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Some applications of special trans function theory in physics

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Abstract. Examples of applications of special trans function theory (STFT) which may be useful in obtaining closed-form analytical solutions of certain transcendental equations found in undergraduate physics is presented. The novelty of the method can be of utmost importance to the academia involved in undergraduate physics.

Keywords. Special trans function theory; Wien's displacement law; thermionic emission; decay constant.

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1. Introduction

Transcendental equations are indispensable in undergraduate physics which are traditionally solved by graphical or numerical methods as they are not solvable with elementary mathematics. However, analytical solutions are always advantageous and preferable as they could facilitate in gaining a deeper insight of the relevant problem with a complete theoretical understanding [1]. In the recent past, several researchers have shown that the application of a special function, Lambert W-function [2], makes it possible to get explicit analytical solutions to many transcendental equations. In fact, Lambert W-function has been used quite effectively in solving transcendental equations associated with several physical problems in physics and engineering domains [2–6].

In the present article, an attempt is made to get exact analytical solutions to some of the transcendental equations in undergraduate physics by the application of a lesser known method – special trans function theory (STFT) [7]. This theory has been applied to find closed-form solutions to many problems which are mathematically expressible in the form of transcendental equations [8–16]. The objective of this article is two-fold. First, to introduce readers with this lesser known theory which has the potential of obtaining closed-form analytical solutions to many of the known physical problems having transcendental nature. Secondly, to apply the theory by selecting some examples from the undergraduate physics to confirm its efficacy. The numerical values can be calculated using mathematical softwares such as Maple and Mathematica.

2. Theory (Special trans function theory (STFT))

STFT originates from the works of Perovich [8]. This is a theory which gives an exact analytical solution to a family of transcendental equations of the form:

$$\psi(\zeta) = U(\zeta)\mathrm{e}^{-\psi(\zeta)}, \qquad U(\zeta) \in R^{+}. \tag{1}$$

STFT solutions are exact, analytical and are easily differentiable. It can also be extended to higher dimensions.

Equation (1) has the analytical closed-form solution

$$\psi(\zeta) = \mathrm{trans}_{+}\left(U(\zeta)\right), \tag{2}$$

where $\mathrm{trans}_{+}\left(U(\zeta)\right)$ is a new special function defined as

$$\mathrm{trans}_{+}\left(U(\zeta)\right) = \lim_{x\to\infty}\left[\ln\left(\frac{\phi(x, U(\zeta)}{\phi(x+1, U(\zeta)}\right)\right] \tag{3}$$

and $\phi\left(x, U(\zeta)\right)$ is defined as

$$\phi\left(x, U(\zeta)\right) = \sum_{n=0}^{[x]} \frac{(U(\zeta))^{n}(x-n)^{n}}{n!}. \tag{4}$$

$[x]$ denotes the greatest integer less than or equal to x. More explicitly, the closed-form analytical solution of eq. (2) can be written as

$$\psi(\zeta) = \mathrm{trans}_{+}\left(U(\zeta)\right) = \lim_{x\to\infty}\left(U(\zeta)\left(\frac{\sum_{n=0}^{[x]}\frac{(U(\zeta))^{n}(x-n)^{n}}{n!}}{\sum_{n=0}^{[x+1]}\frac{(U(\zeta))^{n}(x+1-n)^{n}}{n!}}\right)\right). \tag{5}$$

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The new trans function of eq. (2) defined through eqs (3)–(5) is a solution to a family of transcendental equations defined by eq. (1) and is shown in the Appendix.

For $U(\zeta) \in R^-$, in STFT [14], if we let $U(\zeta) = -D(\zeta)$, then

$$\text{trans}_-(-D(\zeta)) = -\text{trans}_+(U(\zeta)) \tag{6}$$

and the above closed-form solution of eq. (5) remains invariant. More details on STFT and its properties are well documented in [7] that can solve a set of transcendental equations defined by eq. (1). The accuracy in $\psi(\zeta)$ depends upon $[x]$. A numerical comparison between STFT and classical approach using Lambert W-function is given in [7]. Any equation which can be reduced to the form of eq. (1) is solvable analytically by the application of STFT. Solutions based on this theory are exact, analytical and easily differentiable.

