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Generation of entanglement from a two-mode cascade laser

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Abstract. We analyse the entanglement of the light produced by a three-level laser whose cavity contains a parametric amplifier, with the cavity mode driven by the coherent light and coupled to a squeezed vacuum reservoir. Employing stochastic differential equations associated with the normal ordering resulted from the pertinent master equation, the quadrature variances and entanglement of the two-mode light are obtained and photon statistics of the two-light modes is discussed. It is found that the three-level laser generates squeezed light under certain conditions, with maximum intracavity squeezing being 89.7% below the coherent state level. We also show that the parametric amplifier enhances the mean photon number and photon number correlation.

Keywords. Parametric amplifier; squeezing; entanglement; photon number correlations.

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1. Introduction

Three-level cascade lasers have received considerable attention in connection with its potential as a source of light with interesting non-classical features [1–12]. The quantum properties of light, in this device, is attributed to atomic coherence that can be induced either by preparing the atoms initially in a coherent superposition of the top and bottom levels [7,8] or coupling these levels by an external radiation [9–11] or using these mechanisms together [12].

Parametric amplifier is a nonlinear crystal that involves three different modes of radiation field: the signal, the idler and the pump, which are coupled by a nonlinear medium. In this device, a pump photon interacts with the nonlinear crystal inside the cavity and is down-converted into two highly correlated photons of different frequencies [13–16]. Some researchers have also studied quantum properties of light generated by the three-level laser whose cavity contains parametric amplifier [17]. Thus, it has been shown that the cavity radiation is in squeezed and entangled states under certain conditions. In addition, the mean and variance of the photon number for a degenerate [17] and non-degenerate [18,19] three-level cascade laser whose cavity contains parametric amplifier have been determined for different cases.

In most of the previous analyses, the laser cavity is coupled to a vacuum environment in which the effect of an external noise on the quantum properties and photon statistics is completely neglected. For instance, a squeezing of 93% has been realised in a degenerate three-level laser with parametric amplifier and coupled to vacuum reservoir. However, it is quite challenging for real physical situations to be free from the effect of external environments which unavoidably decouple the atomic correlations responsible for the quantum and statistical properties [20–22]. In this regard, Tesfa [22] has considered the effect of thermal noise on squeezing, entanglement and photon statistics of the cavity radiation generated by a correlated emission laser in the absence of parametric amplifier and external pumping radiation that couple the top and bottom levels of the three-level atom. He has found that the thermal noise entering the cavity degrades the squeezing and entanglement but enhances the mean number of photon pairs of the cavity light. On the other hand, the squeezing, entanglement and statistical properties of the cavity radiation get enhanced with the introduction of the parametric amplifier. It is with this motivation that we investigate these non-classical properties for a radiation generated by a non-degenerate three-level laser whose cavity contains a non-degenerate parametric amplifier and coupled to a two-mode thermal reservoir.

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In this study, we analyse the squeezing and entanglement properties and photon statistics of a two-mode cavity light produced by a non-degenerate three-level laser with a non-degenerate parametric amplifier (NDPA), and coupled to a two-mode squeezed vacuum reservoir via a single port mirror. In order to carry out our analyses, we first derive the master equation in the good cavity limit, linear and adiabatic approximations. Employing the master equation, the stochastic differential equations, then solutions for c-number cavity mode variables and correlation property of the noise forces associated with the normal ordering are determined. Using the resulting solutions, the mean number of photon pairs, quadrature fluctuations and EPR-type operators [23] of the cavity radiation are determined. We investigate the effects of parametric amplifier and thermal noise on squeezing, entanglement and mean photon number of the cavity radiation. Moreover, based on the criterion for a continuous variable entanglement developed by Duan *et al* [24], the relation between squeezing and entanglement has been established.

2. The model

Here we want to drive the master equation for a non-degenerate three-level laser whose cavity contains a parametric amplifier, with the cavity modes driven by a two-mode coherent light and coupled to a two-mode squeezed vacuum reservoir. We represent the top, intermediate and bottom levels of a three-level atom in a cascade configuration by $|a\rangle$, $|b\rangle$ and $|c\rangle$, respectively, as shown in figure 1. In addition, we assume the two modes a and b to be at resonance with the two transitions $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$, respectively and direct transition between level $|a\rangle$ and level $|c\rangle$ to be dipole forbidden and also we consider the case in which three-level atoms in the cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity at a constant rate r_a (rate of atomic injection into the cavity) and removed after some time. The interaction of a pumped non-degenerate three-level cascade atom with a resonant two-mode cavity radiation can be described in the rotating-wave approximation and the interaction picture by the Hamiltonian of the form

$$\hat{H}_S = ig\big[|a\rangle\langle b|\hat{a}_1 - \hat{a}_1^\dagger|b\rangle\langle a| + |b\rangle\langle c|\hat{a}_2 - \hat{a}_2^\dagger|c\rangle\langle b|\big] + i\varepsilon_1[\hat{a}_1^\dagger - \hat{a}_1 + \hat{a}_2^\dagger - \hat{a}_2] + i\varepsilon_2[\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2], \quad (1)$$

where g is the coupling constant, ε_1 is proportional to the amplitude of the driving light modes and ε_2 is proportional to the amplitude of the pump mode. Here, $\hat{a}_1(\hat{a}_1^\dagger)$

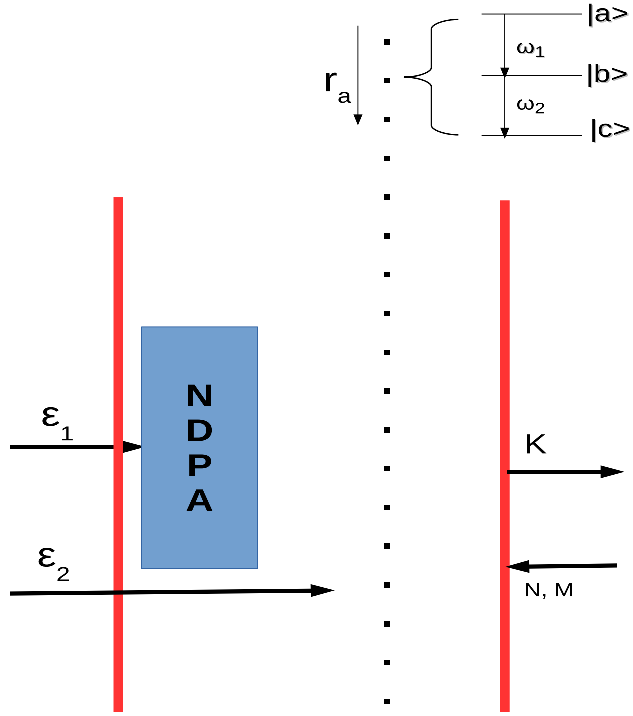

Figure 1. Schematic representation of non-degenerate three-level laser with a non-degenerate parametric amplifier (NDPA) and coupled to a two-mode squeezed vacuum reservoir. Here, ε_1 is a real constant proportional to the amplitude of the pump mode that drives the NDPA while ε_2 describes the amplitude of the coherent light. In addition, r_a is the rate of atomic injection into the cavity and K (kappa) is the cavity damping constant.

