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An optimised stability model for the magnetohydrodynamic fluid

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Abstract. Magnetohydrodynamics (MHD) is a very challenging problem which affects the stability of Poiseuille flow. Therefore, in this work we investigate the instability of an electrically conductive fluid between two parallel plates under the influence of a transverse magnetic field. We apply the Chebyshev collocation method to solve the generalised Orr–Summerfield equations to determine wave number, growth rates and spatial modes of the eigenmodes. To get the neutral curves of MHD instability, the QZ method is used. It is observed that the magnetic field has a stabilising effect on the flow and the stability increases as we increase the Hartmann number and for various wave numbers, magnetic field put down the growth of perturbation. It is concluded that effect of perturbations is little in span-wise direction for different Hartmann numbers that increase the critical values of Reynolds numbers.

Keywords. Instability; magnetohydrodynamics; Chebyshev collocation method; electrically conductive fluid.

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1. Introduction

The stability of an electrically conducting viscous resistive magnetohydrodynamic (MHD) fluid between two infinite parallel plates was first studied theoretically by Lock [1], when a uniform magnetic field was applied perpendicular to the plates. He used spectral method for the numerical simulations of Orr–Summerfield equations. Many studies have been proposed on the stability of magnetic and electrically conductive fluid for Poiseuille flow [2]. Stability of Couette and Poiseuille flows of Oldroyd-B fluid was examined by Nabil [3] under normal magnetic field. Two-dimensional MHD channel flows were extensively studied using numerical solutions. The temporal linear stability analysis was done by Taghavi [4] to study the temporal evolution of an infinitesimal two-dimensional perturbation imposed on the basic flow.

Under the influence of an external magnetic field, an electrically conductive fluid was ananlysed by Muller and Buhler [5] to study the instability of two-dimensional flows of electrically conductive fluid heated from below and exposed to a constant magnetic field. He found that magnetic field has different effects in different unstable modes.

Hossain and Khan [6] studied that, under the influence of magnetic field, flow becomes unstable which was shown by critical Hartmann number and critical Reynolds number. Such critical points of instability for both Hartmann and Reynolds values were explored by using the diagrams of bifurcation. In their work they also discussed the effect of two dimensionless parameters on flow stability and flow velocity. The effects of thermophoresis and Brownian motion on the three-dimensional flow and interpretation of chemical reactions for cross magnetofluid were discussed by many researchers [7–10]. By numerical simulation, the inability of plane Poiseuille flow under a uniform magnetic field by using liquid metal was explored by Tagawa [11]. It was assumed that the basic flow direction turbulence

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is in the form of periodicity. Linear stability of the basic flow which depends on Hartmann number is studied by a finite difference method with the help of HSMAC algorithm and fourth-order central difference scheme. He predicted that phase velocity of Tollmien–Schlichting and Reynolds number formed neutral stability state when input values of wave number, Hartmann number and aspect ratio were provided. It was observed that the critical values at neutral stability curve at wave number is about 1.15, when critical Reynolds number is about 4.0×10^5 and Hartmann number approaches 5. Opanuga [12] presented the entropy formation study of buoyancy impact with induced heat generation on non-Newtonian, viscous and incompressible hydromagnetic Poiseuille flow with transversal isothermal walls. With the help of rapidly convergent semi-analytical approach of Adomian disintegration, solution of nonlinear boundary value problems was investigated by using governing equations. The viscoelastic fluid was analysed to describe natural convection in two layers problem under the influence of uniform flux heating and plane shear flows by Chen [13]. The assessment of heat source/sink and melting effects on crossfluid flow was discussed in refs [14–22]. With the help of numerical methods and theoretical analysis of physical phenomena of thermal instability, results are explained and predicted. For a plane shear flow linear stability technique is applied to obtain eigenvalue problems using Chebyshev collocation method to find neutral stability curves in thermal convection by using viscoelastic fluids and shear flows. By using numerical method, it was concluded that neutral stability curves have a bi-modal nature for a porous fluid. Longitudinal rolls developed by shear flows as well as transverse rolls are studied. For Newtonian fluids, transverse rolls are always fixed after longitudinal rolls. In viscoelastic fluids transverse rolls may be obtained early in the flow system. In this paper, we investigate the instability of MHD fluid of plane Poiseuille flow by using numerical technique of Chebyshev collocation of Orr–Sommerfeld equation. The eigenvalue problem is obtained numerically by using the QZ algorithm. The results of the linear instability analysis in the presence of transverse magnetic field are presented. The physical properties of relevant parameters are explained with the help of graphs. We show that due to the suppression of growing disturbance of Joule dissipation, basic Poiseuille flow is stabilised in the presence of magnetic field. Disturbance in the spanwise direction has a small influence on the instability of Poiseuille flow.

