



An update on coherent scattering from complex non-PT-symmetric Scarf II potential with new analytic forms

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Abstract. The versatile and exactly solvable Scarf II potential has been predicting, confirming and demonstrating interesting phenomena in complex PT-symmetric sector, most impressively. However, for the non-PT-symmetric sector, it has gone underutilised. Here, we present the most simple analytic forms for the scattering coefficients ($T(k)$, $R(k)$, $|\det S(k)|$). On the one hand, these forms demonstrate earlier effects and confirm the recent ones. On the other hand, they make new predictions – all simple and analytical. We show the possibilities of both self-dual and non-self-dual spectral singularities (NSDSS) in two non-PT sectors (potentials). The former one is not accompanied by time-reversed coherent perfect absorption (CPA) and gives rise to the parametrically controlled splitting of spectral singularity (SS) into a finite number of complex conjugate pairs of eigenvalues (CCPEs). NSDSS behave just oppositely: CPA but no splitting of SS. We demonstrate a one-sided reflectionlessness without invisibility. Most importantly, we bring out a surprising coexistence of both real discrete spectrum and a single SS in a fixed potential. Nevertheless, so far, the complex Scarf II potential is not known to be pseudo-Hermitian ($\eta^{-1}H\eta = H^\dagger$) under a metric of the type $\eta(x)$.

Keywords. Non-Hermitian potential; coherent scattering; coherent perfect absorption; lasing; spectral singularity; real; complex conjugate; single non-real eigenvalue.

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1. Introduction

An exactly solvable Hermitian potential called Scarf II potential, proposed by Gendenshtein [1] is written as

$$\begin{aligned} V_{S_1}(x) &= P \operatorname{sech}^2 x + Q \operatorname{sech} x \tanh x, \\ P &= (B^2 - A^2 - A), \quad Q = B(2A + 1). \end{aligned} \quad (1)$$

After the discovery [2] of real spectrum of complex PT-symmetric Hamiltonians, this potential became the most interesting exactly solvable complexified model ($B = iB'$) [3] of the new quantum mechanics. In another complexification, $P = -V_1$, $Q = iV_2$, $V_1, V_2 \in \mathcal{R}$, $V_1 > 0$, it additionally had the distinction of having both the exact and broken regimes of PT-symmetry according to the condition $|V_2| \leq V_1 + 1/4$ [4]. Remarkably, the exactly solvable Scarf II (1) potential is a localised potential which is a very useful scattering potential. In 1988, Khare and Sukhatme have obtained

beautiful expressions of its transmission $t(k)$ and reflection $r(k)$ amplitudes in terms of gamma functions with complex argument, where the parameters are real and $V(x)$ is Hermitian [5]. For about 20 years or so the complexified Scarf II potential has been re-enforced for the novel features of a new kind of scattering from it.

In 2001, it was stated and proved [6] that if a scattering potential is complex and asymmetric, the reflection probability show left/right-handedness. Later, this handedness became an essential feature of complex PT-symmetric potentials and the PT-sector of (1) presented a very interesting analytic demonstration [7] of the now called non-reciprocal reflection. This opened up the scope of studying coherent scattering where two identical beams of particles are sent from left and right at the non-Hermitian potential (optical medium). By PT-sector we mean the parametric regime of (1) wherein ($V(-x, -i) = V(x, i)$) (figure 1).

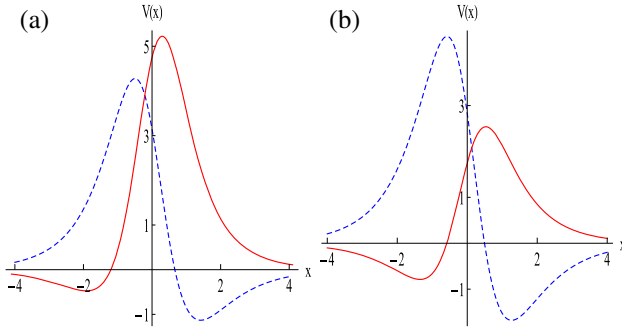


Figure 1. (a) Scarf II $V(x)$ (1) for $A = -ic$, $B = d + i/2$, $c = \sqrt{2}$, $d = \sqrt{3}$ (see §2) and (b) Scarf II $V(x)$ (1) for $A = 1 - ic$, $B = c - i/2$, $c = \sqrt{2}$ (see §3). Red line is for the real part and blue line is for the imaginary part.

