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Generalised conservation laws, reductions and exact solutions of the $K(m, n)$ equations via double reduction theory

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Abstract. In this article, we present the general form of conservation laws for the nonlinear Rosenau–Hyman compacton $K(m, n)$ equations using multiplier's approach. General formulas for some new conservation laws are established for the $K(m, n)$ equations. We describe three different cases where Lie symmetries are associated with these generalised conservation laws. The double reduction theory is utilised to construct some new reductions and exact solutions for different values of m and n.

Keywords. Symmetries; generalised conservation laws; $K(m, n)$ equations; double reduction.

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1. Introduction

Symmetries and conservation laws play vital roles in reducing the number of independent variables and order as well as in constructing exact solutions for certain system of differential equations. There exist different approaches that can be used in the construction of conservation laws. Laplace [1] presented the direct method to compute conservation laws. The conservation laws for the Euler–Lagrange equations can be constructed by the well-known Noethers theorem [2]. Kara *et al* [3] discussed the partial Noether approach for the construction of first integrals.

A conservation law of a partial differential equation can be written in the characteristic form. The characteristics, called multipliers, can be computed by taking the variational derivative of the characteristic form. This approach is called multipliers or variational derivative method (see [4,5]). For a discussion on the comparison of different approaches to compute first integrals for differential equations, interested reader is referred to [6].

Kara and Mahomed [7] derived a fundamental relationship between Lie–Bäcklund symmetry operators and conservation laws without taking into account Lagrangian formulation of differential equations. This enables us to associate symmetries to a given conservation law of a differential equation. Using the symmetry conservation law association, Sjöberg [8,9] proposed the double reduction theory which states that a qth-order partial differential equations (PDE) with two independent variables can be reduced to an ODE of order $q-1$. A generalisation of the double reduction theory to PDEs of higher dimensions is given by Bokhari *et al* [10].

Rosenau and Hyman [11] introduced a class of solitary waves with compact support called compactons that are solutions of a nonlinear dispersive PDE. In order to study the role of nonlinear dispersion in the formation of patterns in liquid drops, they introduced the real nonlinear dispersive $K(m, n)$ equation with compacton solutions. For $m = 2$ and $n = 1$, the $K(m, n)$ equation reduces to the well-known KdV equation.

The most dominant aspect of the $K(m, n)$ equations is the existence of compactly supported solutions. When $m = n = 2,\ 3$, the dispersion is nonlinear and $K(m, n)$ equations were studied to derive compacton solutions (see [11–13]). Ismail and Taha [14] utilised finite difference and finite element methods to describe the compactons' interaction of $K(2, 2)$ and $K(3, 3)$ equations. He and Wu [15] used variational iteration method to construct compacton-like solutions for the $K(m, n)$ equations. Wazwaz [16] employed Adomian decomposition method to derive new compacton solutions for the $K(n, n)$ equations. Bruzón and Gandarias [17] derived solitons' and compactons' travelling wave solutions of

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a $K(m,n)$ equation with generalised evolution term. Several variants of the $K(m,n)$ equations which are used to explore compact structures in higher dimensions have been considered in [18–20]. The Lie point symmetries of the $K(m,n)$ equation, being a special case of $C_1(m, a+b)$ equation, were reported in [21].

In this article, we present conservation laws in the most general form of compacton $K(m,n)$ equations via multiplier's approach. Conservation laws based on different values of m and n are discussed in three different cases. The double reduction theory is utilised for the reduction of order of the nonlinear dispersive $K(m,n)$ equation. Reductions in particular for $m = n+1$, $m = n+2$, and $m = 3n+2$ presented in this paper are not known to the best of our knowledge. For an existing work on conservation laws of $K(n,n)$ equation, interested reader is referred to [22,23].

The article is organised as follows. Basic definitions, notations and theorems are described in §2. In §3, we present different cases of generalised conservation laws for the compacton $K(m,n)$ equation. In §4, corresponding to these conservation laws found in §3, we present exact solutions and reductions for different cases of compacton $K(m,n)$ equation. Finally, the concluding remarks are presented in §5.

2. Preliminaries

In this section, we present some fundamental operators, definitions, notations and theorems which are well known in literature (see [6–10]).