and $\hat{a}_2(\hat{a}_2^\dagger)$ are the annihilation(creation) operator of the two-mode cavity light. In this paper, we take the initial state of a three-level atom to be

$$|\psi_a(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle \quad (2)$$

and hence the initial density operator for a single atom has the form

$$\hat{\rho}_A(0)\rangle = \rho_{aa}(0)|a\rangle\langle a| + \rho_{ac}(0)|a\rangle\langle c| + \rho_{ca}(0)|c\rangle\langle a| + \rho_{cc}(0)|c\rangle\langle c|, \quad (3)$$

where $\rho_{aa}^{(0)} = |C_a|^2$ and $\rho_{cc}^{(0)} = |C_c|^2$ are, respectively, the probabilities for the atom to be initially in the upper and lower levels, and $\rho_{ac}^{(0)} = C_a C_c^*$ and $\rho_{ca}^{(0)} = C_c C_a^*$, represent the initial atomic coherence of the atom. Using the interaction of the cavity modes with the vacuum reservoir, we obtain the master equation for the laser cavity modes in the good-cavity limit ($\gamma \gg \kappa$) and in the linear and adiabatic approximation schemes. In view of this, employing the linear and adiabatic approximation schemes in the good-cavity limit, the equation of evolution of the density operator for the cavity modes is found to be

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}(t) = \varepsilon_1[\hat{\rho}\hat{a}_1 - \hat{a}_1\hat{\rho} + \hat{a}_1^\dagger\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger + \hat{\rho}\hat{a}_2 - \hat{a}_2\hat{\rho} + \hat{a}_2^\dagger\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger] + \varepsilon_2[\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2\hat{\rho} + \hat{\rho}\hat{a}_1\hat{a}_2] + \frac{1}{2}\big[(A\rho_{aa}^{(0)} + \kappa N)\big(2\hat{a}_1^\dagger\hat{\rho}\hat{a}_1 - \hat{a}_1\hat{a}_1^\dagger\hat{\rho} - \hat{\rho}\hat{a}_1\hat{a}_1^\dagger\big) + (A\rho_{cc}^{(0)} + \kappa(N+1))\big[2\hat{a}_2\hat{\rho}\hat{a}_2^\dagger - \hat{\rho}\hat{a}_2^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_2\hat{\rho}\big] + \frac{1}{2}\big[(A\rho_{ac}^{(0)} + \kappa M)\big[2\hat{a}_1^\dagger\hat{\rho}\hat{a}_2^\dagger - \hat{a}_2^\dagger\hat{a}_1^\dagger\hat{\rho} - \hat{\rho}\hat{a}_2^\dagger\hat{a}_1^\dagger\big] + (A\rho_{ca}^{(0)} + \kappa M)\big[2\hat{a}_2\hat{\rho}\hat{a}_1 - \hat{\rho}\hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_2\hat{\rho}\big)\big] + \frac{\kappa}{2}(N+1)\big[2\hat{a}_1\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_1\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_1\big] + \frac{\kappa}{2}N\big[2\hat{a}_2^\dagger\hat{\rho}\hat{a}_2 - \hat{a}_2\hat{a}_2^\dagger\hat{\rho} - \hat{\rho}\hat{a}_2\hat{a}_2^\dagger\big] - \frac{1}{2}\kappa M\big[2\hat{a}_2^\dagger\hat{\rho}\hat{a}_1^\dagger - \hat{a}_1^\dagger\hat{a}_2^\dagger\hat{\rho} - \hat{\rho}\hat{a}_1^\dagger\hat{a}_2^\dagger + 2\hat{a}_1\hat{\rho}\hat{a}_2 - \hat{\rho}\hat{a}_2\hat{a}_1 - \hat{a}_2\hat{a}_1\hat{\rho}\big], \tag{4}$$

where for a squeezed vacuum reservoir $N = \sinh^2 r$, $M = \sinh r \cosh r$ and r is the squeeze parameter. Here $A = 2gr_a/\gamma^2$ is the linear gain coefficient and γ being the spontaneous atomic decay rate is assumed to be the same for all the three-levels.

It proves to be useful to introduce a new parameter which relates the probabilities of the atom to be in the upper and lower levels. We define the parameter η such that

$$\rho_{aa}^{(0)} = \frac{1-\eta}{2}$$

with $-1 < \eta < 1$. For three-level atoms initially in a coherent superposition of the top and bottom levels, one obtains

$$\rho_{cc}^{(0)} = \frac{1+\eta}{2}$$

and in view of the relation

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)},$$

one easily finds

$$\rho_{ac}^{(0)} = \frac{1}{2}\sqrt{1-\eta^2}.$$

Using this master equation, the evolution of the two-mode cavity radiation in terms of c-number variables associated with the normal ordering $\alpha_1(t)$ and $\alpha_2^*(t)$ can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_1(t) = -\frac{1}{2}\mu_-\alpha_1(t) + \frac{1}{2}\nu_-\alpha_2^*(t) + \varepsilon_1 + f_1(t), \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha_2^*(t) = -\frac{1}{2}\mu_+\alpha_2^*(t) + \frac{1}{2}\nu_+^*\alpha_1(t) + \varepsilon_1 + f_2^*(t), \tag{6}$$

where

$$\mu_\mp = \kappa \mp \frac{A(1\mp\eta)}{2}, \tag{7}$$

$$\nu_\pm = 2\varepsilon_2 \pm \frac{A}{2}\sqrt{1-\eta^2} \tag{8}$$

and $f_1(t)$ and $f_2(t)$ are noise forces, the properties of which remain to be determined. Following the straightforward procedure outlined in [10,14,19], it is possible to obtain