Two crucially important concepts/tools have been developed, one is the determinant of two port S-matrix [8]

$$|\det S(k)| = |t^2(k) - r_{\text{left}}(k) r_{\text{right}}(k)|, \quad E = k^2 \quad (2)$$

and the other one is the spectral singularity (SS) [8], wherein at the incident energy $E = E_* = k_*^2$ all three scattering coefficients $R_{\text{left}}(k)$, $R_{\text{right}}(k)$ and $T(k)$ become infinite. Complex PT-symmetric Scarf II potential turned out to be an exactly solvable model [9,10] of SS explicitly. The most interesting physical interpretation of SS is given by the fact that the SS at $E = E_*$ in $T(k)$ leads to a zero in the determinant of the time-reversed S-matrix (2) as $|\det S(-k_*)| = 0$.

$$t_{A,B}(k) = \frac{\Gamma[-A - ik]\Gamma[1 + A - ik]\Gamma[\frac{1}{2} + iB - ik]\Gamma[\frac{1}{2} - iB - ik]}{\Gamma[-ik]\Gamma[1 - ik]\Gamma^2[\frac{1}{2} - ik]}. \quad (3)$$

This phenomenon is called coherent perfect absorption (CPA) which in turn leads to time-reversed lasers [11]. When the medium is PT-symmetric and this symmetry is broken, another phenomenon called CPA with lasing has been observed [12,13]. If SS occurs in both $T(k)$ and $T(-k)$, it is called self-dual. This was first discovered [8] in the PT-sector, later it has been proposed to occur in the non-PT sector as well [14]. Scarf II potential (1) had already been found [14] to have both SDSS and NSDSS in PT and non-PT sectors, respectively. In the PT-sector, if the potential is set for an SS then a slight increase in the strength of the imaginary part of $V(x)$ gives rise to the splitting [16] of SS into a complex conjugate pair of eigenvalues (CCPEs). It is worth noting that discovery of this phase transition [16] owes it to an interesting hint coming from Scarf II (1) [10]. Further increase of this strength causes more and more number of CCPEs. In PT-sector, real-discrete spectrum and SS are found to be mutually exclusive. SS and CCPEs occur strictly in the

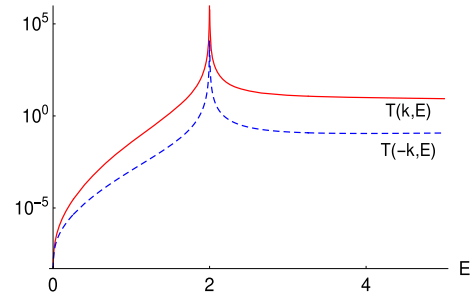


Figure 2. Depiction of scenario for the self-dual spectral singularity in non-PT-sector ($A = 1 - ic$, $B = c - i/2$, $c = \sqrt{2}$). Both $T(k, E)$ and $T(-k, E)$ are unequal but infinite at the same point $k = c$ (see eq. (13)). In PT-sector, one always gets $T(k, E) = T(-k, E)$ [15,22].

domain of broken PT-symmetry, then SS is conjectured to be single which sets an upper bound to the finite number of CCPEs: $E_* > (\approx) \Re(E_l)$ [17], where E_l is the last of CCPEs when arranged with respect to their real parts. Recently, the splitting of SS has been found to occur also in a non-PT sector [18]. Earlier, two NSDSS [15] were found in a non-PT Scarf II potential (1). A construction [18] of numerically solved non-PT potentials having two parametrically controlled SSs (one SDSS and the other NSDSS) has been proposed. Designing of lasing and perfectly absorbing potentials is being searched [19] meticulously both theoretically and experimentally.

