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Realisation of Snyder operators in quantum mechanics

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Abstract. The lack of experimental results of theories concerning the ultimate space/space–time structure, leads one to think about some simple experiments that can help demonstrate the main prediction: Space–time is discrete. In this paper, an implementation of Snyder operators is applied to simple geometries, to demonstrate that the spectrum of the position operator, in standard quantum mechanics (QM), is discrete when a parameter measuring the non-commutativity of quantum operators is introduced. The geometries are specially suitable for experiments in search of the behaviour of ultracold neutrons falling in the Earth's gravitational potential.

Keywords. Quantum mechanics; Snyder algebra.

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1. Introduction

Nowadays, the most important task in theoretical physics is the construction of a theory that accounts for the ultimate space–time structure. Many efforts are on to develop string theory or loop quantum gravity. But, despite these efforts, such theories are still in a speculative stage and it is disappointing that no breakthrough has come yet.

The most conspicuous feature is the hypothesis that space–time is discrete. In fact, mainstream research on very high energy physics and on possible candidates for a suitable quantum gravity theory has led to the idea of the existence of a 'fundamental length'. All theories propose the existence of a minimal length that is usually identified as the minimal size of their fundamental elements.

However, the existence of such fundamental minimal length defies the Lorentz invariance that is solely based on the invariance of the speed of light. Hence, there are a number of effective theories such as double special relativity (DSR) [1] and gravity's rainbow [2], where a universal, fundamental longitude is also considered.

Hence, a fundamental minimal length is a problem where the focus is on the proper space–time rotation symmetries. In order to skip the Lorentz symmetry breaking issue, a solution was proposed in the 1940s by Snyder [3], who postulated a Lorentz-invariant modification of the Heisenberg algebra which implies discrete spectra of the space–time operators.

The classical version of Snyder algebra is based on the non-canonical Poisson brackets between new variables $\bar{x}_i, \bar{p}_j$ that should be the true space and momentum variables:

$$\{\bar{x}_i, \bar{x}_j\} = l^2 L_{ij}, \tag{1}$$

$$\{\bar{x}_i, \bar{p}_j\} = \delta_{ij} + l^2 \bar{p}_i \bar{p}_j, \tag{2}$$

$$\{\bar{p}_i, \bar{p}_j\} = 0, \tag{3}$$

where i, j are general space–time indices and l is a parameter measuring the deformation introduced in the canonical Poisson brackets, having dimensions of inverse of momentum, and L_{ij} is the angular momentum matrix.

It is expected that l would be of the order of Planck length. This classical version has been used in many models applied to planetary movement, usually in modelling the Mercury orbit. In [4], the value obtained for l is about 10^{-32} m^2, using an application of Kepler's model to the Mercury perihelion shift. In a totally different approach and implementation of the model in [5], a similar value was reached. Mignemi and Strajn, using a Snyder model applied to Schwarzchild space–time in [6], found that after the experimental data, the violations of the equivalence principle are less than one part in 10^{12}.

In the quantum realm, the realisation of operators fulfilling a Snyder algebra usually has two paths: a new product between operators can be defined as in [7–9], or a realisation of usual quantum operators can be used, while maintaining the standard product. In this paper, the

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second path has been chosen, mainly because of its simplicity. After a recipe to construct the Snyder operators in a classic version, the realisation is quantised and the Snyder relations verified. Then, a couple of simple applications are performed in order to verify that the space is discrete when Snyder deformations are introduced.

2. The classical version of Snyder algebra and its operators

The classical Poisson Snyder algebra means that we are working with different variables that do not commute in the normal way. To yield this algebra, it is necessary to introduce new variables to observe space–time characteristics. These new variables can be constructed through some recipes that appear in literature. Here we follow the prescription from [10] to construct non-commutative field variables:

$$\tilde{x}_i = x_i \varphi_1(A) + l^2 (xp) p_i \varphi_2(A), \tag{4}$$

$$\tilde{p}_i = p_i, \tag{5}$$

where i, j are space–time indices and φ_1 and φ_2 are functions of the dimensionless quantity $A = l^2 p^2$. The function φ_2 depends on φ_1 by

$$\varphi_2 = \frac{1 + 2\dot{\varphi}_1 \varphi}{\varphi_1 - 2A\dot{\varphi}_1}. \tag{6}$$

It is also possible to set two realisations for the non-commutative Snyder geometry. One is from Snyder himself where $\varphi_1 = 1$, implying that $\varphi_2 = 1$, and the second one is from Maggiore [11], where $\varphi_1 = \sqrt{1 - sp^2}$ and $\varphi_2 = 0$.