The remaining paper is organized as follows: Mathematical formulation is given in §2. Section 3 explains Chebyshev polynomial expansion, and §4 presents results and discussion. Section 5 presents conclusions and remarks.

2. Mathematical formulation

Let us consider the MHD instability of incompressible, electrically conductive, viscous and fully developed fluid flowing between planes under pressure p under the impact of the vertical magnetic field. Planes are placed at a distance $2L$. Flow is along the y-axis, and magnetic field is applied perpendicular to the direction of flow, i.e. magnetic field is acting in the direction of the x-axis. The magnetic field is supposed to be constant as shown in figure 1.

The dimensionless governing system is considered in the following form:

$$\nabla \cdot \vec{V} = 0 \tag{1}$$

$$Re\left(\frac{D\vec{V}}{Dt}\right) = -\nabla P + \Delta\vec{V} + Ha^2\text{curl}\,(B) \times B \tag{2}$$

$$\nabla \cdot \vec{J} = 0 \tag{3}$$

$$\vec{J} = -\nabla\phi + \vec{V} \times \vec{B}. \tag{4}$$

The boundary conditions can be written as

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial \varphi}{\partial x} = u = 0 \quad \text{at } x = \pm L. \tag{5}$$

Here $V = (u, v, w)$ is the three-dimensional velocity vector, $j = \left(j_x, j_y, j_z\right), \phi, p$ and $B = (1, 0, 0)$ represent the current density, electric potential, pressure and external magnetic field along the x-axis, respectively. In this work, it is assumed that all the variables are independent of the coordinate z and are only functions of x and y. The operators are represented as $\nabla = (\partial_x, \partial_x)$ and $\Delta = \left(\partial_x^2 + \partial_y^2\right).B$ is the magnitude of the magnetic field along the x direction.

The above system is defined by two parameters, the interaction parameter $N = Ha^2/Re$ and Reynolds number, where Ha represents Hartmann number which is defined as $Ha = BL\sqrt{\sigma/\rho\upsilon_0}$. The parameters appearing in eq. (2) are the Reynold value $Re = \upsilon_0 L/\upsilon$ which is the ratio between viscous force and inertial force and $N = \sigma LB^2/\rho\upsilon_0$ is the Stuart number which is represented as the ratio of electromagnetic forces and inertial forces. υ_0 is the kinematic viscosity.

The two-dimensional disturbance of the plane Poiseuille flow is expressed as

$$u = u', v = v_0 + v', p = p', \phi = \phi'. \tag{6}$$

On substituting these values in eqs (1)–(4) we get

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0. \tag{7}$$

Now x-component of eq. (2) is

$$\frac{\partial u'}{\partial t} + v_0\frac{\partial u'}{\partial y} = -\frac{\partial p'}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 u'}{\partial y^2}\right), \tag{8}$$