The beautiful transmission amplitude of Scarf II (1) potential found by Khare and Sukhatme [5] is given as

It can readily be checked that $t_l = t_{A,B}$ and $t_r = t_{A,-B}$ are equal representing the reciprocity of transmission. The reflection amplitude is given as

$$r_{A,B}(k) = t_{A,B}(k) f_{A,B}(k), \quad (4)$$

$$f_{A,B}(k) = \left[\frac{\cos \pi A \sinh \pi B}{\cosh \pi k} + i \frac{\sin \pi A \cosh \pi B}{\sinh \pi k} \right]. \quad (5)$$

So in view of the non-reciprocity of reflection [6,7], we find that $r_l = r_{A,B}$, $r_r = r_{A,-B}$. Let us also introduce $F_l = f_{A,B}$ and $F_r = f_{A,-B}$ to be used in the sequel. When $|F_l| = |F_r|$ the reflection is reciprocal: $R_l = R_r$, this happens when A and B are real ($V(x)$ is Hermitian) or when they are purely imaginary. The second possibility is surprising as $V(x)$ (1) would be both non-Hermitian and spatially non-symmetric (see eq. (19) in §4). Otherwise, in general the reflection

for non-Hermitian and non-symmetric potential is non-reciprocal $R_l \neq R_r$ as stated and proved in [6].

When either A or $B = iB'$ are purely imaginary $f_{A,B}$ can become zero, for instance at

$$k_z = \pm \pi^{-1} \tanh^{-1} (\tan \pi A \coth B'),$$

$$\text{if } -1 \leq \tan \pi A \coth B' \leq 1. \tag{6}$$

This unidirectional reflectionlessness turns out to be unidirectional invisibility [19] provided $T(k_z) = 1$.

We would like to explain as to how we could successfully eliminate $\Gamma(z)$ functions with complex arguments while calculating simple analytic forms of $T(k) = |t(k)|^2 = t(k)\bar{t}(k)$ in this paper. We do this by using

$$T(k) = \left| \frac{(k+c)(k+d)}{(k-c)(k-d)} \frac{(\sinh^2 \pi k \cosh^2 \pi c)}{(\cosh^2 \pi k - \cosh^2 \pi c)(\cosh^2 \pi k - \cosh^2 \pi d)} \right|. \tag{8}$$

$\Gamma(z)\Gamma(1-z) = \pi \csc(\pi z)$. For instance, $|\Gamma[-A-ik]\Gamma[1+A-ik]|^2 = \Gamma[-A-ik]\Gamma[-A+ik]\Gamma[1+A-ik]\Gamma[1+A+ik] = \Gamma[-z_1]\Gamma[-z_2]\Gamma[1+z_2]\Gamma[1+z_1]$. Combining first term with fourth and second with third, we eliminate gamma functions. Here $z_1 = A+ik$ and $z_2 = A-ik$. This method succeeds in parametrisations, $\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3$ and \mathbf{P}_4 discussed below where A and B are parametrised specially. For other cases it may not be possible to get simple analytic forms for $T(k)$.

In this paper, we present new simple analytic forms of $R(k), T(k), |\det(S(k))|$ in various non-PT sectors of Scarf II by various complex parametrisations of A and B in (1). Earlier, this has been done for PT-sectors of (1) [22]. In this paper, we reveal the contrasting features of SDSS and NSDSS. The former can be made to split by perturbing the potential but it does not give rise to CPA. We show that the NSDSS that occurs in non-PT sector behaves just oppositely. We show one-sided reflectionlessness devoid of invisibility [20,21]. Most importantly, we show a coexistence of real discrete spectrum and the NSDSS in a fixed non-PT complex Scarf II potential (1). Nevertheless, this potential (1) in non-PT-sectors is not known to be pseudo-Hermitian [24,25]: $\eta^{-1}H\eta = H^\dagger$, so far, under a local metric: $\eta(x)$.

In §2–5, we study and present four different parametric parametrisation (\mathbf{P}_1 – \mathbf{P}_4) of non-PT-sector of complex Scarf II (1), to bring out various features of coherent scattering simply and analytically. In §6 and 7, we present the discussions and conclusions.