$$\alpha_1(t) = E_+(t)\alpha_1(0) + F_-(t)\alpha_2^*(0) + G_+(t) + \chi_1(t), \tag{9}$$

$$\alpha_2^*(t) = E_-(t)\alpha_2^*(0) + F_+(t)\alpha_1(0) + G_-(t) + \chi_2(t), \tag{10}$$

in which

$$E_\pm(t) = \frac{A_+}{2\lambda}\mathrm{e}^{-\frac{1}{2}\lambda_\mp t} - \frac{A_-}{2\lambda}\mathrm{e}^{-\lambda_\pm t}, \tag{11}$$

$$F_\mp(t) = \frac{\nu_\mp}{\lambda}\mathrm{e}^{-\frac{1}{2}\lambda_- t} - \frac{\nu_\mp}{\lambda}\mathrm{e}^{-\lambda_+ t}, \tag{12}$$

$$G_+(t) = \int_0^t \big[E_+(t-t')f_1(t') + F_-(t-t')f_2^*(t')\big]\mathrm{d}t', \tag{13}$$

$$G_-(t) = \int_0^t \big[E_-(t-t')f_2^*(t') + F_+(t-t')(t')f_1\big]\mathrm{d}t', \tag{14}$$

$$\chi_1(t) = \frac{\varepsilon_1}{\lambda}[\zeta_+(1-\mathrm{e}^{-\frac{1}{2}\lambda_- t}) - \zeta_-(1-\mathrm{e}^{-\frac{1}{2}\lambda_+ t})], \tag{15}$$

$$\chi_2(t) = \frac{\varepsilon_1}{\lambda}[\Gamma_+(1-\mathrm{e}^{-\frac{1}{2}\lambda_+ t}) - \Gamma_-(1-\mathrm{e}^{-\frac{1}{2}\lambda_- t})], \tag{16}$$

with

$$A_\pm = A \pm \lambda, \tag{17}$$

$$\lambda = \sqrt{A^2 + 4\nu_+\nu_-}, \tag{18}$$

$$\lambda_\pm = \frac{1}{2}(2\kappa + A\eta \pm \lambda), \tag{19}$$

$$\zeta_\pm = \frac{A_\pm + 2\nu_-}{\lambda_\mp}, \tag{20}$$

$$\Gamma_\pm = \frac{A_\pm - 2\nu_+}{\lambda_\pm}. \tag{21}$$

On the other hand, the correlation properties of the noise forces described in eqs (5) and (6) satisfy

$$\langle f_1(t)\rangle = \langle f_2(t)\rangle = 0, \tag{22}$$

$$\langle f_1^*(t')f_1(t)\rangle = (A\rho_{aa}^{(0)} + \kappa N)\delta(t-t'), \tag{23}$$

$$\langle f_2^*(t')f_2(t)\rangle = \kappa N\delta(t-t'), \tag{24}$$

$$\langle f_1(t')f_2(t)\rangle = \frac{1}{2}(\nu_+ + 2\kappa M)\delta(t-t'), \tag{25}$$

$$\langle f_1(t')f_1(t)\rangle = \langle f_2(t')f_2(t)\rangle = 0, \tag{26}$$

$$\langle f_1^*(t')f_2(t)\rangle = \langle f_2^*(t')f_1(t)\rangle = 0. \tag{27}$$

3. Intracavity quadrature squeezing

In this section, we seek to study the quadrature squeezing of light produced by a non-degenerate three-level laser with a non-degenerate parametric amplifier and coupled to squeezed vacuum reservoir via a single-port mirror. In general, the squeezing properties of a two-mode cavity radiation can be described by two quadrature operators of the cavity mode operator.

$$\hat{c}_+ = (\hat{c}^\dagger + \hat{c}), \tag{28}$$

$$\hat{c}_- = i(\hat{c}^\dagger - \hat{c}), \tag{29}$$

in which $\hat{c} = (\hat{a}_1 + \hat{a}_2)/\sqrt{2}$ where $\hat{a}_1$ and $\hat{a}_2$ represent the separate modes cavity light emitted from the three-level atoms. Employing the commutation relation $[\hat{c}, \hat{c}^\dagger] = 1$, the Hermitian and non-commuting quadrature operators, $\hat{c}_+$ and $\hat{c}_-$, satisfy the relation

$$[\hat{c}_+, \hat{c}_-] = 2i. \tag{30}$$

On the basis of these definitions, a two-mode light is said to be in a squeezed state if either $\Delta c_+^2 < 1$ and $\Delta c_-^2 > 1$ or $\Delta c_+^2 > 1$ and $\Delta c_-^2 < 1$, such that $\Delta c_+ \Delta c_- \geq 1$. The variances of the quadrature operators can be expressed as

$$\Delta c_\pm^2 = \langle \hat{c}_\pm^2\rangle - \langle \hat{c}_\pm\rangle^2. \tag{31}$$

It is possible to express the variance of the quadrature operators (28) and (29) in terms of the c-number variables associated with the normal ordering taking the cavity modes to be initially in a two-mode squeezed vacuum state, as

$$\Delta c_\pm^2 = 1 + \langle \alpha_1^*(t)\alpha_1(t)\rangle + \langle \alpha_2^*(t)\alpha_2(t)\rangle \pm 2\langle \alpha_1(t)\alpha_2(t)\rangle. \tag{32}$$

Hence, on account of eqs (9)–(16) along with (22)–(27), the steady-state quadrature variances of the cavity radiation turn out to be

$$\Delta c_\pm^2 = 1 + \frac{2\kappa A(1-\eta)(2\kappa + 2A\eta + A)}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} + \frac{16\varepsilon_2^2 A\eta - 4\kappa A^2\eta^2 N}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} \pm \frac{2\kappa(4\varepsilon_2 + A\sqrt{1-\eta^2})(2\kappa + A(\eta+1) + \pm\varepsilon_2)}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} + \frac{4[(2\kappa + A\eta)(2\kappa + A\eta \pm 4\varepsilon_2)(N \pm M)]}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)} + \frac{A^2(1 \pm \sqrt{1-\eta^2})(N \mp M)}{4[\kappa(\kappa + A\eta) - 4\varepsilon_2^2](2\kappa + A\eta)}. \tag{33}$$

Equation (33) represents the steady-state quadrature variances of the cavity radiation. It has been verified that squeezing occurs in the minus quadrature variance. Moreover, we notice in eq. (33) that the parameter ε_1 does not appear in this equation, that is the cavity driving coherent light has no effect on the quadrature variances or it does not contribute to the quantum noise reduction which in turn affects the squeezing of the cavity light. This must be related to the fact that we have treated the pump amplitude of the coherent driving light classically to eliminate its depletion. Thus, we analyse the dependence of the squeezing of the squeeze parameter, linear gain coefficient and parametric amplifier.

