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Investigation of entanglement entropy in cyclic bipartite graphs using computer software

REZA AHMADI and SUSAN NAMI*

Institute of Fundamental sciences, University of Tabriz, Tabriz, Iran
*Corresponding author. E-mail: S.Nami@tabrizu.ac.ir

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Abstract. We investigate the entanglement of the ground state in the quantum cyclic graphs whose nodes are considered as quantum harmonic oscillators. To this end, the Schmidt numbers and entanglement entropy between two arbitrary partitions with equal nodes of a cyclic graphs, are calculated. For that, the local operation is used to build singular value decomposition of potential matrix of cyclic graphs. Then the maximum value of entanglement entropy among all bipartite cyclic graphs is obtained.

Keywords. Entanglement; entropy; cyclic graphs; Schmidt number.

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1. Introduction

Entanglement plays a crucial role in quantum information processing, including quantum communication [1,2] and quantum computation [3–5]. It is one of the remarkable features that distinguishes quantum mechanics from classical mechanics.

For decades, entanglement has been the focus of much work in the foundations of quantum mechanics, being associated particularly with quantum non-separability and the violation of Bells inequalities [6]. As entanglement has been regarded as such an important resource, there is a need for a means of quantifying it. For the case of bipartite entanglement, a recent exhaustive review was written by the Horodecki family [7] and entanglement measures have been reviewed in detail by Plenio and Virmani [8]. One of the operational entanglement criteria is the Schmidt decomposition [9–11]. The Schmidt decomposition is a very good tool to study entanglement of bipartite pure states. The Schmidt number provides an important variable to classify entanglement. The entanglement of a partly entangled pure state can be naturally parametrised by its entropy of entanglement, defined as the von Neumann entropy, or equivalently as the Shannon entropy of the squares of the Schmidt coefficients [9,11]. The situation simplifies if only so called 'Gaussian states' of the harmonic oscillator modes are considered [12–17]. The importance of Gaussian states is two-fold; firstly, its structural mathematical description makes them much more amenable than any other continuous variable system (continuous variable systems are those described by canonical conjugated coordinates x and p endowed with infinite-dimensional Hilbert spaces). Secondly, its production, manipulation and detection with current optical technology can be done with a very high degree of accuracy and control. Cardillo *et al* [18] quantified the amount of information that a single element of a quantum network shares with the rest of the system. They considered a network of quantum harmonic oscillators and analysed its ground state to compute the entropy of entanglement that vacuum fluctuations creates between single nodes and the rest of the network by using the Von Neumann entropy. Jafarizadeh *et al* [20] quantified the entanglement entropy between two parts of the network. To this aim, they computed the vacuum state of bosonic modes harmonically coupled through the specific adjacency matrix of a given network.

In this paper, we first rewrite the adjacency matrix of cyclic graph. Then we calculate the Schmidt numbers and entanglement entropy between two equal subsets of cyclic graph and we give the maximum values of entanglement entropy. In §2, we give some preliminaries such

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as definitions about Hamiltonian and our method. In §3, we calculate bipartite entanglement entropy between two equal parts of the cyclic graphs for different partitions.

2. Entanglement entropy in ground-state wave function

We define a network as a set of N nodes and E edges (or links) accounting for their pairwise interactions. The network backbone is usually encoded in the adjacency matrix, A, such that $A_{ij} = 1$ if an edge connects nodes i and j while $A_{ij} = 0$ otherwise. In this paper, we restrict our studies to undirected networks so that $A_{ij} = A_{ji}$. The Laplacian of a network is defined from the adjacency matrix as $L_{ij} = k_i \delta_{ij} - A_{ij}$, where $k_i = \sum_j A_{ij}$ is the connectivity of node i, i.e., the number of nodes connected to i. We consider nodes as identical quantum oscillators, interacting as dictated by the network topology encoded in L. In [21], the Hamiltonian of the quantum network thus reads as