Consider a qth-order system of p partial differential equations of n independent variables $\mathbf{x} = (x^1, x^2, \ldots, x^n)$ and m dependent variables $\mathbf{u} = (u^1, u^2, \ldots, u^m)$

$$E_\sigma\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(q)}\right), \quad \sigma = 1, 2, \ldots, p, \tag{2.1}$$

where $u_{(q)}$ denote all the qth-order partial derivatives.

DEFINITION 1

The total derivative operator with respect to x^i is defined as

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, 2, \ldots, n.$$

DEFINITION 2

The Euler Lagrange operator is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 \cdots i_s}}, \quad \alpha = 1, \ldots, m. \tag{2.2}$$

DEFINITION 3

The Lie–Bäcklund or generalised operator is defined as

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s\geq 1} \zeta^\alpha_{i_1,\ldots,i_s} \frac{\partial}{\partial u^\alpha_{i_1,\ldots,i_s}}, \quad \xi^i, \eta^\alpha \in \mathcal{A}, \tag{2.3}$$

where $\mathcal{A}$ is the universal space of differential functions and the additional coefficients are determined uniquely by the following formulas:

$$\zeta_i^\alpha = D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j),$$
$$\zeta^\alpha_{i_1\cdots i_s} = D_{i_s}(\zeta^\alpha_{i_1\cdots i_{s-1}}) - u^\alpha_{j i_1 \cdots i_{s-1}} D_{i_s}(\xi^j), \quad s > 1.$$

DEFINITION 4

A conserved vector of (2.1) is n-tuple $T = (T^1, T^2, \ldots, T^n)$ satisfying the relation

$$D_i T^i|_{(2.1)} = 0,$$

where $T^i = T^i\left(x, u, u_{(1)}, u_{(2)}, \ldots, u_{(q)}\right) \in \mathcal{A}$, $i = 1, 2, \ldots, n$.

A conservation law can be expressed in characteristic form as (see [4,5])

$$D_i T^i = \Lambda^\sigma E_\sigma, \quad \sigma = 1, 2, \ldots, p, \tag{2.4}$$

where Λ^σ are the characteristics or multipliers for the PDE system (2.1). Corresponding to each multiplier there exists a local conservation law if and only if

$$\frac{\delta}{\delta u^\alpha}\left[\sum_{\sigma=1}^{p} \Lambda^\sigma E_\sigma\right] = 0, \qquad \alpha = 1, 2, \ldots, m. \tag{2.5}$$

Equation (2.5) will be used to find determining equations for the multipliers.

DEFINITION 5

A Lie–Bäcklund operator X given in (2.3) is said to be associated with the conserved vector T of (2.1) if the following relation is satisfied (see [7]):

$$X(T^i) + D_j(\xi^j) T^i - T^j D_j(\xi^i) = 0, \quad i = 1, 2, \ldots, n. \tag{2.6}$$

Using symmetries and the associated conservation laws satisfying (2.6), Sjöberg [8,9] developed the double reduction theory which states that a PDE with two independent variables can be reduced to an ODE of order one less than the order of the PDE. Suppose we have a scalar PDE $G = 0$ which admits a symmetry X associated with a conserved vector (T^t, T^x). In terms of the similarity variables r, s obtained by mapping X to $Y = \partial/\partial s$, the conservation laws can be written as [8]

$$D_r T^r + D_s T^s = 0,$$

where T^r and T^s expressed in terms of old variables are

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)} \tag{2.7}$$

and

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \tag{2.8}$$

Theorem 1 (see [8]). *A PDE $G = 0$ of order q with two independent variables, which admits a symmetry X that is associated with a conserved vector T, can be reduced to an ODE of order $q-1$, namely $T^r = k$, where T^r is defined in (2.7).*

3. Lie symmetries and generalised conservation laws of compacton $K(m, n)$ equation

The general compacton $K(m, n)$ equation is [11]

$$u_t + (u^m)_x + (u^n)_{xxx} = 0, \quad m, n > 1. \tag{3.1}$$

The expanded form of (3.1) is

$$u_t + mu^{m-1}u_x + n(n-1)(n-2)u^{n-3}(u_x)^3 \\ +3\,n(n-1)u^{n-2}\,u_x\,u_{xx} + n\,u^{n-1}\,u_{xxx} = 0.$$