Next we need to investigate the explicit dependence of squeezing of the two-mode cavity radiation on these parameters. It is worth stressing that the possibility for generating highly squeezed light by altering various parameters will make this system a reliable and an attractive source of squeezed light.

It can be seen from figure 2 that squeezing of the cavity light strongly depends on η and A. It is shown that the degree of squeezing increases with linear gain coefficient for smaller values of η, but decreases for larger values. The effect of linear gain coefficient in increasing the degree of squeezing is more significant when more atoms are initially prepared in the top level. Moreover, the minimum value of quadrature variance described by eq. (33) for $A = 1000$, $\kappa = 0.8$, $\varepsilon_2 = 0.45$ and $r = 0.5$, is found to be $\Delta c_-^2 = 0.1027$ and occurs when

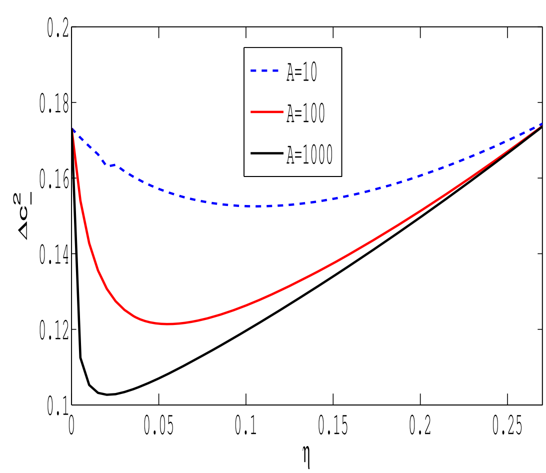

Figure 2. Plots of the quadrature variance (eq. (33)) vs. η for $\kappa = 0.8$, $r = 0.5$, $\varepsilon_2 = 0.45$ and for different values of linear gain coefficient.

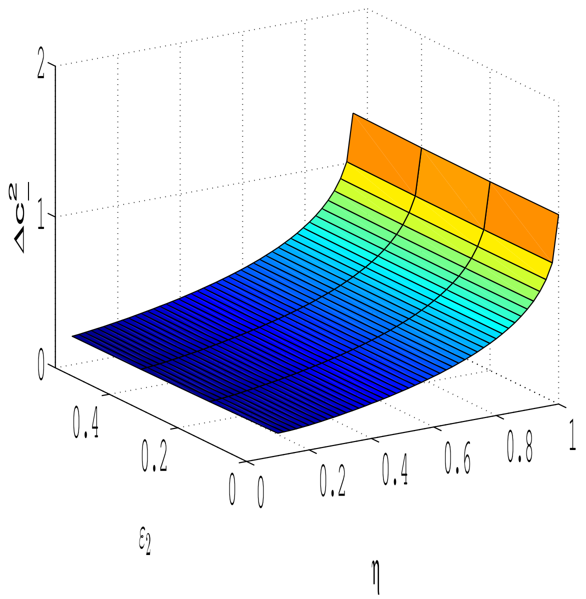

Figure 3. A plot of the quadrature variance (eq. (33)) of the two-mode cavity radiation vs. η and ε_2 for $A = 100$, $\kappa = 0.8$, $r = 0.5$.

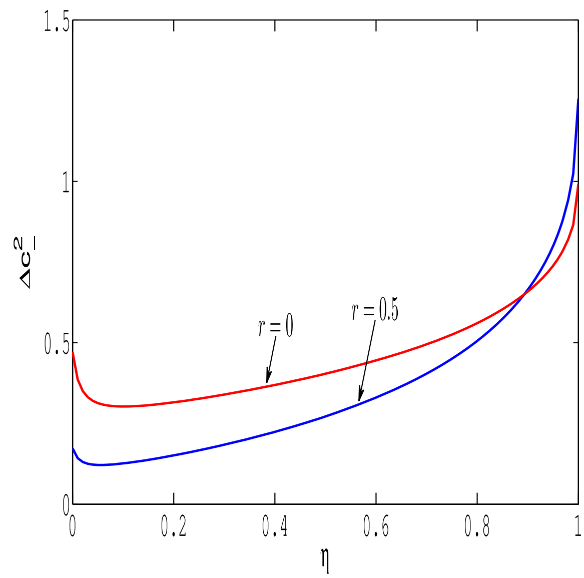

Figure 4. Plots of the quadrature variance (eq. (33)) vs. η for $A = 1000$, $\kappa = 0.8$, $\varepsilon_2 = 0.45$ and for different values of r.

$\eta = 0.0202$. This result implies that the maximum intracavity squeezing for the above values is 89.7% below the coherent state level.

On the other hand, figure 3 represents the variance of the minus quadrature of the cavity modes for a nondegenerate three-level laser with a parametric amplifier. It clearly shows that the squeezing decreases with the increase in atomic coherence and for small values of the amplitude of parametric amplifier. Squeezing in general increases with the amplitude of parametric amplifier for certain values of atomic coherence.

We clearly see that squeezing occurs for values of η between 0 and 1. This corresponds to the case when the atoms are initially prepared in such a way that there are more atoms in the bottom level than in the upper level. It is clearly indicated in figure that the degree of squeezing increases with the squeeze parameter r for smaller values of η but decreases for larger values η. Thus we can argue that the squeezed vacuum reservoir has significant effect on the squeezing of the cavity modes.

As shown in figure 4, the minimum value of quadrature variance for small values of η is found to be $\Delta c_-^2 = 0.1487$ and occurs at $\eta = 0.1$ for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.45$ and $r = 0.5$. This result indicates that the maximum intracavity squeezing is 85.1% below the coherent-state level and the degree of squeezing increases with the squeezing parameter for $0 \leq \eta \leq 0.9$ and decreases for other values of η. Moreover, we can clearly see from figure 4 that the squeezing property of the cavity radiation can be significantly enhanced by the squeeze parameter and the nonlinear crystal introduced into the laser cavity.