Then, choosing the Snyder realisation, we have

$$\tilde{x}_i = x_i + l^2 (xp) p_i, \tag{7}$$

$$\tilde{p}_j = p_j. \tag{8}$$

3. Quantisation of the Snyder operators

As stated in the Introduction, many efforts have been done to construct quantum operators that yield the Snyder algebra. In this paper, the pathway for constructing the operators from their classical field version in the standard quantum mechanics form which mimics the field form is being chosen. For doing this, the standard differential form, usual product of operators and some results can be used in simple applications. Since there is a product between x_i and p_j which does not commute, a recipe to treat with this must be used, and the simplest way to do it is by symmetrising the product. So, after symmetrisation, we have

$$\hat{\hat{x}}_i = \hat{x}_i + \frac{l^2}{4}((\hat{x}\,\hat{p} + \hat{p}\,\hat{x})\hat{p}_i + \hat{p}_i(\hat{x}\,\hat{p} + \hat{p}\,\hat{x})) \tag{9}$$

$$\hat{\hat{p}}_j = \hat{p}_j. \tag{10}$$

Using the standard quantum mechanics (QM) algebra between $\hat{x}_i$ and $\hat{p}_j$, it can be proven that the Snyder algebra is reproduced.

$$[\hat{\hat{x}}_i, \hat{\hat{p}}_j] = i\hbar(\delta_{ij} + l^2 \hat{p}_i \hat{p}_j), \tag{11}$$

$$[\hat{\hat{x}}_i, \hat{\hat{x}}_j] = i\hbar l^2 L_{ij}, \tag{12}$$

$$[\hat{\hat{p}}_i, \hat{\hat{p}}_j] = 0. \tag{13}$$

Now, we have Snyder versions of quantum operators that have been constructed from standard QM operators. The advantage of doing this is that the usual product and differential versions of $\hat{x}$ and $\hat{p}$ can be used in calculations. In order to do this and in search of simple examples, calculations shall be restricted to one dimension only. The position operator can be expressed as

$$\hat{\hat{x}} = -\hbar^2 l^2 x \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \hbar^2 l^2 \frac{\mathrm{d}}{\mathrm{d}x} + x. \tag{14}$$

Now, the space operator $\hat{\hat{x}}$ is a second-order differential operator.

On the other hand, a very important result is the Heisenberg uncertainty principle after the modification of the space operator. In fact, we now have

$$(\triangle\hat{\hat{x}})(\triangle\hat{\hat{p}}) \geq \frac{\hbar}{2}(1 + l^2 \langle \psi, \hat{p}^2 \psi \rangle). \tag{15}$$

If a standard plane wave of the general form $\psi = A\mathrm{e}^{i(kx - \omega t)}$ is chosen, the uncertainty principle becomes

$$(\triangle\hat{\hat{x}})(\triangle\hat{p}) \geq \frac{\hbar}{2}(1 - l^2 k^2). \tag{16}$$

The uncertainty has an additional term that depends on the energy of the particle. In other words, it depends on the idea that the observable is actually the deformed operator and the usual space operator is just an approximation. Compare this with [12,13] and principles used to develop results on cold neutrons falling in the Earth's gravitational potential and the introduction of the generalised uncertainty principle (GUP) [14].

4. 1D box

As previously mentioned, we are interested in determining the characteristic of the spectra of Snyder quantum operators in simple cases. The simplest case is, evidently, the one with a particle in a 1D box.

Here we are interested in the position operator as momentum remains the same. In the standard QM problem, x has a simple functional representation:

$$x\psi = \lambda\psi \tag{17}$$

which means that λ belongs to $\mathbb{R}$.