$$H = \frac{1}{2}(P^T P + X^T (I + 2gL)X), \tag{1}$$

where I is the $N \times N$ identity matrix, g is the coupling strength between connected oscillators while $p^T = (p_1, p_2, \ldots, p_N)$ and $x^T = (x_1, x_2, \ldots, x_N)$ are the operators corresponding to the momenta and positions of nodes respectively, satisfying the usual commutation relations: $[x, p^T] = i\hbar I$ (we set $\hbar = 1$ in the following). Then the ground state of this Hamiltonian is

$$\psi(X) = \frac{(\det(I + 2gL))^{1/4}}{\pi^{N/4}} \times \exp\left(-\frac{1}{2}(X^T (I+2gL)X)\right), \tag{2}$$

where

$$A_g = \frac{(\det(I + 2gL))^{1/4}}{\pi^{N/4}}$$

is the normalisation factor for the wave function.

The potential matrix of $I + 2gL$ in bipartite graphs can be written in the form

$$V = \begin{pmatrix} I + 2gL_{11} & -2gL_{12} \\ -2gL_{12}^T & I + 2gL_{22} \end{pmatrix}, \tag{3}$$

where L_{11} is the sub-Laplacian of the first part and L_{22} is the sub-Laplacian of second part. L_{12} is the Laplacian between two parts. We know that any local operation does not change the entanglement between the nodes, and so we apply the following operation to calculate the entanglement entropy for bipartite graphs. After applying this operation, we have

$$F = (I + 2gL_{11})^{-1/2}(-2gL_{12})(I + 2gL_{22})^{-1/2}. \tag{4}$$

The Schmidt number is the Singular Value Decomposition (SVD) of F is

$$d = SVD(F), \tag{5}$$

then the final form of the wave function is

$$\begin{aligned}
&\psi(q_1^x, q_2^x, \ldots, q_m^x, q_1^y, q_2^y, \ldots, q_{N-m}^y) \\
&= A_g \exp\Bigg(-\frac{1}{2}(q_1^x, q_2^x, \ldots, q_m^x, q_1^y, q_2^y, \ldots, q_{N-m}^y) \\
&\times \begin{pmatrix}
1 & 0 & \ldots & 0 & d_1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 & 0 & d_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & d_m \\
d_1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
0 & d_2 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & d_{N-m} & 0 & 0 & \ldots & 1
\end{pmatrix}
\begin{pmatrix} q_1^x \\ q_2^x \\ \vdots \\ q_m^x \\ q_1^y \\ q_2^y \\ \vdots \\ q_{N-m}^y \end{pmatrix} \Bigg).
\end{aligned} \tag{6}$$

From the above equation, it is clear that the node q_i^x is just entangled with q_i^y, and so we can use following identity to calculate the Schmidt number of this wave function:

$$\frac{1}{\pi^{1/2}} \exp\left(-\frac{1+t^2}{2(1-t^2)}((q_i^x)^2 + (q_i^y)^2)) + \frac{2t}{1-t^2} q_i^x q_i^y\right) = (1-t^2)^{1/2} \sum_n t^n \psi_n(q_i^x)\psi_n(q_i^y). \tag{7}$$

Then, we apply a change of variable as

$$1 - t^2 = \frac{2}{\nu + 1},$$
$$t^2 = \frac{\nu - 1}{\nu + 1}.$$

So, the above identity becomes

$$\frac{1}{\pi^{1/2}} \exp\left(-\frac{\nu}{2}((q_i^x)^2 + (q_i^y)^2)) + (\nu^2 - 1)^{1/2} q_i^x q_i^y\right) = \left(\frac{2}{\nu+1}\right)^{1/2} \sum_n \left(\frac{\nu-1}{\nu+1}\right)^{n/2} \psi_n(q_i^x)\psi_n(q_i^y) \tag{8}$$

and the reduced density matrix is

$$\rho = \frac{2}{\nu + 1} \sum_n \left(\frac{\nu - 1}{\nu + 1}\right)^n |n\rangle\langle n|. \tag{9}$$