It is not difficult to see from figures 2–4 that the quadrature variance of the two-mode cavity radiation strongly depends on the linear gain coefficient A, the atomic coherence η, the squeeze parameter r and amplitude of the parametric amplifier ε_2.

4. Entanglement properties of the two-mode light

To investigate the properties of CV entanglement produced by this quantum optical system, we need an entanglement criterion for the CV system. Recently, several sufficient criteria have been proposed for the continuous variable systems. The preparation and manipulation of these entangled states that have non-classical and nonlocal properties lead to a better understanding of the basic quantum principles [25–27]. The summation of the quantum fluctuation proposed by Duan *et al* [24] is a sufficient condition to predict the presence of entanglement for Gaussian states. Here we employ this criterion to determine the amount entanglement generated in the quantum system. According to Duan *et al* [24], a quantum state of the system is entangled provided that the sum of the variances of the two EPR-type operators $\hat{u}$ and $\hat{v}$ [23] satisfies the condition

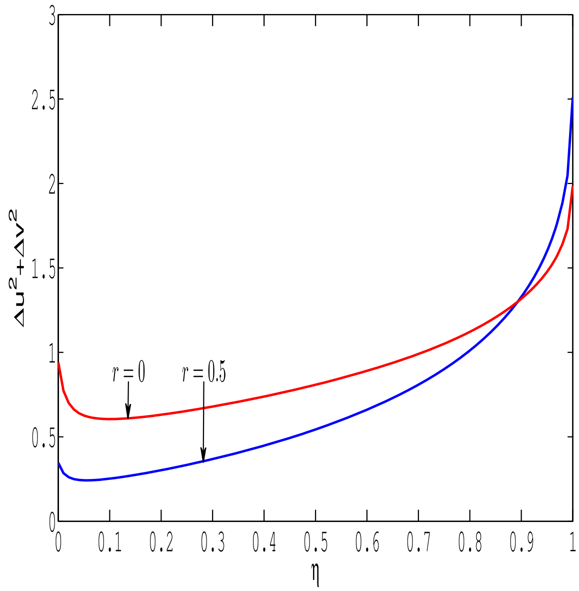

Figure 5. Plots of $\Delta u^2 + \Delta v^2$ of the two-mode cavity radiation vs. η for $\kappa = 0.8$, $A = 100$, $\varepsilon_2 = 0.45$ and for different values r.

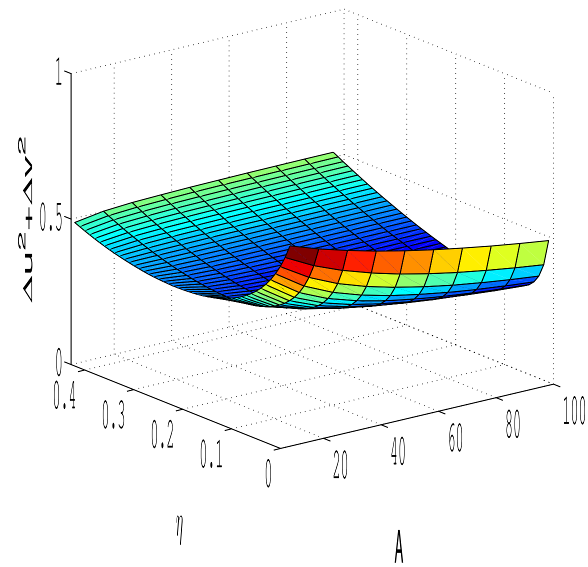

Figure 6. Plots of $\Delta u^2 + \Delta v^2$ of the two-mode cavity radiation vs. η and A for $\kappa = 0.8$, $r = 0.5$ and $\varepsilon_2 = 0.45$.

$$\Delta u^2 + \Delta v^2 < 2, \tag{34}$$

where

$$\hat{u} = \hat{x}_1 - \hat{x}_2, \tag{35}$$

$$\hat{v} = \hat{p}_1 + \hat{p}_2, \tag{36}$$

with

$$\hat{x}_1 = \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + \hat{a}_1), \quad \hat{x}_2 = \frac{1}{\sqrt{2}}(\hat{a}_2^\dagger + \hat{a}_2), \tag{37}$$

$$\hat{p}_1 = \frac{i}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_1), \quad \hat{p}_2 = \frac{i}{\sqrt{2}}(\hat{a}_2^\dagger - \hat{a}_2), \tag{38}$$

being the quadrature operators for modes $\hat{a}_1$ and $\hat{a}_2$. Thus, the sum of the variances of $\hat{u}$ and $\hat{v}$ is easily found to be

$$\Delta u^2 + \Delta v^2 = 2[1 + \langle \alpha_1^*(t)\alpha_1(t)\rangle + \langle \alpha_2^*(t)\alpha_2(t)\rangle - 2\langle \alpha_1(t)\alpha_2(t)\rangle]. \tag{39}$$

It then follows that

$$\Delta\hat{u}^2 + \Delta\hat{v}^2 = 2\Delta c_-^2, \tag{40}$$

where Δc_-^2 is given by eq. (33). We see from this result that the degree of entanglement is directly proportional to the degree of squeezing of the two-mode light. Similar result is also reported in refs [3] and [19].

In figure 5, we plot eq. (40) vs. η for different values of squeeze parameter, r. It is easy to see from this plot that the entanglement criterion given by (34) is satisfied for certain values of η. As the two-mode squeezed vacuum reservoir introduces additional squeezing to the system, it enhances the degree of entanglement between the two modes. We also note that the entanglement disappears when squeezing vanishes. This is due to the fact that the entanglement is directly related to squeezing as given by eq. (40).

It is clearly indicated in figure 6 that the cavity radiation is found to be entangled for all parameters under consideration except for the initial atomic coherence $\eta = 1$. It can be observed that the degree of entanglement increases for smaller values of the initial preparation of atoms, but decreases for larger values. The maximum degree of entanglement is found to be 75% and it occurs at $\eta = 0.18$ and $A = 100$ for $\kappa = 0.8$, $r = 0.5$ and $\varepsilon_2 = 0.45$. Moreover, figure 7 shows that other than large values of η, the degree of detectable entanglement increases with the parametric amplifier, ε_2. This indicates that the presence of parametric amplifier enhances the degree of entanglement.