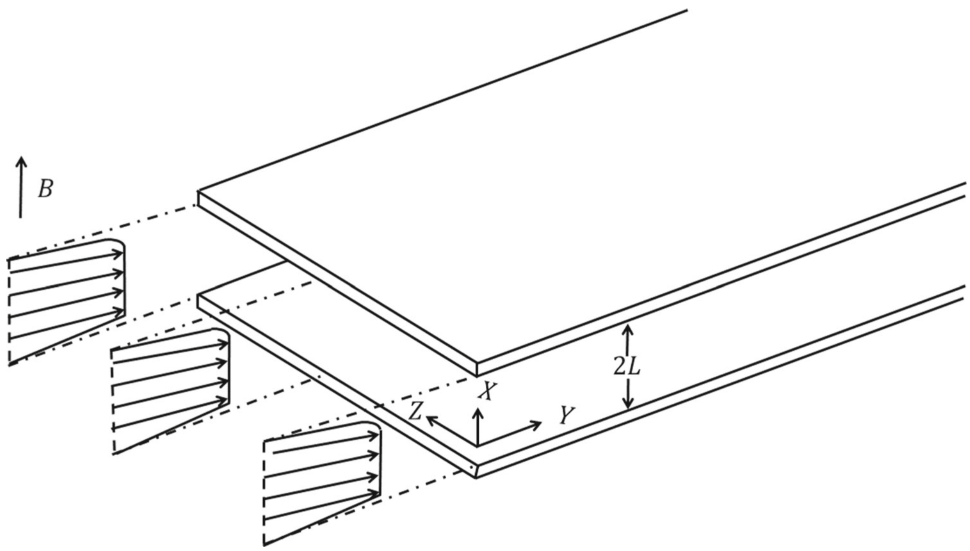

Figure 1. Geometry of the problem.

and y-component of eq. (2) is

$$\frac{\partial v'}{\partial t} + u'\frac{\partial v_0}{\partial x} + v_0\frac{\partial v'}{\partial y} = -\frac{\partial p'}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2}\right) - N\mu_0 v', \tag{9}$$

$$\frac{\partial^2 \phi'}{\partial x^2} + \frac{\partial^2 \phi'}{\partial y^2} = 0. \tag{10}$$

The characteristic magnitude of velocity is expressed in the following form:

$$v_0 = \left(\frac{\cosh(Ha) - \cosh(Ha \cdot x)}{\cosh(Ha) - 1}\right). \tag{11}$$

Small perturbations in terms of normal modes u', v', p', ϕ' are defined as

$$\left(u', v', p', \phi'\right) = \left[\hat{u}(x), \hat{v}(x), \hat{p}(x), \hat{\phi}(x)\right] \mathrm{e}^{i(kx-\omega t)}, \tag{12}$$

where $\omega = \omega r + \omega i$ and k are complex eigenvalue and an arbitrary real wave number, respectively. The real and imaginary parts of the complex eigenvalue is defined as the amplification rate and oscillation frequency. All variations with the parallel coordinate y and time t are assumed to be contained in the factor $\mathrm{e}^{i(ky-\omega t)}$. Substituting (12) into (7)–(9) we get

$$D\hat{u} + ik\hat{v} = 0, \tag{13}$$

$$-i\omega\hat{u} + ikv_0\hat{u} = -D\hat{p} + \frac{1}{Re}\left(D^2 - k^2\right)\hat{u}, \tag{14}$$

$$-i\omega\hat{v} + ikv_0\hat{v} + \hat{u}Dv_0 = -ik\hat{p} + \frac{1}{Re}\left(D^2 - k^2\right)\hat{v} + N\hat{v}, \tag{15}$$

where $D = \mathrm{d}/\mathrm{d}x$. The linear stability equations (13)–(15) reduce to

$$D^4\hat{u} + \left(-ReN - 2k^2 + ikRev_0\right)D^2\hat{u} + \left(k^4 + ikReD^2v_0 + ik^3Rev_0\right)\hat{u} = -\omega\left(iReD^2\hat{u} - ik^2Re\hat{u}\right), \tag{16}$$

$$\left(D^2 - k^2\right)\hat{\phi} = 0. \tag{17}$$

The defined linear boundary conditions reduce to the following equation:

$$\hat{u} = 0 \text{ and } D\hat{u} = D\hat{\phi} = 0 \text{ at } x = \pm L. \tag{18}$$

The linear equations (16) and (17) are ordinary differential equations in terms of $\hat{u}$ and $\hat{u}\hat{\phi}$, which are used as a two-point boundary value system. With the help of numerical simulation, we have to solve the above-mentioned eigenvalue system.

