)}{\Gamma(3-\upsilon)}x^2 t^{2-\upsilon} + (x^2-x-1)t^2 \right)(1-x)\sinh(x) + 2t^2(1+x)\cosh(x).$$

In table 4, the MAEs, convergence order and computational time with $\upsilon = 0.85$ for $M = 2$ and various values of N are listed. The MAEs at $\alpha = -0.49, 0.5$ and various time levels with $M = 2$, $N = 8$ and $\upsilon = 1$ are compared with the meshless local radial point interpolation (MLRPI) method [26] with space and time step size $\Delta x = 0.0125$ and $\Delta t = 0.001$ in table 5. In addition, the exact and approximate solution curves at $t = 0.1, 0.4, 0.7, 1.0$ are plotted in figure 2.

Table 3. MAEs at $\alpha = 1$ with $N = 9$, $M = 2$ for Example 2.

Case	$p(x)$	$q(x)$	$r(x)$	η	$\upsilon = 1.05$	$\upsilon = 1.5$	$\upsilon = 1.95$
I	1	1	1	$\upsilon - 1$	2.176×10^{-9}	5.796×10^{-10}	3.379×10^{-10}
II	x	x	x	$\upsilon - 1$	4.841×10^{-10}	2.100×10^{-10}	1.326×10^{-10}
III	0	1	1	–	2.320×10^{-9}	1.790×10^{-9}	1.475×10^{-9}
IV	0	0	1	–	1.278×10^{-9}	4.165×10^{-9}	1.052×10^{-9}
V	1	0	x	1	1.489×10^{-6}	1.187×10^{-6}	1.011×10^{-6}
					$\upsilon = 0.05$	$\upsilon = 0.5$	$\upsilon = 0.95$
VI	0	x	x	–	1.952×10^{-9}	6.553×10^{-9}	1.099×10^{-9}

Table 4. MAEs at $\alpha = 0.5$, $\upsilon = 0.85$, $M = 2$ and various choices of M for Example 3.

N	MAEs(x, t)	CO	CPU time
2	1.83×10^{-2}	–	8.797
4	1.19×10^{-4}	7.257	17.313
6	1.36×10^{-7}	16.715	33.953
8	8.50×10^{-10}	17.639	69.344

Table 5. A comparison of MAEs at various time levels with $\upsilon = 1$ for Example 3.

	MLRPI method [26]	The proposed method	
	$\Delta x = 0.0125$, $\Delta t = 0.001$	$M = 2$, $N = 8$	
t		$\alpha = -0.49$	$\alpha = 0.5$
0.1	1.024×10^{-8}	9.690×10^{-12}	2.320×10^{-11}
0.2	1.624×10^{-8}	3.876×10^{-11}	9.279×10^{-11}
0.3	1.793×10^{-8}	8.721×10^{-11}	2.088×10^{-10}
0.4	1.752×10^{-8}	1.550×10^{-10}	3.712×10^{-10}
0.5	1.707×10^{-8}	2.422×10^{-10}	5.799×10^{-10}
0.6	1.555×10^{-8}	3.488×10^{-10}	8.351×10^{-10}
0.7	1.096×10^{-8}	4.748×10^{-10}	1.137×10^{-9}
0.8	4.664×10^{-9}	6.201×10^{-10}	1.485×10^{-9}
0.9	5.918×10^{-9}	7.849×10^{-10}	1.879×10^{-9}

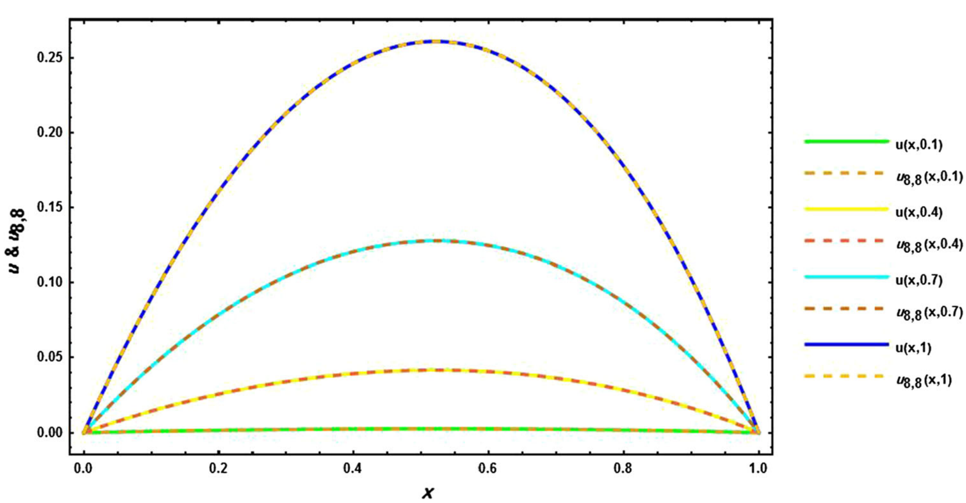

Figure 2. Exact and approximate solutions for various values of t when $\upsilon = 0.55$ and $N = M = 8$ for Example 3.