The entropy is

$$S(\rho) = -\sum_n p_n \log(p_n), \quad (10)$$

where

$$p_n = \frac{2}{\nu+1}\left(\frac{\nu-1}{\nu+1}\right)^n,$$

$$\sum_n p_n \log(p_n) = \log\left(\frac{2}{\nu+1}\right) + \langle n\rangle \log\left(\frac{\nu-1}{\nu+1}\right) \quad (11)$$

and

$$\langle n\rangle = \frac{\nu-1}{2},$$

$$S(\rho) = \frac{\nu+1}{2}\log\left(\frac{\nu+1}{2}\right) - \frac{\nu-1}{2}\log\left(\frac{\nu-1}{2}\right). \quad (12)$$

By the comparing the wave function (7) and the identity (8) and defining the scale μ^2, we conclude that

$$\nu_i = 1\times\mu^2, \quad (\nu_i^2-1)^{1/2} = -d_i\times\mu^2.$$

After some straightforward calculations we obtain

$$\nu_i = \left(\frac{1}{1-d_i^2}\right)^{1/2}. \quad (13)$$

By the above discussion we conclude that

$$\mathrm{e}^{-\frac{(q_1^x)^2}{2}-\frac{(q_1^y)^2}{2}-d_1 q_1^x q_1^y} = \sum_n \lambda_{1,n}\psi_n(q_1^x)\psi_n(q_1^y),$$

$$\mathrm{e}^{-\frac{(q_2^x)^2}{2}-\frac{(q_2^y)^2}{2}-d_2 q_2^x q_2^y} = \sum_n \lambda_{2,n}\psi_n(q_2^x)\psi_n(q_2^y),$$

$$\vdots$$

$$\mathrm{e}^{-\frac{(q_m^x)^2}{2}-\frac{(q_m^y)^2}{2}-d_m q_m^x q_m^y} = \sum_n \lambda_{m,n}\psi_n(q_m^x)\psi_n(q_m^y),$$

where

$$\lambda_{i,n} = \left(\frac{2}{\nu_i+1}\right)^{1/2}\left(\frac{\nu_i-1}{\nu_i+1}\right)^{n/2}.$$

Therefore, the entropy of each part can be written as

$$S(\rho_i) = \frac{\nu_i+1}{2}\log\left(\frac{\nu_i+1}{2}\right) - \frac{\nu_i-1}{2}\log\left(\frac{\nu_i-1}{2}\right)$$
$$= \frac{\left(\frac{1}{1-d_i^2}\right)^{1/2}+1}{2}\log\left(\frac{\left(\frac{1}{1-d_i^2}\right)^{1/2}+1}{2}\right)$$
$$- \frac{\left(\frac{1}{1-d_i^2}\right)^{1/2}-1}{2}\log\left(\frac{\left(\frac{1}{1-d_i^2}\right)^{1/2}-1}{2}\right). \quad (14)$$

So the total entropy is

$$S(\rho) = \sum_i S(\rho_i). \quad (15)$$

3. Bipartite cyclic graph

A closed walk consists of a sequence of vertices starting and ending at the same vertex, with each two consecutive vertices in the sequence adjacent to each other in the graph. Now, we consider bipartite cyclic graph, where half of the nodes are in the first part and other nodes are in the second part. So, the number of nodes are $2n$ and the number of nodes in each part are n. The matrix $I+2gL$ will be in the following form:

$$I+2gL = \begin{pmatrix} A & B \\ B^T & C\end{pmatrix} = \begin{pmatrix} I+4g & B \\ B^T & I+4g\end{pmatrix}. \quad (16)$$

Due to the partitioning, we have different partitions as follows:

First case: n = odd

Type 1: In this type, block B can be written as follows:

$$\frac{-2g}{1+4g}(I+s),$$

where s is the circulant shift matrix:

$$s = \begin{pmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & 0 & \dots & 0 & 1\\ 1 & 0 & 0 & \dots & 0\end{pmatrix}. \quad (17)$$

Therefore, d_j is calculated by using the discrete Fourier transform (DFT):

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j}{n}+\cos\frac{2\pi j}{n}^{n-1}}, \quad (18)$$

where $j = 0, 1, \dots, n-1$.