3. Chebyshev polynomial expansion

Here $T_n(x)$ is the first type of nth degree polynomial in the Chebyshev method which is described by the following equation:

$$T_n(\cos\theta) = \cos(n\theta). \tag{19}$$

For all positive integers n from Hamming [23] and Fox [24], expansion of the above equation in a Chebyshev series can be written as

$$u_j = \sum_{n=0}^{N_1} a_n T_{nj}, \tag{20}$$

Table 1. Comparisons of numerical results.

	Re_c				k_c				
Ha	Present	Zakir	Lock [1]	Takashema [25]	Present	Zakir	Lock [1]	Takashema [25]	N_1
1.0	9805.89	10072.84	10450	10016.26	0.9755	1.0	0.98	0.971828	35
2.0	28039.13	28666.65	30000	28603.64	0.9248	0.93	0.93	0.927773	40
3.0		65765.21	69000	65155.21		0.95	0.96	0.958249	45
4.0	109484.3	113291.7	119000	112394	1.04	1.04	1.04	1.035454	55
5.0	158289.9	165134.7	175000	164089.99	1.143	1.14	1.15	1.134248	60

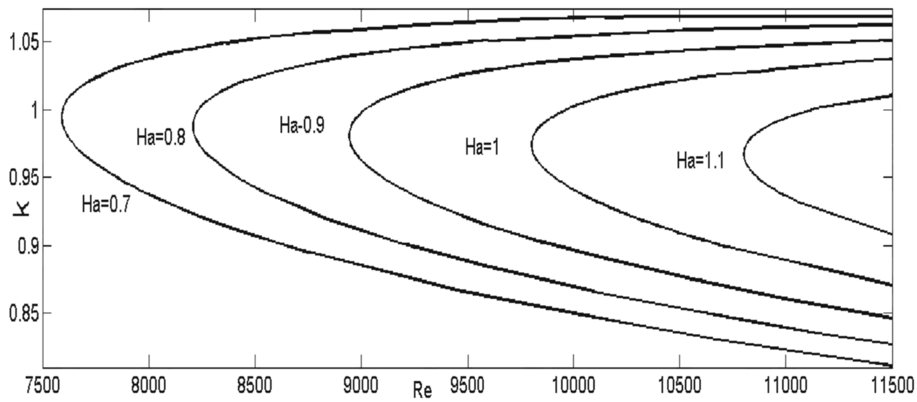

Figure 2. Stability curves for various values of Ha.

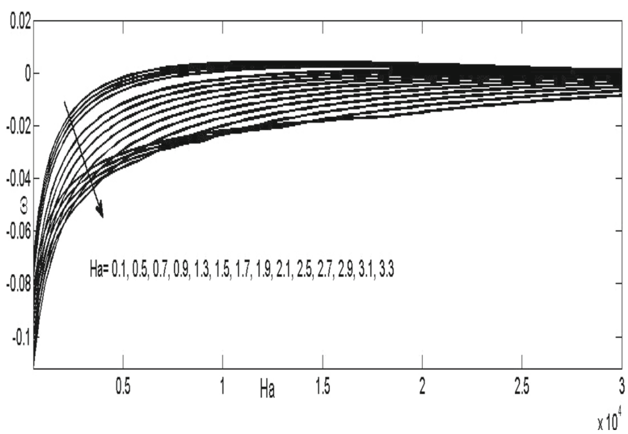

Figure 3. Growth rate curve ω vs. Re for various values of Ha.

$$u'_j = \sum_{n=0}^{N_1} a_n T'_{nj}, \tag{21}$$

$$u''_j = \sum_{n=0}^{N_1} a_n T''_{nj}, \tag{22}$$

$$u^{(4)}_j = \sum_{n=0}^{N_1} a_n T^{(4)}_{nj}, \tag{23}$$

where $j = 1, 2, 3..., N$ and $T_{nj}, T'_{nj}, T''_{nj}, ..., T^{(4)}_{nj}$ represent the Chebyshev polynomial and its derivatives at collocation point j and a_n are the coefficients of series. So, eq. (15) can be written as

$$\sum_{n=0}^{N_1} \left\{ T^{(4)} + \left[-RN - 2k^2 - ikv_0 Re\right] T'' + \left[k^4 + ik^3 Rev_0 + ikReD^2 v_0\right] T \right\} a_n = \omega \sum_{n=0}^{N_1} \left\{-iRe\left[T'' - kT\right]\right\} a_n. \tag{24}$$

Equation (23) is the generalised Sommerfeld equation for our present work. The above system can be verified by the coefficient matrix $X = (a_0, a_1, a_2, ..., a_n)$ which is obtained as

$$AX = \omega BX. \tag{25}$$

This spectral discretisation process yields a generalised eigenvalue problem with non-symmetrical $(N_1 + 1)$ by $(N_1 + 1)$ square matrices $X$and Y.