Now the properties of the new space operator are evaluated. We have

$$\hat{\hat{x}} : \mathcal{D}_{\hat{\hat{x}}} \to \mathcal{L}^2([a, b]), \tag{18}$$

where $\mathcal{L}^2([a, b])$ are the set of square integrable functions on $[a, b]$.

The action of the operator on the wave function is

$$\psi \to -\hbar^2 l^2 x \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \hbar^2 l^2 \frac{\mathrm{d}\psi}{\mathrm{d}x} + x\psi, \tag{19}$$

where the operator has the domain

$$\mathcal{D}_{\hat{\hat{x}}} = \{\psi : \psi \in C^2(a, b) \cap \mathcal{L}^2([a, b]) | \psi(a) = \psi(b) = 0\}. \tag{20}$$

The eigenvalue equation

$$-\hbar^2 l^2 x \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \hbar^2 l^2 \frac{\mathrm{d}\psi}{\mathrm{d}x} + x\psi = \lambda\psi. \tag{21}$$

It is worth noting that due to the presence of l^2, the eigenvalue equation has the form of a Sturm–Liouville problem. It must be kept in mind that a Sturm–Liouville problem is a problem with edge values which yield

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right] + r(x)y + \lambda w(x)y = 0, \quad a < x < b, \tag{22}$$

$$a_1 y(a) + a_2 y'(a) = 0, \tag{23}$$

$$b_1 y(b) + b_2 y(b) = 0, \tag{24}$$

where λ is a parameter to be determined when $p(x)$, $r(x)$ and $w(x)$ are continuous on $[a, b]$, $p(x) > 0$ and $w(x) > 0$ in $[a, b]$. This is a regular Sturm–Liouville problem with valued boundaries.

On the other hand, it is known that the operator $\hat{\hat{x}}$ is itself adjoint because it is of the form

$$Ay(x) = p(x)y''(x) + q(x)y'(x) + r(x)y(x),$$

where $p(x)$, $q(x)$ and $r(x)$ are real functions.

From the Sturm–Liouville problem, it can be said that λ belongs to a set $\{\lambda_i\}$ which is a countable set, meaning that the spectra of the position operator is discrete.

This is a simple problem, but can offer a very interesting theoretical approximation to simple experiments. Again, research concerning cold neutrons falling in a gravity potential might also be of interest (see also [15,16]).

5. The ring

Similarly, another simple but interesting case is that of a particle constrained to moving in a ring. In this case, the domain of the operators is chosen as

$$\hat{\hat{x}} : \mathcal{D}_{\hat{\hat{x}}} \to \mathcal{L}^2([0, L]), \tag{25}$$

where

$$\mathcal{D}_{\hat{\hat{x}}} = \left\{ \psi = A\sin\left(\frac{2n\pi x}{L} + \alpha\right) \setminus \psi(0) = \psi(L)\ \forall x \in [0, L] \right\}$$

and L is the perimeter of the ring.

The eigenvalue equation is formally the same as in the 1D box case:

$$-\hbar^2 l^2 x \frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \hbar^2 l^2 \frac{\mathrm{d}\psi}{\mathrm{d}x} + x\psi = \lambda\psi. \tag{26}$$

Facing a Sturm–Liouville problem, the λ's are once again a discrete set. So, a ring is a very suitable scenario to develop real experiments to prove the ultimate space structure.

6. Final remarks

The focus of this paper was to find a reliable and simple realisation for Snyder quantum variables that could lead to real experiments searching for the verification of the predicted discreteness of space. Evidently, experiments capable of detecting such tiny effects are still in teething stage, but in principle they are attractive. The prediction that space is discrete in simple geometries is, theoretically and experimentally, a more affordable and controllable situation than those that concern with astrophysics effects, specially when experiments with ultracold neutrons are considered. It should be noted that the introduction of a fundamental length changes the structure of the eigenvalue equation of the space operator, but the energy aspects of the problems remain untouched, meaning that all aspects of energy levels remain the same, and the main change is that there is an infinite discrete set of positions which are possible due to the characteristics of the new position operator.

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