Now, this method will be applied to three transcendental equations we come across in undergraduate physics. The general methodology adopted is to first transform the transcendental equation of interest into the form of eq. (1) so that we can apply STFT to obtain its closed-form solution which is given by eq. (5). The relevant explicit solution of the concerned variable can be retracted from this solution. We shall illustrate it through the following three examples from undergraduate physics.

3. Applications of STFT and the closed-form solutions

3.1 *Wien's displacement law*

The most satisfactory formula that describes the spectral distribution of a blackbody radiation both on theoretical and experimental grounds is given by Planck's law which is written in many forms. One particular form gives the density of the radiation between wavelengths λ and $\lambda + \mathrm{d}\lambda$ [17]

$$u_\lambda \mathrm{d}\lambda = \frac{8\pi ch}{\lambda^5}\frac{d\lambda}{\left(\mathrm{e}^{\frac{ch}{\lambda k_\mathrm{B} T}} - 1\right)}. \tag{7}$$

A characteristic of the blackbody radiation is that the wavelength of the most strongly emitted radiation in the blackbody spectra is inversely proportional to the temperature of the body in Kelvin and is known as Wien's displacement law. Wien arrived at it through thermodynamic arguments. The same law can be deduced from Planck's law also. If λ_m is the wavelength corresponding to the maximum radiation emitted, then

$$\left(\frac{\partial u_\lambda}{\partial \lambda}\right)_{\lambda_m} = 0.$$

On solving one gets a transcendental equation of the form

$$-5\mathrm{e}^{\frac{ch}{\lambda_m k_\mathrm{B} T}} + \frac{ch}{\lambda_m k_\mathrm{B} T}\mathrm{e}^{\frac{ch}{\lambda_m k_\mathrm{B} T}} + 5 = 0$$

or

$$(x-5)\,\mathrm{e}^x = -5, \tag{8}$$

where

$$x = \frac{ch}{\lambda_m k_\mathrm{B} T}. \tag{9}$$

Equation (8) can be rearranged as

$$(x-5) = -(5\mathrm{e}^{-5})\mathrm{e}^{-(x-5)}. \tag{10}$$

This takes the form of eq. (1) where

$$\psi(x) = x - 5 \tag{11}$$

and

$$U(\zeta) = -5\mathrm{e}^{-5}. \tag{12}$$

By the application of STFT, the closed-form expression is given by eq. (5)

$$x = 5 + \text{trans}_+(U(\zeta)).$$

That is,

$$x = 5 + \lim_{x\to\infty}\left((-5\mathrm{e}^{-5})\left(\frac{\sum_{n=0}^{[x]}\frac{(-5\mathrm{e}^{-5})^n(x-n)^n}{n!}}{\sum_{n=0}^{[x+1]}\frac{(-5\mathrm{e}^{-5})^n(x+1-n)^n}{n!}}\right)\right). \tag{13}$$

The value of x is calculated to be 4.98089 when $[x] = 5$. With this value, the Wien constant is determined as

$$\begin{aligned}\lambda_m T &= \frac{ch}{4.98 k_\mathrm{B}} \\ &= \frac{(3\times 10^8)\times(6.62\times 10^{-34})}{4.98\times(1.38\times 10^{-23})} \\ &= 2.89\times 10^{-3}\,\mathrm{mK}.\end{aligned} \tag{14}$$