The Lie point symmetries of eq. (3.1) can be determined by solving [5]

$$X^{[3]}[u_t + (u^m)_x + (u^n)_{xxx}]|_{(3.1)} = 0, \tag{3.2}$$

where $X^{[3]}$ is the third prolongation and can be computed from (2.3). Expansion of (3.2) and then separation with respect to different combinations of derivatives of u result in the following overdetermined system of linear PDEs:

$$\xi^1_u = 0, \quad \xi^1_x = 0, \quad \xi^1_{tt} = 0, \quad \xi^2_t = 0, \\ \xi^2_u = 0, \quad \xi^2_x = \frac{(m-n)\,\xi^1_t}{3\,m-n-2}, \quad \eta = \frac{2\,u\,(m-n)\,\xi^1_t}{3\,m-n-2}. \tag{3.3}$$

Solving (3.3), we obtain the following three Lie symmetries of (3.1):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \\ X_3 = t\frac{\partial}{\partial t} + \left(\frac{(m-n)\,x}{3\,m-n-2}\right)\frac{\partial}{\partial x} \\ -\left(\frac{2u}{3\,m-n-2}\right)\frac{\partial}{\partial u}. \tag{3.4}$$

Note that the Lie symmetries (3.4) of $K(m, n)$ equation can be obtained from the classification of C_1 $(m, a+b)$ equation (see [21]).

The determining equation for the multipliers of $K(m, n)$ equation (3.1) is

$$\frac{\delta}{\delta u}[\Lambda(u_t + mu^{m-1}u_x + n\,(n-1)\,(n-2)\,u^{n-3}\,(u_x)^3 \\ + 3n\,(n-1)\,u^{n-2}\,u_x\,u_{xx} + n\,u^{n-1}\,u_{xxx})] = 0, \tag{3.5}$$

where $\Lambda = \Lambda\,(t, x, u)$ and $\delta/\delta u$ is the Euler operator given in (2.2). The following cases need to be considered in order to discuss eq. (3.5):

3.1 *Case* I, *when $m = n$*

Expanding (3.5) for $m = n$ and separating with respect to various combinations of derivatives of u result in the following overdetermined system:

$$\Lambda_{xxx} = -\Lambda_x, \quad \Lambda_{ux} = 0, \\ \Lambda_{uu} = \frac{(n-1)\Lambda_u}{u}, \quad \Lambda_t = 0. \tag{3.6}$$

Solving (3.6), we have

$$\Lambda(t, x, u) = c_1 + c_2\sin(x) + c_3\cos(x) + c_4\,u^n. \tag{3.7}$$

Equation (2.4) with multipliers in (3.7) takes the following form:

$$[\Lambda(u_t + nu^{n-1}u_x + n(n-1)(n-2)u^{n-3}(u_x)^3 \\ + 3\,n(n-1)u^{n-2}\,u_x\,u_{xx} + n\,u^{n-1}\,u_{xxx})] \\ = D_tT^t + D_xT^s,$$

where

$$T_t = c_1u + c_2\,\sin(x)u + c_3\,\cos(x)\,u + \left(\frac{c_4}{n+1}\right)u^{n+1}$$

and

$$T_x = c_1(n(n-1)u^{n-2}(u_x)^2 + nu^{n-1}u_{xx} + u^n) \\ + c_2(n(n-1)u^{n-2}(u_x)^2\sin(x) \\ + nu^{n-1}u_{xx}\sin(x) - nu^{n-1}u_x\cos(x)) \\ + c_3(n(n-1)u^{n-2}(u_x)^2\cos(x) \\ + nu^{n-1}u_{xx}\cos(x) + nu^{n-1}u_x\sin(x)) \\ + \frac{1}{2}c_4(u^{2n} + n(n-2)u^{2n-2}(u_x)^2 \\ + 2nu^{2n-1}u_{xx}).$$

For different choices of constants, we obtain four different multipliers and the corresponding conservation laws are listed in table 1.

The conservation laws presented in table 1 derived by multiplier's approach for the general $K(m, n)$ equation

Table 1. Multipliers and conserved vectors for $K(m, n)$ equations when $m = n$.