Comparisons of figures 2, 4 and 5 with figures 6 and 7 clearly indicate that when squeezing is good, the entanglement is also good. Such relation between the squeezing and entanglement is attributed to eq. (40), and the cause of these quantum properties is the correlation initiated by the injected atomic coherence.

5. Photon statistics

Apart from the quantum features of the cavity radiation, classical properties of the photon number that can generally be designated as photon statistics have been the subject of investigation recently [3,4,10,19,22]. This is mainly because the intensity of the cavity radiation

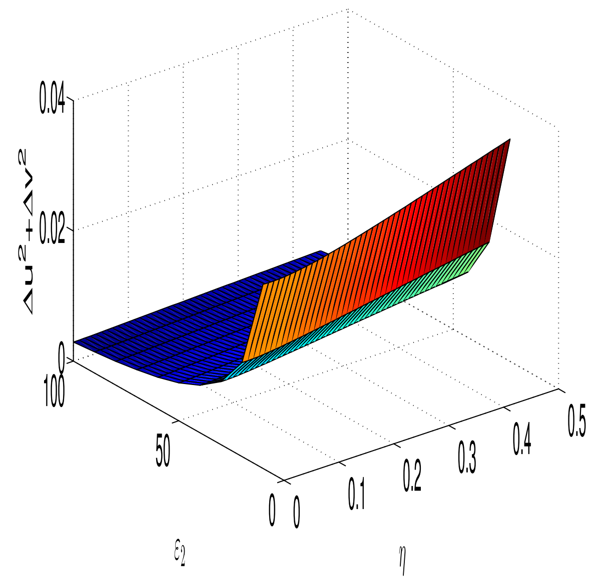

Figure 7. Plots of $\Delta u^2 + \Delta v^2$ of the cavity radiation vs. η and ε_2 for $\kappa = 0.8$, $A = 100$ and $r = 0.5$.

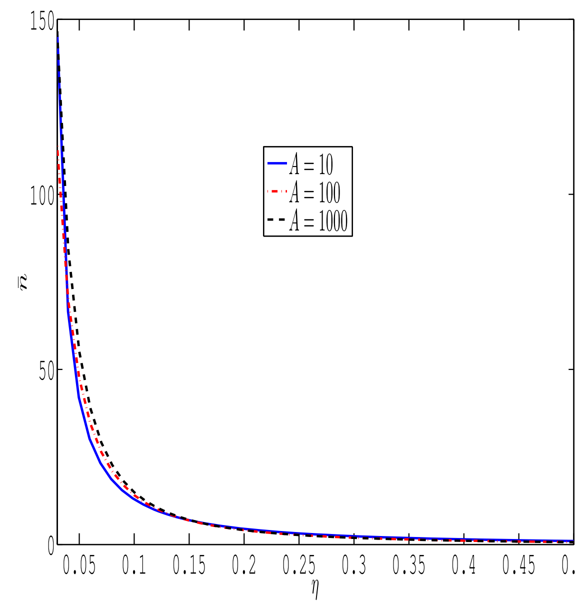

Figure 8. Plots of the mean photon number pairs (eq. (42)) vs. η for $\kappa = 0.8$, $\varepsilon_2 = 0.45$, $r = 0.5$ and for different values of A.

and the correlation of the photon number are very vital resources in describing the emitted radiation. In this respect, the mean number of photon pairs and photon number correlation, have been analysed for different schemes of the non-degenerate three-level cascade laser. It has been essentially found that the mean number of photon pairs is larger when the degree of entanglement is significant [4,19,22] which is a promising signature for utilising this system as a source of entangled light.

5.1 *Mean number of photon pairs*

In order to learn about the brightness of the generated light, it is necessary to study the mean number of photon pairs describing the two-mode cavity radiation that can be defined as [22]

$$\bar{n} = \langle \hat{c}^{\dagger}(t)\hat{c}(t)\rangle, \tag{41}$$

where $\hat{c}(t)$ is the annihilation operator for the two-mode cavity light. It is possible to rewrite it in terms of c-number variables associated with the normal ordering as

$$\bar{n} = \frac{1}{2}[\langle \alpha_1^*(t)\alpha_1(t)\rangle + \langle \alpha_2^*(t)\alpha_2(t)\rangle]. \tag{42}$$

Since $\langle \alpha_1^*(t)\alpha_1(t)\rangle$ and $\langle \alpha_2^*(t)\alpha_2(t)\rangle$ represent the mean photon numbers in mode a_1 and mode a_2, respectively, $\bar{n}$ can be interpreted as the mean number of photon pairs. It is easy to verify that eq. (42) represents the mean number of photon pairs of the system. In what follows, we analyse the dependence of the mean number of photon pair on the squeeze parameter, linear gain coefficient and parametric amplifier.

It is not difficult to see from figure 8 that the mean photon number pairs of the cavity radiation increases with the linear gain coefficient for $\eta \leq 0.2$ but it slowly decreases otherwise. It can also be verified that the cavity radiation becomes more intense where squeezing and entanglement are stronger.

In figure 9, we plot the mean photon number of the two-mode light vs. η in the absence and presence of the the parametric amplifier. It is easy to see from this figure that the presence of parametric amplifier increases the mean photon number in region where there is strong squeezing and entanglement. Hence, this system generates a bright and highly squeezed as well as entangled light.

In figures 8 and 9, the mean photon number increases when η decreases. Thus, the more the atoms are initially prepared in the upper energy level, the stronger the intensity of the radiation would be. Moreover, comparing the results obtained here with the discussion on entanglement and two-mode squeezing, we find that the mean photon number is significantly higher for the case in which there is a strong non-classical correlation. Hence, based on the possibility of increasing the intensity by a proper selection of the involved parameters, this may lead to the conclusion that the system under consideration can be a source of bright, entangled and squeezed light.