2. One/two non-self-dual spectral singularity (NSDSS), no coherent perfect absorption (CPA), one reflectivity zero (RZ)

\mathbf{P}_1 : $A = -ic, B = d + i/2$. From eq. (1), the potential becomes (figure 1)

$$V(x) = (c^2 + d^2 + i(c+d) - 1/4) \operatorname{sech}^2 x$$

$$+ (c+d - 2icd + i/2) \operatorname{sech} x \tanh x, c, d \in R. \tag{7}$$

By eliminating $\Gamma(z)$ functions using identities like $\Gamma(z)\Gamma(1-z) = \pi z \operatorname{cosec} \pi z$, trigonometric and hyperbolic identities in eq. (3), we get

From eqs (4) and (5), we can write

$$F_{l,r}(k) = i \left[\pm \frac{\cosh \pi c \cosh \pi d}{\cosh \pi k} + \frac{\sinh \pi c \sinh \pi d}{\sinh \pi k} \right]. \tag{9}$$

Clearly, two spectral singularities exist at $k = k_{*1} = c$ and $k = k_{*2} = d$. Next, there exists a single reflectivity zero as

$$k_z = \pi^{-1} \tanh^{-1} [\tanh \pi c \tanh \pi d]. \tag{10}$$

When c is large such that $\tanh \pi c \approx 1, k_z \approx d$ and if d is large then $k_z \approx c$. In these cases, the SS will appear to coincide with the single reflectivity zero. In the other cases k_{*1}, k_{*2} and k_z will be distinct. The modulus of the determinant of two-port S-matrix (2) after interesting simplifications becomes

$$|\det S(k)| = \left| \frac{(k+c)(k+d)}{(k-c)(k-d)} \right|, \tag{11}$$

which becomes ∞ for $k = c, d$ and in time-reversed setting $T(-k)$ at $k = c, d$ it attains finite values. So these two are NSDSS [14].

If $c > 0$ and $d < 0$. From eqs (9) and (11), it clearly turns out that there will be one SS in $T(E)$ at $k = k_* = c$ and one SS in the time-reversed case: $(T(-k))$ at $k = k_* = d$, once again these are NSDSS. Next, when $c = d$, eq. (8) shows that there will be only one SS which will again be NSDSS occurring at $k = k_* = c$ in $T(k)$ but not in $T(-k)$. At $k = c$, the former is infinite but the latter attains a finite limit. The case of $c = -d$, makes the potential (1) PT-symmetric [20,21] and from eqs (9)

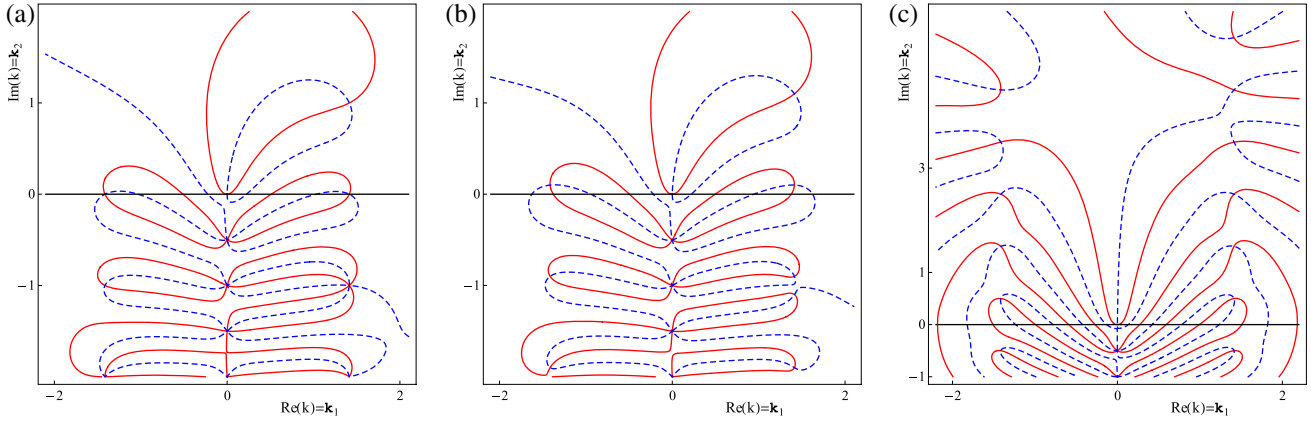


Figure 3. $A = q + 1/2 - ic$, $B = c - iq$ splitting of SDSS, the intersection of the red and blue curves gives CCPEs in complex k plane: $(k_n = -c + i(q - (n + 1/2)))$, $0 \leq n < q + 1/2$, $(E_n = k_n^2)$. We take $c = \sqrt{2}$, (a) for $q = 1/2$, $k = k_* = c$, (b) for $q = 0.6$, notice one pair of CCPE, (c) for $q = 5.0$, five pairs of CCPEs. In all the three cases, see an unpaired complex eigenvalue $k^\dagger = c + (q + 1/2)i$ (18).

and (11) one can readily get the SDSS at $k = \pm c = k_*$ and the CPA with lasing characteristically [11] as $|\det S(\pm c)| = \frac{0}{0}$, but $\lim_{k \rightarrow \pm c} |\det S(k)| = 1$. With this, we can also rule out CPA with lasing in a non-PT sector.