Multipliers	Conserved vectors
$\Lambda_1 = 1$	$T_1^t = u$
	$T_1^x = n(n-1)u^{n-2}(u_x)^2 + nu^{n-1}u_{xx} + u^n$
$\Lambda_2 = \sin(x)$	$T_2^t = \sin(x)u$
	$T_2^x = n(n-1)u^{n-2}(u_x)^2\sin(x) + nu^{n-1}u_{xx}\sin(x) - nu^{n-1}u_x\cos(x)$
$\Lambda_3 = \cos(x)$	$T_3^t = \cos(x)\,u$
	$T_3^x = n(n-1)u^{n-2}\,(u_x)^2\cos(x) + nu^{n-1}u_{xx}\cos(x) + nu^{n-1}u_x\sin(x)$
$\Lambda_4 = u^n$	$T_4^t = \dfrac{u^{n+1}}{n+1}$
	$T_4^x = \frac{1}{2}[u^{2n} + n(n-2)u^{2n-2}(u_x)^2 + 2nu^{2n-1}u_{xx}]$

agree with those described in [23] for $m = n$. In the following two cases, we derive conservation laws of the general $K(m, n)$ equation for different values of m and n.

3.2 *Case* II, *when* $m \neq n$ *and* $m \neq n+1$

In this case, expansion and separation of (3.5) with respect to different combinations of derivatives of u results in

$$\Lambda_t = 0, \quad \Lambda_x = 0, \quad \Lambda_{uu} = \frac{\Lambda_u}{u}. \tag{3.8}$$

Solution of (3.8) yields

$$\Lambda(t, x, u) = c_1 + c_2\,u^n. \tag{3.9}$$

Multipliers in (3.9) also satisfy (2.4) for eq. (3.1), i.e.,

$$\begin{aligned}&[\Lambda(u_t + mu^{m-1}u_x + n(n-1)(n-2)u^{n-3}(u_x)^3\\ &\quad + 3n(n-1)u^{n-2}u_xu_{xx} + nu^{n-1}u_{xxx})]\\ &= D_tT^t + D_xT^s,\end{aligned}$$

where

$$T_t = c_1\,u + \frac{c_2}{n+1}u^{n+1}$$

and

$$\begin{aligned}T_x =& c_1[n(n-1)u^{n-2}(u_x)^2 + nu^{n-1}(u_{xx}) + u^m]\\ &+ c_2\Big[\frac{m}{m+n}\,u^{m+n} + \frac{n(n-2)}{2}\,u^{2\,n-2}\,(u_x)^2\\ &+ nu^{2\,n-1}u_{xx}\Big].\end{aligned}$$

We obtain two different multipliers and the corresponding components of conserved vectors as in table 2 for different choices of constants.

Substitution of any value of m or n in table 2 results in the corresponding conserved vectors for the $K(m, n)$ equations. Conservation laws for $K(0, 1)$ and $K(1, 1)$ equations computed in [24] agree with $\left(T_1^t, T_1^x\right)$.

3.3 *Case* III, *when* $m = n + 1$

In this case, the determining equation (3.5) yields the following overdetermined system:

$$\begin{aligned}&\Lambda_{xx} = 0, \quad \Lambda_{ux} = 0, \quad \Lambda_{uu} = \frac{(n-1)\,\Lambda_u}{u},\\ &\Lambda_t = -(n+1)\,\Lambda_x\,u^n.\end{aligned} \tag{3.10}$$

Solving (3.10), we find

$$\Lambda(t, x, u) = c_1 + c_2\,u^n + c_3(x - (n+1)\,t\,u^n). \tag{3.11}$$

From eqs (2.4) and (3.11), we have

$$\begin{aligned}&[\Lambda(u_t + (n+1)u^nu_x + n(n-1)(n-2)u^{n-3}(u_x)^3\\ &\quad + 3n(n-1)u^{n-2}\,u_xu_{xx} + nu^{n-1}u_{xxx})]\\ &= D_tT^t + D_xT^s,\end{aligned}$$

where

$$T_t = c_1u + c_2\frac{u^{n+1}}{n+1} + c_3(xu - tu^{(n+1)})$$

and

$$\begin{aligned}T_x =& c_1(n(n-1)u^{n-2}(u_x)^2\\ &+ nu^{n-1}u_{xx} + u^{n+1})\\ &+ c_2\left(\frac{n+1}{2n+1}u^{2n+1} + \frac{n(n-2)}{2}u^{2n-2}(u_x)^2\right.\\ &\left. + nu^{2\,n-1}u_{xx}\right)\\ &+ c_3\left(-\frac{t\,(n+1)^2u^{2n+1}}{2n+1}\right.\\ &- \frac{(n-2)n(n+1)}{2}t\,u^{2\,n-2}(u_x)^2 + u^{n+1}x\\ &- (n+1)ntu^{2\,n-1}u_{xx} + n(n-1)xu^{n-2}(u_x)^2\\ &\left. + nxu^{n-1}u_{xx} - nu^{n-1}u_x\right).\end{aligned}$$