Table 6. A comparison of MAEs with $t = 1$ and $\beta = 1$ for Example 4.

	RWM [27]	CTM [14]	BCM [28]	JCM [15]	The proposed method $N = 2, M = 7$	
x	$N = 6, M$=9	$N = M = 7$	$N = 7$	$N = 4, M = 7$	$\alpha = 0.00001$	$\alpha = 0.5$
0.125	1.2×10^{-6}	5.8×10^{-8}	1.25×10^{-7}	1.22×10^{-9}	3.54×10^{-9}	1.23×10^{-10}
0.250	1.9×10^{-6}	1.0×10^{-7}	1.06×10^{-8}	1.89×10^{-9}	6.06×10^{-9}	2.10×10^{-10}
0.375	2.4×10^{-6}	1.2×10^{-7}	3.72×10^{-7}	2.29×10^{-9}	7.58×10^{-9}	2.63×10^{-10}
0.500	2.6×10^{-6}	1.3×10^{-7}	6.21×10^{-7}	2.57×10^{-9}	8.08×10^{-9}	2.80×10^{-10}
0.625	2.4×10^{-6}	1.2×10^{-7}	3.71×10^{-7}	2.73×10^{-9}	7.58×10^{-9}	2.63×10^{-10}
0.750	1.9×10^{-6}	1.0×10^{-7}	1.08×10^{-8}	2.59×10^{-9}	6.06×10^{-9}	2.10×10^{-10}
0.875	1.2×10^{-6}	5.8×10^{-8}	1.25×10^{-7}	1.84×10^{-9}	3.54×10^{-9}	1.23×10^{-10}

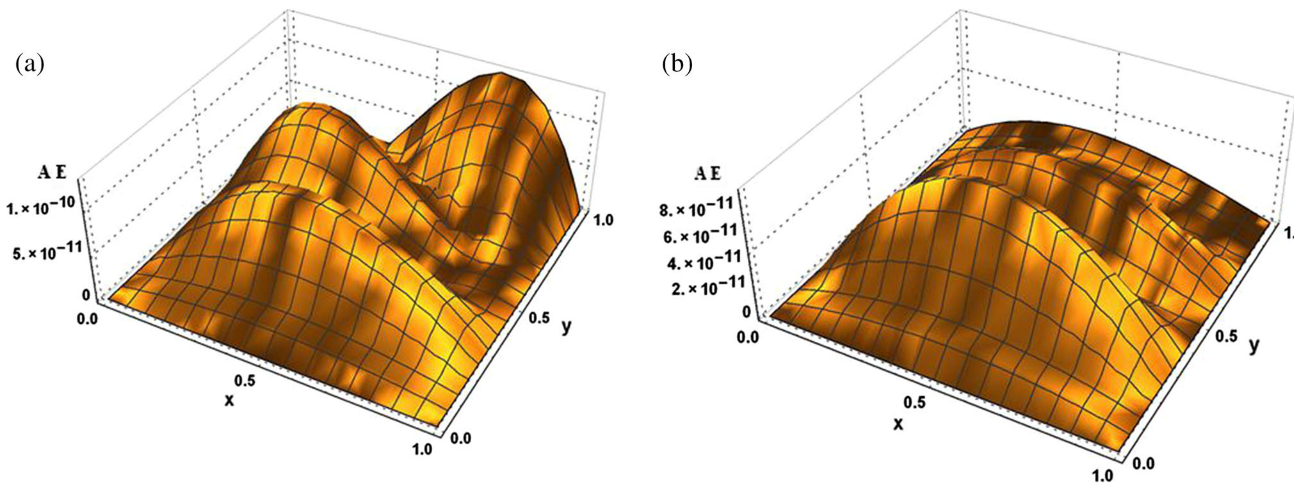

Figure 3. The surfaces of absolute errors when $N = 3$ and $M = 9$ for (**a**) $\alpha = 0.001$ and $\beta = 0.55$ and (**b**) $\alpha = 0.00001$ and $\beta = 0.9$ for Example 4.

Example 4. Consider the following space FTE [15]:

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} + u(x,t) = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} + f(x,t),$$
$$0 < x < 0, 0 < t \leq 1, \quad (67)$$

with the initial and Dirichlet boundary conditions:

$$u(x,0) = 0, \quad \frac{\partial u(x,0)}{\partial t} = 0, \quad 0 < x < 1, \quad (68)$$
$$u(0,t) = 0, \quad u(1,t) = 0, \quad 0 < t \leq 1, \quad (69)$$

With the exact solution $u(x,t) = t^2(x - x^2)\mathrm{e}^{-t}$

$$f(x,t) = \left((2 - 2t + t^2)(x - x^2) + \frac{\Gamma(3)}{\Gamma(3 - 2v)} x^{2-2v} t^2\right) \mathrm{e}^{-t}.$$

In table 6, at fixed time $t = 1$, $\beta = 1$ and various space levels, we compare the proposed SG tau method at $\alpha =$ 0.00001, 0.5 with Chebyshev tau method (CTM) [14], Jacobi collocation method (JCM) [15], Rothe wavelet method (RwM) [27] and Bessel collocation method (BCM) [28]. Among all the previous methods, it is noted that the proposed method presents the best approximated solution of the given problem with a small number of Gegenbauer functions. Figure 3 depicts the accuracy of the proposed method at space-fractional derivative parameter $\beta = 0.9$ by plotting the absolute error function with $N = 3$ and $M = 9$.