3. One self-dual SS, no CPA and one RZ

P₂: $A = 1 - ic$, $B = c - i/2$. The potential (1) becomes (figure 2)

$$V(x) = \left(2c^2 + 2ic - \frac{9}{4}\right) \text{sech}^2 x + \left(2c - 2ic^2 - \frac{3i}{2}\right) \text{sech} x \tanh x, \quad c \in \mathbb{R}. \quad (12)$$

Further, we can convert eqs (3)–(5) to simple analytic forms as

$$T(k) = \left| \frac{(1 + (k + c)^2)}{(1 + (k - c)^2)} \frac{(\sinh^2 \pi k \cosh^2 \pi c)}{(\cosh^2 \pi k - \cosh^2 \pi c)^2} \right|. \quad (13)$$

Notice that $T(k) \neq T(-k)$, but they are singular at the same point $E = c^2$ (see figure 3). In PT-sector, one gets $T(k) = T(-k)$ [14,20,21].

$$F_{l,r}(k) = i \left[\pm \frac{\cosh^2 \pi c}{\cosh \pi k} + \frac{\sinh^2 \pi c}{\sinh \pi k} \right] \quad (14)$$

and (2) as

$$|\det S(k)| = \frac{1 + (k + c)^2}{1 + (k - c)^2}, \quad (15)$$

which rules out both CPA with [9] or without [11] lasing.

$$k_z = \pm \frac{1}{\pi} \tanh^{-1} [\tanh^2(\pi c)]. \quad (16)$$

4. Splitting of SDSS into complex conjugate pairs of eigenvalues (CCPEs), no CPA, one RZ

P₃: $A = q + 1/2 - ic$, $B = c - iq$, $q \geq 1/2$, $c > 0$. For this phenomenon, we have introduced a new parameter p to control the splitting of the self-dual spectral singularity. In this regard, the previous parametrisations, **P₁**, **P₂** can be seen to be the cases of $q = -1/2, 1/2$, respectively. For **P₃**, the potential becomes

$$V(x) = (2c^2 - 2q^2 + 2ic - 2q - 3/4) \text{sech}^2 x + [2c - 2ic^2 - 2iq - 2iq^2] \text{sech} x \tanh x. \quad (17)$$

Complex PT-symmetric potentials have been discussed to have three types [17] of discrete spectrum which come as complex k -poles of $t(k)$. The physical poles are of the types: $k = \pm k_x, ik_y, \pm k_x + ik_y$ ($k_y > 0$) for SS, bound-states and complex conjugate eigenvalues, respectively. Other poles are unphysical. The recent phenomenon of splitting of SS into CCPEs in non-PT-sector gives the opportunity to discuss the presence of a finite number of CCPEs in non-PT sector of eq. (1) with **P₃**. In this regard, the four parameter-dependent gamma functions in the numerator of (3) are interesting. We use $\Gamma -n = \infty$, $n = 0, 1, 2, \dots$, the first and last one gives finite number of poles as $k_m = c + (q + 1/2 - m)i$, $k_n = -c + (q - 1/2 - n)i$, $m, n = 0, 1, 2, \dots, m \leq q + 1/2$ and $n \leq q - 1/2$, such that $k_y \geq 0$. These pairs of poles can be synchronised as