Table 2. Multipliers and conserved vectors for (3.1) when $m \neq n$ and $m \neq n+1$.

Multipliers	Conserved vectors
$\Lambda_1 = 1$	$T_1^t = u$ $T_1^x = n(n-1)u^{n-2}(u_x)^2 + nu^{n-1}u_{xx} + u^m$
$\Lambda_2 = u^n$	$T_2^t = \dfrac{u^{n+1}}{n+1}$ $T_2^x = \dfrac{m}{m+n}u^{m+n} + \dfrac{n\,(n-2)}{2}u^{2n-2}(u_x)^2 + nu^{2n-1}u_{xx}$

Table 3. Multipliers and conserved vectors for $K(m,n)$ equations when $m = n+1$.

Multipliers	Conserved vectors
$\Lambda_1 = 1$	$T_1^t = u$ $T_1^x = n(n-1)u^{n-2}(u_x)^2 + nu^{n-1}u_{xx} + u^{n+1}$
$\Lambda_2 = u^n$	$T_2^t = \dfrac{u^{n+1}}{n+1}$ $T_2^x = \dfrac{n+1}{2n+1}u^{2n+1} + \dfrac{n\,(n-2)}{2}u^{2n-2}(u_x)^2 + nu^{2n-1}u_{xx}$
$\Lambda_3 = x - t\,(n+1)\,u^n$	$T_3^t = xu - t\,u^{n+1}$ $T_3^x = n\,xu^{n-1}u_{xx} - nu^{n-1}u_x + u^{n+1}x$ $-\dfrac{t\,(n+1)^2\,u^{2n+1}}{2n+1} - \dfrac{(n-2)\,n\,(n+1)}{2}\,t\,u^{2n-2}(u_x)^2$ $-(n+1)n\,tu^{2n-1}u_{xx} + n(n-1)xu^{n-2}(u_x)^2$

In this case, we obtain three conserved vectors which are presented in table 3.

Thus, we obtained three generalised conservation laws for the $K(m,n)$ equations when $m = n+1$, where the conservation law $T_3 = (T_3^t, T_3^x)$ is new and different from the four conservation laws obtained in the case $m = n$. The zeroth-order multipliers were used to construct the conservation laws for $K(m,n)$ equation. Note that the determining equation corresponding to the first-and higher-order multipliers becomes too difficult to separate. However, one can use the computer packages [25,26] to find the higher-order multipliers. We noticed that higher-order multipliers yield the same conservation laws for $K(m,n)$ equation.

4. Reductions and exact solutions

In this section, we present reductions and exact solutions corresponding to different cases of conserved vectors given in §3 via double reduction theory.

4.1 *Case* I, *when* $m = n$

Using symmetry conservation law relation (2.6), we observe that $T_1 = (T_1^t, T_1^x)$ and $T_4 = (T_4^t, T_4^x)$ are associated with X_1 and X_2, but not related with X_3. Similarly, $T_2 = (T_2^t, T_2^x)$ and $T_3 = (T_3^t, T_3^x)$ are associated with X_1. There is no conservation law associated with X_3.

Thus, we can have reduction via T_1 and T_4 by using combination of symmetries X_1 and X_2. Let $X = X_1 + \alpha X_2$, which gives the following canonical coordinates:

$$s = t, \quad r = x - \alpha t, \quad v\,(r) = u. \tag{4.1}$$

Thus, the conservation law $T = (T^t, T^x)$ can be rewritten as $D_r T^r + D_s T^s = 0$ (see [8]).