3.2 *Operating temperature in thermionic emission*

The physics of thermionic emission can be summarised in the form of an equation through Richardson–Dushmann equation. The equation gives the expression of the electric current density J from the emitter in terms of the operating temperature T and other constants [18]

$$J = AT^2\mathrm{e}^{-\frac{\varphi}{k_\mathrm{B} T}} \tag{15}$$

Here,

$$A = \frac{4\pi e m_e k_\mathrm{B}^2}{h^3} = 1.2\times 10^6\,\mathrm{Am^{-2}K^{-2}}$$

is Richardson constant, φ is the work function of the emitter material and k_B is Boltzmann's constant. Even though A is treated as a universal constant, for most of the metals with clean surfaces, it is found to be in the range of (0.6–1.2) $\times 10^6 \text{A m}^{-2} \text{K}^{-2}$ [18]. As the purpose of this article is the application of STFT, we shall not go into the details of the nature of constant A. Traditionally, eq. (15) is used to obtain φ of the emitter material experimentally from the values of J and T. On the other hand, application of STFT in eq. (15) enable us to determine T by measuring the current density J, provided the value of φ is known (say from photoelectric effect measurements). To do so, we can rewrite eq. (15) as

$$\left(\frac{J}{A}\right)^{1/2} = T\mathrm{e}^{-\frac{\varphi}{2k_B T}}$$

or

$$\frac{1}{2}\frac{\varphi}{k_B T} = \left(\frac{1}{2}\frac{\varphi}{k_B}\left(\frac{A}{J}\right)^{1/2}\right)\mathrm{e}^{-\frac{\varphi}{2k_B T}}. \tag{16}$$

This also takes the form of eq. (1) where

$$\psi(\zeta) = \frac{1}{2}\frac{\varphi}{k_B T} \tag{17}$$

and

$$U(\zeta) = \frac{1}{2}\frac{\varphi}{k_B}\left(\frac{A}{J}\right)^{1/2}. \tag{18}$$

By the application of STFT, the explicit analytical closed-form expression for the temperature at which the thermionic current density is produced is given by

$$T = \frac{\varphi}{2k_B}\frac{1}{\text{trans}_+(U(\zeta))}, \tag{19}$$

where $\text{trans}_+(U(\zeta))$ is defined by eq. (5). One can use eq. (19) in the experimental determination of temperature T at which the emitter is emitting thermionic electrons by measuring the current density J. For a physical analysis and practical calculations, eq. (19) may be written as

$$T = \frac{\varphi}{2k_B}\frac{1}{(U(\zeta))}\left(\frac{\sum_{n=0}^{[x+1]}\frac{(U(\zeta))^n(x+1-n)^n}{n!}}{\sum_{n=0}^{[x]}\frac{(U(\zeta))^n(x-n)^n}{n!}}\right). \tag{20}$$

To check the validity of the above expression, taking A as a constant, we obtain the variation of temperature as a function of current density for two thermionic emitters: (i) tungsten ($\varphi = 4.54\,\text{eV}$) and (ii) thoriated tungsten ($\varphi = 2.63\,\text{eV}$) as shown in figure 1.

3.3 *Decay constant in radioactivity*

Radioactivity discovered by Henry Becquerel in 1896 [19] is the single phenomenon that plays the most vital

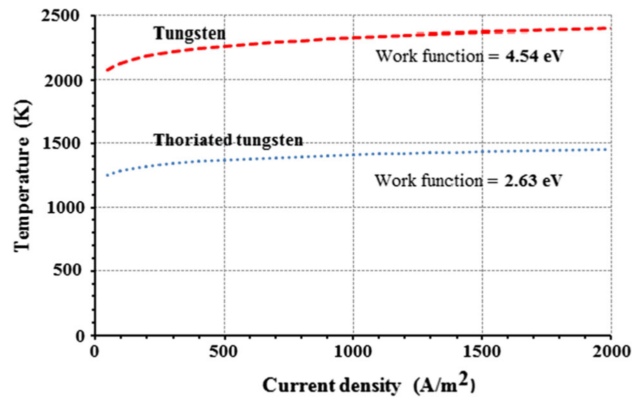

Figure 1. Temperature as function of current density for two thermionic emitters: tungsten (work function = 4.54 eV) and thoriated tungsten (work function = 2.63 eV).

role in the development of nuclear physics. The activity law gives the time variation of the activity that follows the formula

$$R(t) = R_0\mathrm{e}^{-\lambda t}, \tag{21}$$

where λ is the decay constant having different values for different radionuclides. Conventionally, in physics textbooks, the decay constant is calculated as

$$\lambda = \frac{\ln 2}{T_{1/2}},$$

where $T_{1/2}$ is the half-life of the radioactive sample at which the activity R drops to $\frac{1}{2}R_0$ and ln is the natural logarithm.