$$k_n = \pm c + (q - 1/2 - n),$$

$$n = 0, 1, 2, \dots \leq [q - 1/2]$$

and

$$k^\uparrow = c + (q + 1/2)i, \quad k_0(q = 1/2) = c = k_*. \quad (18)$$

With k^\uparrow being single (unpaired), the middle two gamma functions in (3) give rise to infinitely many pairs of unphysical poles ($k_y < 0$). Unlike the poles in (18), $|\psi_n(x)|$ diverges to infinity. When $q = 1/2, n = 0$, we get $k_0 = \pm c = k_*$. So with SDSS at $E = c^2$, for $q > 1/2$, we get one or more pairs of CCPEs. These k_n are complex poles of $t(k)$ or complex zeros of $f(k) = 1/t(k)$. In figure 3, as an alternate method, we plot the contours of $\Re(f(k_x + ik_y)) = 0$ (red lines) and $\Im(f(k_x + ik_y)) = 0$ (blue lines) in the plane (k_x, k_y) and note k_n as the point of their intersection for $k_y > 0$. In figure 3a for $q = 1/2$, we show unsplit SS at $k = \sqrt{2}$, in figure 3b splitting of SS to a single pair when $q = 0.6$ and in figure 3c splitting SS to five pairs when $q = 5$. In figures 3a–3c, see the single unpaired k^\uparrow which is the signature of non-PT-symmetry. Otherwise, for the split-

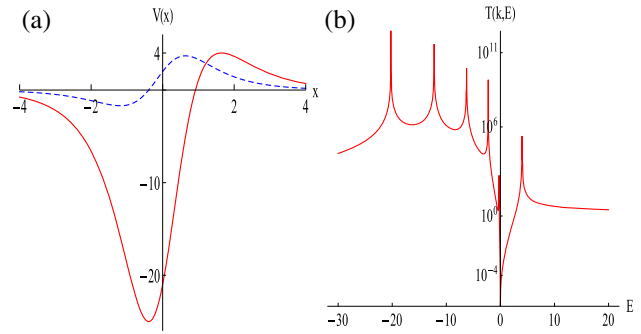


Figure 4. Depiction of the coexistence of real discrete spectrum and an NSDSS in a fixed non-PT potential when $A = -ic, B = id, c = 2, d = 5$. (a) $V(x)$, (b) $T(E < 0)$ (20), $T(E > 0)$ (21). Note the real discrete energy poles (see eq. (20)) at $E = -(2n + 1)^2/4$ and an SS at $E = c^2 = 4$.

Using the properties of gamma functions such as $\Gamma(z)\Gamma(-z) = \pi z \operatorname{cosec} \pi z$, trigonometric and hyperbolic functions, we get the interesting form for the transmission probability as

$$T(k) = \left| \frac{(k + c)}{(k - c)} \frac{\sinh^2 \pi k \cosh^2 \pi c}{(\cosh^2 \pi k - \cosh^2 \pi c)(\sinh^2 \pi k + \cosh^2 \pi d)} \right|, \quad E > 0, \quad (21)$$

ting SS in PT-sector [16,17] $k^\uparrow = \alpha + i\beta, \alpha, \beta > 0$ can be seen to be absent because there the discrete eigenvalues are known [2] to be either real or complex conjugate pairs (figure 4).

5. One NSDSS, real discrete spectrum and reciprocal reflection

Now, we propose the parametric domain for (1) as **P₄**: $A = -ic$ and $B = id$ where $c, d > 0$. The Scarf II potential (1) becomes

$$V(x) = -(d^2 + c^2 - ic) \operatorname{sech}^2 x + (2cd + id) \operatorname{sech} x \tanh x \quad (19)$$

which is non-PT-symmetric. Using this parametrisation in $t(k)$ when we set $k = k_* = c$, the first gamma function becomes $\Gamma[0] = \infty$. This gives rise to a single SS in $t(k)$ or on $T(k) = |t(k)|^2$. This SS is real positive discrete energy embedded in the positive energy continuum. Next, if we set $k = i\kappa_n$ ($\kappa = \sqrt{-E}$) in the argument of the third gamma function in the numerator of (3) and set $1/2 - d + \kappa_n = -n$, where $n \in I^+ + \{0\}$, we get the real discrete spectrum ($E < 0$) of (19) as

$$E_n = \kappa_n^2 = -(d - 1/2 - n)^2, \quad n = 0, 1, 2, 3, \dots, [d - 1/2], \quad \kappa_n = d - 1/2 + n > 0. \quad (20)$$

clearly showing SS at $E = c^2$ and negative energy bound states at E_n is given by eq. (20). For the reflection probabilities, from eqs (4) and (5), we can write

$$F_{l,r}(k) = \left[\mp \frac{\sinh \pi c \cos \pi d}{\sinh \pi k} + i \frac{\cosh \pi c \sin \pi d}{\cosh \pi k} \right] \Rightarrow |F_l| = |F_r|. \quad (22)$$