4.1.1 *Reduction by* T_1. Using eqs (2.7), (2.8) and (4.1), the conserved vector $T_1 = (T_1^t, T_1^x)$ given in table 1 is reduced to

$$T_1^r = \alpha v - n(n-1)v^{n-2}(v_r)^2 - n\,v^{n-1}\,v_{rr} - v^n, \tag{4.2}$$

$$T_1^s = -v. \tag{4.3}$$

As (4.3) is independent of s, the reduced conserved vector becomes $D_r T_1^r = 0$, resulting in

$$\alpha v - n(n-1)v^{n-2}(v_r)^2 - n\,v^{n-1}\,v_{rr} - v^n = k_1, \tag{4.4}$$

where k_1 is an arbitrary constant.

In order to solve eq. (4.4), we employ the sine–cosine method. The method suggests that the solutions can be expressed in terms of either sine or cosine functions. For detailed analysis of sine–cosine method, the interested reader is referred to [27–29].

Let

$$v(r) = A\cos^{\beta}(\lambda r) \tag{4.5}$$

be the solution of the ODE (4.4). Substituting (4.5) in (4.4) and simplifying for $k_1 = 0$, we have

$$\begin{aligned}&\alpha A\cos^{\beta}(\lambda r) - n^2A^n\beta^2\lambda^2\cos^{\beta n-2}(\lambda r)\\ &\quad + n^2A^n\beta^2\lambda^2\cos^{\beta n}(\lambda r)\\ &\quad + nA^n\beta\lambda^2\cos^{\beta n-2}(\lambda r) - A^n\cos^{\beta n}(\lambda r) = 0.\end{aligned} \tag{4.6}$$

Equating the exponents and the corresponding coefficients of the cosine functions, we obtain the following system of equations:

$$\begin{aligned}&\beta = \beta n - 2,\\ &\alpha A - n^2A^n\beta^2\lambda^2 + nA^n\beta\lambda^2 = 0,\\ &n^2A^n\beta^2\lambda^2 - A^n = 0.\end{aligned} \tag{4.7}$$

Solving (4.7) and substituting values of A, β and λ in eq. (4.4), we obtain the following exact solution:

$$v(r) = \mathrm{e}^{\frac{\ln\left(\frac{2\alpha n}{n+1}\right)}{(n-1)}}\left[\cos\left(\frac{(n-1)r}{2n}\right)\right]^{\frac{2}{n-1}}. \tag{4.8}$$

Write (4.8) in terms of the original variables as

$$u(t,x) = \mathrm{e}^{\frac{\ln\left(\frac{2\alpha n}{n+1}\right)}{(n-1)}}\left[\cos\left(\frac{(n-1)(x-\alpha t)}{2n}\right)\right]^{\frac{2}{n-1}}. \tag{4.9}$$

Equation (4.9) provides compacton solution of $K(m,n)$ equation (3.1) when $0 \le x - \alpha t \le (2n\pi/n-1)$ via reduction by T_1 for $m = n$.

4.1.2 *Reduction by* T_4. Again using eqs (2.7), (2.8) and (4.1), the reduced form of the conserved vector $T_4 = (T_4^t, T_4^x)$ given in table 1 is

$$\begin{aligned}T_4^r &= \frac{\alpha v^{n+1}}{n+1} - \frac{1}{2}v^{2n} - \frac{n(n-2)}{2}v^{2n-2}(v_r)^2\\ &\quad - nv^{2n-1}v_{rr},\\ T_4^s &= -\frac{v^{n+1}}{n+1}.\end{aligned} \tag{4.10}$$

Solving the reduced equation $T_4^r = k_4$ from (4.10) by using sine–cosine method for $m = n$, yields the same exact solution presented in (4.9).

4.2 *Case* II, *when* $m \neq n$ *and* $m \neq n+1$

Using symmetry conservation law relation (2.6), we noticed that both $T_1 = (T_1^t, T_1^x)$ and $T_2 = (T_2^t, T_2^x)$ are associated with X_1 and X_2. Also $T_1 = (T_1^t, T_1^x)$ is associated with X_3 only when $m = n+2$ and $T_2 = (T_2^t, T_2^x)$ is associated with X_3 whenever $m = 3n+2$. Using $X = X_1 + \alpha X_2$, the canonical coordinates are

$$s = t, \quad r = x - \alpha t, \quad v(r) = u. \tag{4.11}$$

4.2.1 *Reduction by* T_1. From (2.7), (2.8) and (4.11), the reduced form of the conserved vector $T_1 = \left(T_1^t, T_1^x\right)$ in table 2 is

$$\begin{aligned}T_1^r &= \alpha v - n(n-1)v^{n-2}(v_r)^2 - nv^{n-1}v_{rr} - v^m,\\ T_1^s &= -v.\end{aligned}$$