If a radioactive sample contains N undecayed nuclei at time t, then the radioactive decay formula is written in terms of the decay constant λ and the number of undecayed nuclei N_0 at $t = 0$ as

$$N = N_0\mathrm{e}^{-\lambda t}. \tag{22}$$

Then the activity of the sample is given by

$$R = \frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N_0\mathrm{e}^{-\lambda t}. \tag{23}$$

Equation (23) may be rewritten as

$$-\lambda t = \frac{Rt}{N_0}\mathrm{e}^{\lambda t}. \tag{24}$$

If

$$\begin{aligned}\psi(\zeta) &= -\lambda t\\ U(\zeta) &= Rt/N_0\end{aligned} \tag{25}$$

then eq. (24) takes the form of eq. (1) and so by applying STFT, we have the closed-form analytical expression for the decay constant as

$$\psi(\zeta) = -\lambda t = \text{trans}_+(U(\zeta))$$

or

$$\lambda = -\frac{1}{t}\left[\lim_{x\to\infty}\left(\left(\frac{Rt}{N_0}\right)\right.\right.$$
$$\left.\left.\times\left(\frac{\sum_{n=0}^{[x]}\frac{(Rt/N_0)^n(x-n)^n}{n!}}{\sum_{n=0}^{[x+1]}\frac{(Rt/N_0)^n(x+1-n)^n}{n!}}\right)\right)\right]. \tag{26}$$

If the activity R is known from experiments at any time t then we can determine the decay constant.

4. Conclusion

This communication presents the use of special trans function theory (STFT) in solving certain transcendental equations found in undergraduate physics analytically. The physical significance of the obtained results is unchanged but their mathematical formulation by the method of STFT is exact and a novel one. Consequently, method of STFT can be a useful theoretical technique to the academia involved in undergraduate physics.

A. Appendix

Here, we shall show how eq. (2) is a solution to a family of transcendental equations represented by eq. (1). The proof of the same is presented in [7]. We reproduce it here so that it can be useful to the readers. According to STFT [7], eq. (1) has the analytical closed-form solution

$$\psi(\zeta) = \text{trans}_+(U(\zeta)), \tag{A1}$$

where $\text{trans}_+(U(\zeta))$ is a new special function defined as

$$\text{trans}_+(U(\zeta)) = \lim_{x\to\infty}\left[\ln\left(\frac{\phi(x, U(\zeta)}{\phi(x+1, U(\zeta)}\right)\right] \tag{A2}$$

and $\phi(x, U(\zeta))$ is defined as

$$\phi(x, U(\zeta)) = \sum_{n=0}^{[x]}\frac{(U(\zeta))^n(x-n)^n}{n!}, \tag{A3}$$

where $[x]$ denotes the greatest integer less than or equal to x.

Proof. The transcendental equation (1) can be identified with a partial differential equation of the following type:

$$\frac{\partial\phi(x, U(\zeta))}{\partial x} = U(\zeta)\phi(x-1, U(\zeta)). \tag{A4}$$

Making use of the Laplace transform, we shall solve the partial differential equation (A4). On taking Laplace transform of eq. (A4), we get

$$sF(s, U(\zeta)) - U(\zeta)F(s, U(\zeta))\,\mathrm{e}^{-s} = \phi(0), \tag{A5}$$

where

$$F(s, U(\zeta)) = L\{\phi(x, U(\zeta))\}.$$

A little arrangement of eq. (A5) gives

$$F(x, U(x)) = \frac{\phi(0)}{s - U(\zeta)e^{-s}} = \frac{\phi(0)}{s\left(1 - U(\zeta)\frac{\mathrm{e}^{-s}}{s}\right)} \tag{A6}$$