Moreover, we learned from figure 10 that a quite strong cavity radiation could be generated by further

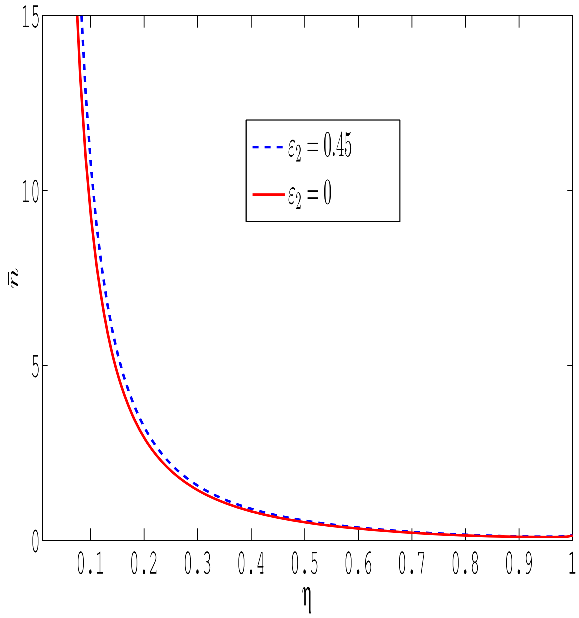

Figure 9. Plots of the mean photon number pairs (eq. (42)) vs. η for $A = 100$, $\kappa = 0.8$, $r = 0.5$ and for different values of ε_2.

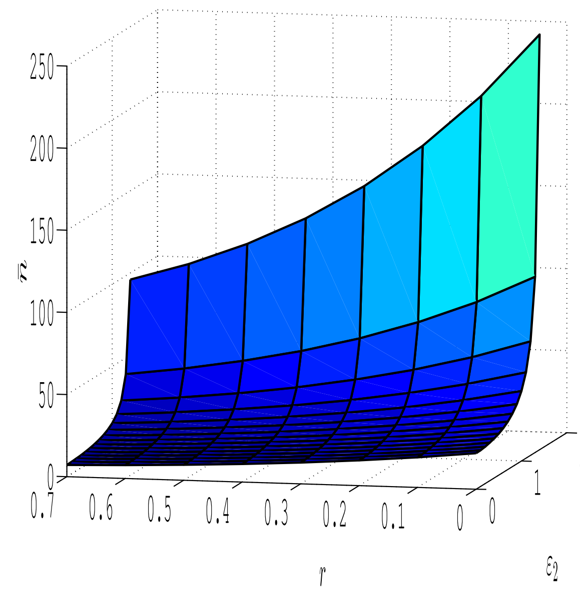

Figure 10. Plots of the mean photon number pairs (eq. (42)) vs. r and ε_2 for $A = 100$, $\kappa = 0.8$ and $\eta = 0.1$.

manipulating the squeeze parameter and amplitude of the pump mode that interacts with the parametric amplifier. From this figure we observe that the mean number of photon pairs increases with the amplitude of the pump mode of the parametric amplifier, which makes sense. It is also clearly demonstrated that the mean number of photon pairs decreases with the squeeze parameter, r.

5.2 *Photon number correlation*

The photon number correlation for two modes of a radiation can be defined as

$$g_{(\hat{n}_a,\hat{n}_b)} = \frac{\langle \hat{n}_a \hat{n}_b \rangle}{\langle \hat{n}_a \rangle \langle \hat{n}_b \rangle}, \tag{43}$$

in which

$$\langle \hat{n}_a \hat{n}_b \rangle = \langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \hat{a}_2 \rangle, \tag{44}$$

$$\langle \hat{n}_a \rangle = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle, \tag{45}$$

$$\langle \hat{n}_b \rangle = \langle \hat{a}_2^\dagger \hat{a}_2 \rangle, \tag{46}$$

and the operators are in the normal order. Therefore, eq. (43) can be expressed in terms of the c number variables associated with the normal ordering as

$$g_{(\hat{n}_a,\hat{n}_b)} = 1 + \frac{\langle \alpha_1(t)\alpha_2(t)\rangle^2}{\langle \alpha_1^*(t)\alpha_1(t)\rangle \langle \alpha_2^*(t)\alpha_2(t)\rangle}. \tag{47}$$

It then follows that

$$g_{(\hat{n}_a,\hat{n}_b)} = 1 + \frac{\Gamma_1^2}{\Gamma_2\Gamma_3}, \tag{48}$$

in which

$$\Gamma_1 = 2\kappa A(1-\eta)(2\kappa + 2A\eta + A) + 16\varepsilon_2^2 A\eta \\ -4\kappa A^2\eta^2 N, \tag{49}$$

$$\Gamma_2 = 2\kappa \left(4\varepsilon_2 + A\sqrt{1-\eta^2}\right)(2\kappa + A\eta + A - 4\varepsilon_2), \tag{50}$$

$$\Gamma_3 = 4\kappa(2\kappa + A\eta)(2\kappa + A\eta + 4\varepsilon_2)(N \pm M)) \\ +A^2\left(1 - \sqrt{1-\eta^2}\right)(N \mp M). \tag{51}$$

Equation (48) describes the steady-state photon number correlation, $g_{(n_a,n_b)}$, for the three-level laser with parametric amplifier and coupled to squeezed vacuum reservoir.

We have found that in figure 11 the dependence of the photon number correlation is evident. However, an increase of ε_2 leads to the decrease of the photon number correlation. Also, from figure 12 we see that the photon number correlation falls below 2 for $\eta = 1$. This is another indication of the fact that squeezing and entanglement vanish in the absence of atomic coherence. Moreover, the photon number correlation demonstrates the intensity of intracavity radiation for $\eta \leq 0.9$.

Furthermore, we have seen in figure 12 that the photon number correlation increases with the squeeze parameter for $\eta \leq 0.9$, but decreases for $\eta > 0.9$. One can easily compare the effect of vacuum environment with

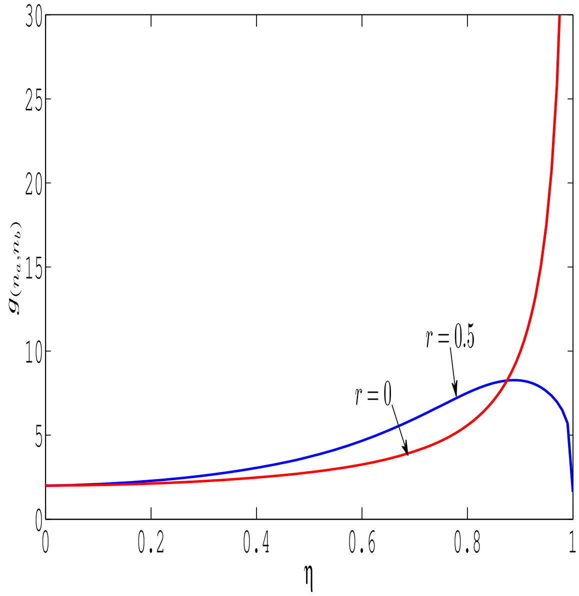

Figure 11. Plots of the photon number correlation (eq. (48)) vs. η for $A = 100$, $\kappa = 0.8$, $\varepsilon_2 = 0.45$ and for different values of r.