As X is related to $T_1 = (T_1^t, T_1^x)$, $T_1^r = k_1$ implies

$$\alpha v - n(n-1)v^{n-2}(v_r)^2 - nv^{n-1}v_{rr} - v^m = k_1. \tag{4.12}$$

Solving (4.12), we obtain the following implicit exact solution:

$$\int \frac{n\sqrt{n+1}\sqrt{m+n}\,v^{n-1}}{\sqrt{2(m+n)(\alpha v^{n+1}n - (k_1v^n - \frac{1}{2}n^2d_1)(n+1)) - 2(n^2+n)v^{m+n}}}\mathrm{d}v = r + d_2,$$

where d_1 and d_2 are constants of integration. Thus, solution of the $K(m,n)$ equation in terms of the original variables is

$$\int \frac{n\sqrt{n+1}\sqrt{m+n}\,u^{n-1}}{\sqrt{2(m+n)(\alpha u^{n+1}n - (k_1u^n - \frac{1}{2}n^2d_1)(n+1)) - 2(n^2+n)u^{m+n}}}\mathrm{d}u = x - \alpha t + d_2. \tag{4.13}$$

4.2.2 *Reduction by* T_2. Again, using (2.7), (2.8) and (4.11), the reduced form of the conserved vector $T_2 = \left(T_2^t, T_2^x\right)$ in table 2 gives rise to

$$T_2^r = \frac{\alpha v^{n+1}}{n+1} - \frac{m v^{m+n}}{m+n} - \frac{n(n-2)}{2} v^{2n-2}(v_r)^2 - n v^{2n-1} v_{rr},$$

$$T_2^s = -\frac{v^{n+1}}{n+1}.$$

As $D_s T_s = 0$, the reduced conserved vector $T_r = k_2$ implies

$$\frac{\alpha v^{n+1}}{n+1} - \frac{m v^{m+n}}{m+n} - \frac{n(n-2)}{2} v^{2n-2}(v_r)^2 - n v^{2n-1} v_{rr} = k_2. \quad (4.14)$$

Solving (4.14), we obtain the following implicit solution:

$$\int \frac{n\sqrt{m+n}\,\sqrt{n+1}\,v^{n-1}}{\sqrt{2(\alpha v^{n+1} n + (n+1)(\frac{1}{2} v^n n^2 d_3 + k_2))(m+n) - 2(n+n^2) v^{m+n}}} \mathrm{d}v = r + d_4,$$

which in terms of the original variables is

$$\int \frac{n\sqrt{m+n}\sqrt{n+1}\,u^{n-1}}{\sqrt{2(\alpha\, u^{n+1} n + (n+1)(\frac{1}{2}\, u^n n^2 d_3 + k_2))(m+n) - 2(n+n^2) u^{m+n}}} \mathrm{d}u = x - \alpha t + d_4. \quad (4.15)$$

Thus, (4.15) provides an exact implicit solution of the $K(m, n)$ equation.

4.2.3 *Reduction by T_1, when $m = n + 2$.* The conserved vector $T_1 = (T_1^t, T_1^x)$ in table 2 is associated with X_3 which yields the following canonical coordinates:

$$r = x t^{-(2n+1)^{-1}}, \quad s = \ln(t), \quad v(r) = u t^{2(2n+1)^{-1}}. \quad (4.16)$$

Equations (2.7) and (2.8) along with (4.16) yield components of the reduced conserved form of $T_1 = (T_1^t, T_1^x)$ in table 2, i.e.

$$T_1^r = \frac{1}{n+2}[r v - n(n-1)(n+2) v^{n-2}(v_r)^2 - n(n+2) v^{n-1} v_{rr} - (n+2) v^{n+2}],$$

$$T_1^s = -v.$$

As $T_1 = (T_1^t, T_1^x)$ is associated with X_3 for $m = n + 2$, $T_1^r = k_1$ implies

$$\frac{1}{n+2}[r v - n(n-1)(n+2) v^{n-2}(v_r)^2 - n(n+2) v^{n-1} v_{rr} - (n+2) v^{n+2}] = k_1,$$

which provides another double reduction by T_1 of the $K(m, n)$ equation (3.1).