Expansion of the denominator on the right-hand side of eq. (A6) gives

$$F(x, U(\zeta)) = \frac{\phi(0)}{s}\sum_{n=0}^{\infty}(U(\zeta))^n\frac{\mathrm{e}^{-ns}}{s^n}. \tag{A7}$$

The series on the right-hand side of eq. (A7) converges for $\left|U(\zeta)\frac{\mathrm{e}^{-s}}{s}\right| \ll 1$. Now, inverting eq. (A7) term by term, we get

$$\phi(x, U(\zeta)) = \sum_{n=0}^{\infty}(U(\zeta))^n\frac{(x-n)^n}{n!}H(x-n), \tag{A8}$$

where $H(x-n)$ is the Heaviside's unit function. The function series on the right-hand side of eq. (A8) is identical to that of eq. (A3) for $x > n$. Finally, we get the analytical solution to the partial differential equation (A4) by applying Laplace transform in a closed-form representation as

$$\phi(x, U(\zeta)) = \sum_{n=0}^{[x]}(U(\zeta))^n\frac{(x-n)^n}{n!}. \tag{A9}$$

According to Lerch's theorem, eq. (A9) is the unique analytical closed-form solution to eq. (A4).

Lemma 1. *For any $x > 1$, the series on the right -hand side of eq.* (A9) *satisfies eq.* (A4).

Substitution of eq. (A9) into eq. (A4) yields

$$\frac{\partial\phi(x, U(\zeta))}{\partial x} - U(\zeta)\phi(x-1, U(\zeta))$$
$$= \frac{\partial}{\partial x}\left(\sum_{n=0}^{[x]}(U(\zeta))^n\frac{(x-n)^n}{n!}\right) - U(\zeta)\sum_{n=0}^{[x-1]}(U(\zeta))^n\frac{(x-1-n)^n}{n!}$$
$$= \sum_{n=0}^{[x]}(U(\zeta))^n\frac{(x-n)^{n-1}}{(n-1)!} - \sum_{n=0}^{[x-1]}(U(\zeta))^{n+1}\frac{(x-(n+1))^n}{n!}$$

$$= \sum_{n=0}^{[x-1]} (U(\zeta))^{n+1} \frac{(x-(n+1))^n}{n!} - \sum_{n=0}^{[x-1]} (U(\zeta))^{n+1} \frac{(x-(n+1))^n}{n!} = 0 \tag{A10}$$

for any $x \notin N$. For $x \in N$ and $x \neq 1$, $\phi(x, U(\zeta))$ is differentiable and the proof follows as well.

Remark 1. It can be easily shown that the function of eq. (A9) is not differentiable at $x = 1$, due to the term of first order. Now, from eq. (A8) we can write

$$\varphi_n(x) = (x-n)^n H(x-n)$$

and

$$\varphi_n(x) = \begin{cases} 0 & \text{for } x < n \\ (x-n)^n & \text{for } x > n \end{cases}. \tag{A11}$$

Also,

$$\varphi_n'(x) = \begin{cases} 0 & \text{for } x < n \\ n(x-n)^{n-1} & \text{for } x > n \end{cases}. \tag{A12}$$

Clearly, $\varphi_n(x)$ is differentiable for $n \neq 1 \quad \forall x > 0$. Also, $\varphi_1(x)$ for $n = 1$ is differentiable $\forall x > 0$ and $x \neq 1$.