So, υ_j is calculated from eq. (13):

$$\upsilon_j = \frac{2(1+4g)}{\sqrt{4(1+4g)^2-16g^2d_j^2}}. \quad (19)$$

Therefore, the entropy from eq. (14) is obtained as:

$$S_j(\rho) = \frac{\frac{2(1+4g)}{\sqrt{4(1+4g)^2-16g^2d_j^2}}+1}{2}$$

$$\times \log \frac{\frac{2(1+4g)}{\sqrt{4(1+4g)^2-16g^2d_j^2}}+1}{2}$$
$$-\frac{\frac{2(1+4g)}{\sqrt{4(1+4g)^2-16g^2d_j^2}}-1}{2}$$
$$\times \log \frac{\frac{2(1+4g)}{\sqrt{4(1+4g)^2-16g^2d_j^2}}-1}{2}, \tag{20}$$

$$S(\rho) = \sum_{j=0}^{n-1} S_j(\rho). \tag{21}$$

Type 2: In this type, block B can be written as one of these forms:

$$\frac{-2g}{1+4g}(I+s^2).$$

$$I + 2gL = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} = \begin{pmatrix} 1 & 0 & \dots & \frac{2g}{1+4g}d_0 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \frac{2g}{1+4g}d_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & \dots & 1 & 0 & \dots & 0 & \frac{2g}{1+4g}d_{n-1} \\ \frac{2g}{1+4g}d_0 & 0 & \dots & 1 & 0 & \dots & 0 \\ 0 & \frac{2g}{1+4g}d_1 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & \dots & \frac{2g}{1+4g}d_{n-1} & 0 & \dots & 1 & \end{pmatrix}, \tag{25}$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j}{n}^2+\cos\frac{2\pi j}{n}^{n-2}}. \tag{22}$$

So, the Schmidt number and entropy are calculated from eqs (13) and (14).

$$v_j = \sqrt{\frac{1}{1+4g\frac{\left(2+\cos\frac{2\pi}{n}j^2+\cos\frac{2\pi}{n}j^{(n-2)}\right)}{(1+4g)^2}}}.$$

Type $(n-1)/2$: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^{\frac{n-1}{2}}).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j}{n}^{\frac{n-1}{2}}+\cos\frac{2\pi j}{n}^{\frac{n+1}{2}}}. \tag{23}$$

So, the Schmidt number and entropy are calculated from eqs (13) and (14).

By calculating of entropies, in all cases, we have:

$S(\text{type 1}) = S(\text{type 2}) = S(\text{type 3}) = \cdots$.

Second case: $n =$ even: Block B is represented in the following form:

$$-2g(I+s). \tag{24}$$

Now, we find the singular value decomposition of B as

But, in the second case we have two cases:

First case: $n/2 =$ even.

In this case, we have $n/4$ different types of BCGs, where d_j is different in any type:

Type 1: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j}{n}+\cos\frac{2\pi j^{n-1}}{n}}. \quad (26)$$

So, the Schmidt number and entropy are calculated from eqs (13) and (14).

Type 2: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^3).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j^3}{n}+\cos\frac{2\pi j^{n-3}}{n}}. \quad (27)$$

So, the Schmidt number and entropy are calculated from eqs (13) and (14).

⋮

Type $n/4$: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^{\frac{n}{2}-1}).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j^{\frac{n}{2}-1}}{n}+\cos\frac{2\pi j^{\frac{n}{2}+1}}{n}}. \quad (28)$$

So, The schmidt number and entropy are calculated from eqs (13) and (14).