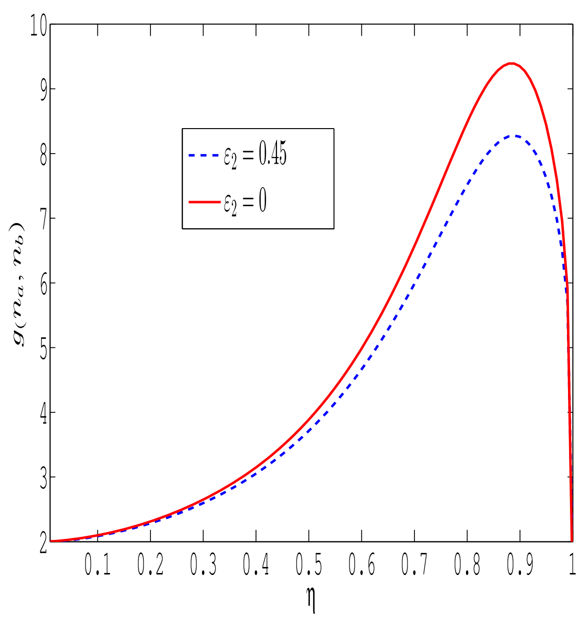

Figure 12. Plots of the photon number correlation (eq. (48)) vs. η for $r = 0.5$, $\kappa = 0.8$, $A = 100$ and for different values of ε_2.

that of squeezed vacuum reservoir from this plot. When contribution of the squeezed vacuum reservoir is turned off ($r = 0$), the mean photon number correlation grows rapidly for $\eta > 0.8$. However, for $r = 0.5$ we have seen that the photon number correlation falls below 2 at $\eta = 1$ indicating that the cavity squeezing and entanglement disappear there.

We see from figure 13 that the correlation of the photon number decreases with increasing injected atomic

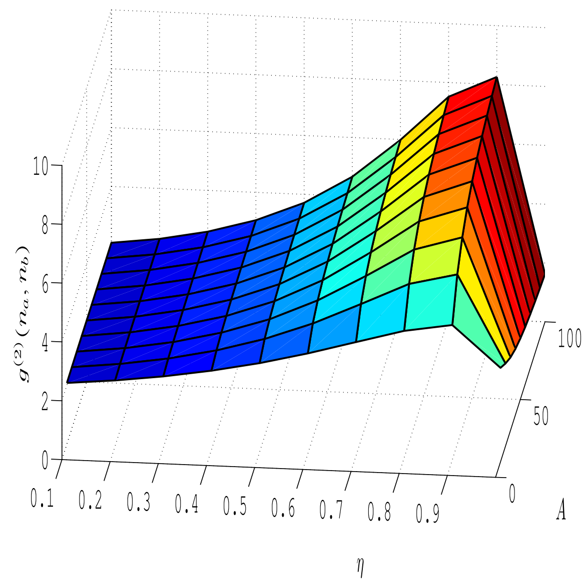

Figure 13. Plots of the photon number correlation (eq. (48)) vs. η and A for $r = 0.5$, $\kappa = 0.8$ and $\varepsilon_2 = 0.45$.

coherence. We also found that for η very close to 1, the correlation of the photon number would be significantly large, as the mean photon number of the light in mode b is very close to zero when initially almost all atoms are populated in the lower level. Furthermore, it clearly shows that the correlation of photon number decreases with linear gain coefficient. However, we found that the degree of squeezing increases with the linear gain coefficient, and so from these results the correlation between the photon numbers get to be minimum in the region where squeezing is maximum.

6. Conclusion

In this paper, we have studied the steady-state two mode squeezing and entanglement of light produced by non-degenerate three-level laser whose cavity contains two degenerate parametric amplifiers and coupled to a squeezed vacuum reservoir. We first obtained the master equation for the system under consideration. Using the master equation, we obtained c-number Langevin equation associated with the normal ordering and the correlation properties of the noise forces. Applying the solutions of the resulting c-number Langevin equations, we determined the quadrature squeezing. In addition, applying the criterion developed by Duan *et al*, the quantum entanglement of the cavity is studied.

We have found that the light produced by the system under consideration exhibits squeezing and entanglement. It is found that the degree of squeezing for the system under consideration increases with the amplitude of the parametric amplifiers. This implies that the presence of parametric amplifiers enhances the squeezing of light generated by the system under consideration. It is also observed that the degree of squeezing for the system under consideration increases with the squeezing parameter of the squeezed vacuum reservoir. It so turns out that the squeezed vacuum reservoir increases the degree of squeezing. As the two-mode squeezed vacuum reservoir introduces additional squeezing to the system, it enhances the degree of entanglement between the two modes. We also note that the entanglement disappears when the squeezing vanishes. This is due to the fact that the entanglement is directly related to squeezing as given by eq. (40).

Furthermore, we found that the mean photon number is significantly higher when there is a strong non-classical correlation. Hence, proper selection of the involved parameters leads to the conclusion that the system under consideration can be a source of bright, entangled and squeezed light. In addition, our calculation of the photon number correlation shows that when correlation between the states of the emitted light is stronger, the correlation between the photon number tends to be smaller.

In general, we conclude after detailed calculations and analysis that the proposed quantum system can be utilised as a source of squeezing, entanglement and other non-classical and statistical features which have potential applications in different fields including modern physics, especially in quantum technologies and applications. The idea presented here is also good and may be useful for most users in various research fields, especially in quantum optics, cavity-QED and quantum information. Moreover, the amount of entanglement and squeezing exhibited is quite robust against decoherence and hence can be used in quantum processing tasks. The introduced parameters have enhanced squeezing, entanglement and intensity of the cavity light simultaneously.

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