Lemma 2. *On the particular solution to the differential eq.* (A4)

By the method of separation of variables, we can write

$$\phi(x, U(\zeta)) = \Theta(U(\zeta))\, \mathrm{e}^{\omega(U(\zeta))x} \tag{A13}$$

and from eqs (A4) *and* (A13)*, we have*

$$\omega(U(\zeta)) = U(\zeta)\mathrm{e}^{-\omega(U(\zeta))}. \tag{A14}$$

From eqs (1) and (A14), we see that

$$\omega(U(\zeta)) = \psi(\zeta) = \psi(\zeta; U(\zeta)).$$

This means that the particular solution of the form

$$\phi_p(x, U(\zeta)) = \Theta(U(\zeta))\, \mathrm{e}^{\omega(U(\zeta))x} = \Theta(U(\zeta))\, \mathrm{e}^{\psi(\zeta;U(\zeta))x} = \Theta(U(\zeta))\, \mathrm{e}^{\psi(\zeta)x} \tag{A15}$$

satisfies the differential equation (A4) under the condition that $\psi(\zeta)$ satisfies eq. (1). Consequently, eq. (A15) is an asymptotic function to eq. (A4) and hence

$$\lim_{x\to\infty} \left(\frac{\phi(x, U(\zeta))}{\phi_p(x, U(\zeta))} \right) = 1 \tag{A16}$$

based on the functional theory. Now, by the principle of unique solution and functional theory, we have

$$\lim_{x\to\infty} \left(\frac{\phi(x+1, U(\zeta))}{\phi(x, U(\zeta))} \right) = \frac{\phi_p(x+1, U(\zeta))}{\phi_p(x, U(\zeta))} = \frac{\Theta(U(\zeta))\,\mathrm{e}^{\omega(U(\zeta))(x+1)}}{\Theta(U(\zeta))\,\mathrm{e}^{\omega(U(\zeta))x}} = \mathrm{e}^{\omega(U(\zeta))} = \mathrm{e}^{\psi(\zeta)} \tag{A17}$$

and

$$\psi(\zeta) = \mathrm{trans}_+(U(\zeta)) = \lim_{x\to\infty} \left[\ln \left(\frac{\phi(x+1, U(\zeta))}{\phi(x, U(\zeta))} \right) \right]. \tag{A18}$$

As $\psi(\zeta) = U(\zeta)\mathrm{e}^{-\psi(\zeta)}$, eq. (A18) can be written as

$$\psi(\zeta) = \lim_{x\to\infty} \left(U(\zeta) \frac{\phi(x, U(\zeta))}{\phi(x+1, U(\zeta))} \right). \tag{A19}$$

More explicitly,

$$\psi(\zeta) = \mathrm{trans}_+(U(\zeta)) = \lim_{x\to\infty} \left(U(\zeta) \frac{\sum_{n=0}^{[x]} \frac{(U(\zeta))^n (x-n)^n}{n!}}{\sum_{n=0}^{[x+1]} \frac{(U(\zeta))^n (x+1-n)^n}{n!}} \right). \tag{A20}$$

Equation (A20) defines a new special function namely, $\mathrm{trans}_+(U(\zeta))$. The essential part of STFT is the existence of eq. (A16). It is clear that by applying unique solution principle, we have eq. (A16). Thus, it is proved that eq. (A1) is the solution to eq. (1).

To illustrate the application of STFT, let us consider the following transcendental equation as a mathematical example:

$$x = \ln(y) + y. \tag{A21}$$

Taking exponential on both sides of eq. (A21), we have

$$\mathrm{e}^x = y\mathrm{e}^y. \tag{A22}$$

We can rewrite eq. (A22) as

$$y = \mathrm{e}^x \mathrm{e}^{-y}. \tag{A23}$$

Equation (A23) takes the form of eq. (1) and hence by the application of STFT, for $\mathrm{e}^x \in R$, we have the analytical closed-form solution of eq. (A21) as

$$y = \mathrm{trans}_+(\mathrm{e}^x) = \lim_{M\to\infty} \left(\mathrm{e}^x \left(\frac{\sum_{n=0}^{[M]} (\mathrm{e}^x)^n \frac{(M-n)^n}{n!}}{\sum_{n=0}^{[M+1]} (\mathrm{e}^x)^n \frac{(M+1-n)^n}{n!}} \right) \right), \tag{A24}$$

where $[M]$ is the greatest integer less than or equal to M. Equation (A24) is a new special function that gives explicit analytical solution to eq. (A21).

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