Notice that for real values of c and d , $F_{l,r}$ cannot become zero for a real value of $k = k_z$. So in this sector we do not get reflectionlessness. One may also check that the reflection will be reciprocal. The reflection amplitudes $r_l(k) = t(k)F_l$ and $r_r(k) = t(k)F_r$ are unequal but their modulus-square being equal, the reflection will be reciprocal ($R_l(k) = R_r(k)$), despite non-Hermiticity and asymmetry of the potential (1). The modulus of the determinant of the S -matrix (2) becomes

$$|\det S(k)| = \left| \frac{k + c}{k - c} \right|. \quad (23)$$

At $E = c^2, |\det S(c)| = 0$ displaying spectral singularity but in time-reversed setting $|\det S(-c)| = \infty$, there occurs CPA without lasing [11] (figure 4).

6. Discussions

We find that we can also introduce integers in the parametrisation of A and B in (1) at the cost of more

involved or unamenable forms of $T(k)$. Also in such cases we have not found any other result that differs qualitatively from the results presented here. Our search of the non-PT parametric regimes may not have been exhaustive. So we would like to encourage one to go for new re-parametrisations and use eqs (3)–(5) directly. There may arise a new scenario.

Section 2 mainly demonstrates the earlier findings [15] analytically. However, the new essence is that NSDSS which gives rise to CPA exists in non-PT sector and they cannot be split into complex conjugate pairs by perturbing the potential parametrically. On the other hand, the new essence coming out from §3 is that a SDSS cannot give rise to CPA but it can be split such that $E_*(q = 0) > \text{Re}(\mathcal{E}_n(q))$. Additionally, §2 and 3 reveal that an SS and discrete set of CCPEs cannot coexist in a non-PT complex potential. This is also true for complex PT-potentials. Unlike the PT-sector, when the SDSS takes place $T(k_*) = T(-k_*) = \infty$ but $T(k) \neq T(-k) \forall k \in \mathcal{R}$ (see figure 3). It may be recalled that in PT-sector, we have $T(k) = T(-k)$. Our demonstration of the coexistence of real discrete spectrum with an SS in a non-PT potential is unlike PT-symmetric potentials. The real discrete spectrum is itself remarkable as the complex Scarf II potential is not known to be pseudo-Hermitian.

Pseudo-Hermitian (PH) Hamiltonians which are such that $\eta^{-1}H\eta = H^\dagger$ [24,25] may have real discrete or complex conjugate pairs of eigenvalues in two regimes, mutually exclusively. Complex PT-symmetric potentials are found to be parity-pseudo-Hermitian. Some non-Hermitian potentials with imaginary shift of x are PH under $\eta = e^{-p\theta}$ and some are so under $\eta = e^{-f(x)}$. Using the idea of PH, very interesting non-Hermitian potentials are constructed [26,27] which have real discrete or complex conjugate pairs of eigenvalues. In this regard, non-PT Scarf II potential may be studied further if it could be PH under a local metric $\eta(x)$.

7. Conclusion

Our simple analytic forms presented in eqs (8)–(11), (13)–(16), (18), (20)–(23) are for the non-PT sectors of Scarf II potential which are new and instructive. We confirm the splitting of self-dual spectral singularity into complex conjugate pairs but unlike the case of PT-sector, here we show a distinct presence of a single (unpaired) discrete complex value $k^\dagger = \alpha + i\beta$, $\alpha, \beta > 0$ (see figure 3).

In non-PT sectors studied here, the new thought provoking possibilities which require a confirmation in other complex non-PT potentials are: (1) the existence of one-sided reflectionlessness, (2) self-dual spectral singularity not giving rise to coherent perfect absorption (CPA) with or without lasing, (3) non-self-dual spectral singularity does not split but gives rise to CPA without lasing, (4) a co-existence of a non-self-dual spectral singularity and real discrete spectrum. In any case, here or elsewhere [18,20,21,23] the existence of real discrete or complex conjugate pairs of eigenvalues hint at replacement of pseudo-Hermiticity with some other more general condition if Scarf II potential is not proved to be pseudo-Hermitian.

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