4.2.4 *Reduction by T_2, when $m = 3n + 2$.* When $m = 3n + 2$, an association of X_3 with the conserved vector $T_2 = (T_2^t, T_2^x)$ in table 2 leads to the following canonical coordinates:

$$s = \ln(t), \quad r = x t^{\frac{-n-1}{4n+2}},$$

$$v(r) = t^{(4n+2)^{-1}} u(t, x). \quad (4.17)$$

Equations (2.7) and (2.8) along with (4.17) yield components of the reduced conserved form of $T_2 = \left(T_2^t, T_2^x\right)$ in table 2, i.e.

$$T_2^r = \frac{1}{4n+2}[r v^{n+1} - n(n-2)(2n+1) v^{2n-2}(v_r)^2 - 2n(2n+1) v^{2n-1} v_{rr} - (3n+2) v^{4n+2}], \quad (4.18)$$

$$T_2^s = -\frac{v^{n+1}}{n+1}. \quad (4.19)$$

Writing $T_2^r = k$ from (4.18) yields another double reduction of the $K(m, n)$ equation (3.1).

4.3 *Case* III, *when $m = n + 1$*

In this case, using symmetry conservation law relation (2.6), we see that $T_1 = (T_1^t, T_1^x)$ and $T_2 = (T_2^t, T_2^x)$ are associated with X_1 and X_2 only. They both are not related with X_3. $T_3 = (T_3^t, T_3^x)$ is the only conservation law associated with X_3.

The reductions and solutions via T_1 and T_2 by symmetry $X = X_1 + \alpha X_2$ are similar to those presented in (4.13) and (4.15) obtained by $m = n + 1$. So we only present reduction via T_3 by symmetry X_3.

4.3.1 *Reduction by* T_3. The conserved vector $T_3 = (T_3^t, T_3^x)$ in table 3 is associated with X_3 which results in the following canonical coordinates:

$$r = xt^{-(2n+1)^{-1}}, \quad s = \ln(t), \quad v(r) = ut^{2(2n+1)^{-1}}. \tag{4.20}$$

Equations (2.7) and (2.8) along with (4.20) yield components of the reduced conserved form of $T_3 = (T_3^t, T_3^x)$ in table 3, i.e.

$$\begin{aligned} T_3^r =& \frac{1}{4n+2}\Bigg[2n(n+1)(n-2)\left(n+\frac{1}{2}\right)v^{2n-2}(v_r)^2 \\ &+ 2(n+1)^2 v^{2n+1} \\ &+ 4n(n+1)\left(n+\frac{1}{2}\right)v^{2n-1}v_{rr} \\ &- 4n\left(n+\frac{1}{2}\right)(r\,v_{rr} - v_r)v^{n-1} \\ &- 4r\left(n(n-1)\left(n+\frac{1}{2}\right)v^{n-2}(v_r)^2 \right. \\ &\left. +(n+1)v^{n+1} - \frac{1}{2}rv\right)\Bigg], \\ T_3^s =& -rv + v^{n+1}, \end{aligned} \tag{4.21}$$

where the reduced conserved form satisfies $D_r T_3^r = 0$.

Thus, $T_3^r = k$ from (4.21) provides another new double reduction of the $K(m, n)$ equation (3.1).

5. Concluding remarks

We presented the generalised form of conservation laws of the nonlinear dispersive $K(m, n)$ equations. Using the multiplier's approach, we obtained three different types of generalised conservation laws based on different values of m and n. In the first case, four conservation laws were presented for $m = n$. The two conservation laws given in the second case hold good for any value of m and n. In the third case, we obtained three conservation laws for $m = n + 1$. The double reduction theory for these generalised conservation laws is utilised to construct some new reductions and exact solutions for different values of m and n of the $K(m, n)$ equations. In future, the ODEs obtained after reduction, in particular for $m = n + 1$, $m = n + 2$ and $m = 3n + 2$, can be used to find some new exact or approximate solutions for the nonlinear compacton $K(m, n)$ partial differential equations. To the best of our knowledge, an analysis of conservation laws and its association with the symmetries of $K(m, n)$ equations presented in this way have not been reported in the literature.

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