7. Conclusion

In this article, we presented an efficient algorithm for solving a class of constant and variable coefficient space–time FTE subject to Neumann, Dirichlet and Robin boundary conditions. The operational matrix of fractional differentiation of SGPs was derived and combined with the spectral tau method to obtain a linear system of algebraic equations in the unknown expansion coefficients which could be easily solved numerically. The convergence of the approximate solution was estimated theoretically in the $\mathcal{L}^2_{\omega_\alpha}$-norm. The proposed method for various values of the Gegenbauer parameter α has the advantages to capture highly accurate solutions, rapid convergence with small increase in SGPs order. Several test examples have been presented and compared with other methods to prove the high accuracy and efficiency of the proposed method. The proposed method can be extended to solve a wide variety of space–time one-dimensional fractional differential equations with space variable coefficients.

References

[1] I Podlubny, *Fractional differential equations* (Academic Press, New York, 1999)
[2] I Podlubny, *Fract. Calc. Appl. Anal.* **5**, 367 (2002)
[3] H Sun, W Chen and Y Chen, *Phys. A Stat. Mech. Appl.* **388(21)**, 4586 (2009)
[4] J H He, *Comput. Meth. Appl. Mech. Eng.* **167(1–2)**, 57 (1998)
[5] M P Singh and T S Chang, *J. Eng. Mech.* **135(6)**, 571 (2009)
[6] H F Ahmed and M B Melad, *J. Fract. Calc. Appl.* **9(2)**, 141 (2018)
[7] H F Ahmed, M S M Bahgat and M Zaki, *Pramana – J. Phys.* **92(3)**: 38 (2019)
[8] M S Hashemi and D Baleanu, *J. Comput. Phys.* **316**, 10 (2016)
[9] W Jiang and W Lin, *Commun. Nonlinear Sci. Numer. Simul.* **16(9)**, 3639 (2011)
[10] Z Zhao and C Li, *Appl. Math. Comput.* **219(6)**, 2975 (2012)
[11] A Mohebbi, M Abbaszadeh and M Dehghan, *Int. J. Numer. Methods Heat Fluid Flow* **24(8)**, 1636 (2014)
[12] A H Bhrawy, M A Zaky and J A T Machado, *J. Opt. Theory Appl.* **174(1)**, 321 (2017)
[13] T Akram, M Abbas, A I Ismail, N H M Ali and D Baleanu, *Adv. Differ. Eq.* **1**, 365 (2019)
[14] A Saadatmandi and M Dehghan, *Numer. Methods Partial Differ. Equ.* **26(1)**, 239 (2010)
[15] R M Hafez, *Comput. Appl. Math.* **37(4)**, 5253 (2018)
[16] M Saadatmandi and M Mohabbati, *Math. Rep.* **17(67)**, 2 (2015)
[17] E H Doha, A H Bhrawy and S S Ezz-Eldien, *Comput. Math. Appl.* **62(5)**, 2364 (2011)
[18] E H Doha, *Comput. Math. Appl.* **21(2–3)**, 115 (1991)
[19] M Usman, M Hamid, R U Haq and W Wang, *Eur. Phys. J. Plus* **133(8)**, 327 (2018)
[20] H F Ahmed, *J. Taibah Univ. Sci.* **12(5)**, 494 (2018)
[21] A A El-Kalaawy, E H Doha, S S Ezz-Eldien, M A Abdelkawy, R M Hafez, A Z M Amin, D Baleanu and M A Zaky, *Rom. Rep. Phys.* **70**, 109 (2018)
[22] H F Ahmed, M S M Bahgat and M Zaki, *Pramana – J. Phys.* **94(1)**: 72 (2020)
[23] G Ben-Yu, *SIAM J. Numer. Anal.* **37(2)**, 621 (1999)
[24] R Adams and J F Fournier, *Sobolev spaces* (Academic Press, New York, 1975)
[25] N J Ford, M M Rodrigues, J Xiao and Y Yan, *J. Comput. Appl. Math.* **249**, 95 (2013)
[26] E Shivanian, S Abbasbandy and A Khodayari, *Int. J. Ind. Math.* **10(2)**, 151 (2018)
[27] M S El-Azab and M El-Gamel, *Appl. Math. Comput.* **190(1)**, 757 (2007)
[28] S Yzba, *Appl. Math. Comput*. **287**, 83 (2016)