Second case: $n/2 =$ odd. In this case, we have $(n+2)/4$ different types of BCGs, where d_j is different in any type:

Type 1: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j}{n}+\cos\frac{2\pi j^{n-1}}{n}}. \quad (29)$$

So, the schmidt number and entropy are calculated from eqs (13) and (14).

Type 2: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^3).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j^3}{n}+\cos\frac{2\pi j^{n-3}}{n}}. \quad (30)$$

So, the Schmidt number and entropy are calculated from eqs (13) and (14).

⋮

Type $(n+2)/4$: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^{\frac{n}{2}}).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+2\cos\frac{2\pi j^{\frac{n}{2}}}{n}}. \quad (31)$$

So, the schmidt number and entropy are calculated from eqs (13), (14) and the above graphs are connected BCGs. But when $n =$ even we have disconnected BCGs. If block B is written as the following types, the graph is disconnected:

First case: $n/2 =$ even

Type 1: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^2).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j^2}{n}+\cos\frac{2\pi j^{n-2}}{n}}. \quad (32)$$

So, the Schmidt number and entropy are calculated from eqs (13) and (14).

⋮

Type $n/4$: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^{\frac{n}{2}-1}).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j^{\frac{n}{2}-1}}{n}+\cos\frac{2\pi j^{\frac{n}{2}+1}}{n}}. \quad (33)$$

Second case: $n/2 =$ odd

Type 1: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^2).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+\cos\frac{2\pi j^2}{n}+\cos\frac{2\pi j^{n-2}}{n}}. \quad (34)$$

So, the Schmidt number and entropy are calculated from eqs (13) and (14).

⋮

Type $(n-2)/4$: In this type, block B can be written in one of these forms:

$$\frac{-2g}{1+4g}(I+s^{\frac{n}{2}}).$$

Therefore, d_j is calculated by using the DFT:

$$d_j = \frac{-2g}{1+4g}\sqrt{2+2\cos\frac{2\pi j}{n}^{\frac{n}{2}}}. \tag{35}$$

By calculating entropies, in all the cases, we have:

$$\begin{aligned} &S(\text{type}(B = I + s^{\frac{n}{2}}) \\ &> S(B = I + s^{\frac{n}{2}-1}, B = I + s^{\frac{n}{2}+1}) \\ &> S(B = I + s^{\frac{n}{2}-2}, B = I + s^{\frac{n}{2}+2}) \\ &> \cdots > S(B = I + s, B = I + s^{n-1}). \end{aligned}$$

We use entanglement entropy as a tool for studying the amount of information stored in quantum complex networks. By considering the ground state of a network of coupled quantum harmonic oscillators, we compute the information that each part of the system has on the rest of the system. If the number of nodes in each part are odd, then the amount of information is independent of the selection of the system configuration. But, if the number of nodes in each part are even, then the amount of information depends on the selection of the system configuration and we can find maximum and minimum amount of information. By using the Von Neumann entropy, we show that the amount of information is limited, regardless of how many connections a node has. The synergy between the field of complex networks and that of information theory has recently appealed to the quantum information community. The advent of network science has influenced the research in many fields of science in general and physics in particular, in a pervasive way. One of the most important avenues of research in network science is its connection with information theory. For instance, the entrophy has been successfully applied to characterise the networks.

4. Conclusion

The entanglement entropy is obtained between two parts in the cyclic graphs in which their nodes are considered as quantum harmonic oscillators. Our method is used to calculate the Schmidt numbers and entanglement entropy between two equal parts of the cyclic graph. Analytically, entanglement entropy in two cases of partitioning are calculated. Then, the ordering of entanglement entropies for these partitioning are given. By using this result, we can conclude which partitionings have the minimum and maximum entanglement entropy.

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