4. Results and discussion

First, the expansion applied to this work is verified by the numerical method and the critical Reynolds number Re_c and critical wave number k_c are obtained. Table 1 shows the solution of these parameters for different values of Ha. A number of terms in eq. (24) represented by N_1

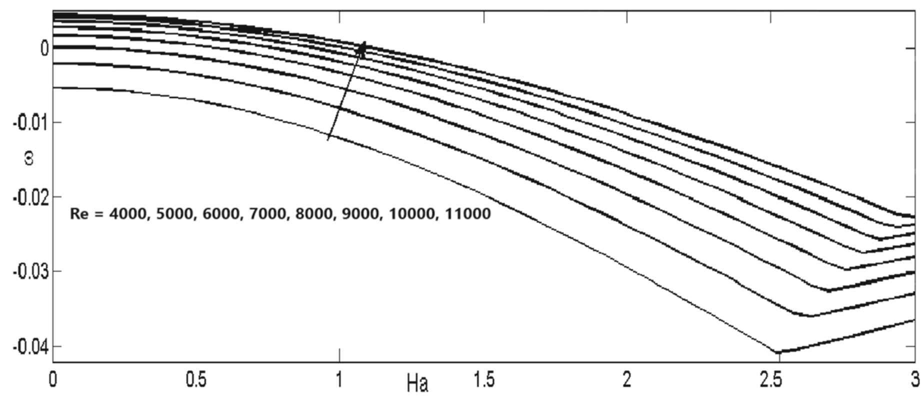

Figure 4. Growth rate curves ω with Ha for various values of Re.

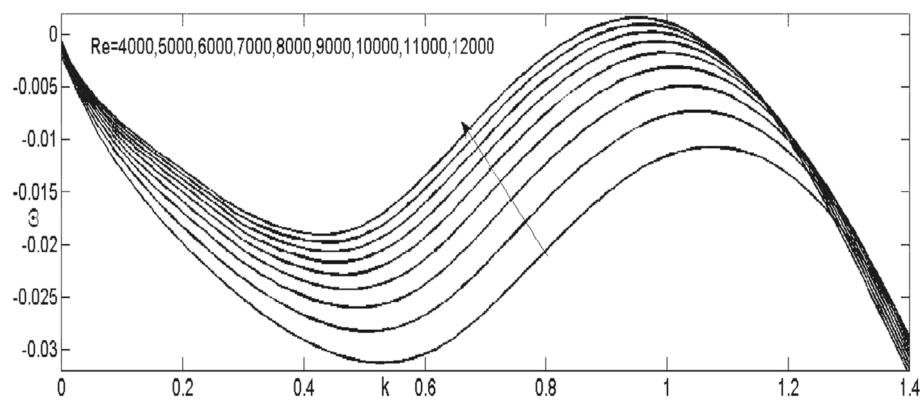

Figure 5. Growth rate curves ω with k for various values of Re.

for evaluating Re_c and k_c. From table 1 we can see that Chebyshev collocation method converges slowly when we increase Ha. The results in this work are approximately comparable with Lock [1], Takashima [25] and Hussain *et al* [26].

Takashima [25] obtained the value of critical Reynold number $Re_c = 10016.26$ and $k_c = 0.97$, as shown in table 1. Lock [1] obtained the value $k_c = 0.98$ and $Re_c = 10450$. In addition, we observed the value of $k_c = 0.975$ and $Re_c = 9805$ as summarised in table 1.

With the aid of the method described above, eigenvalues, k and $Re = 0.7$ to 1.1 are obtained for Ha. The curves k vs. Re when $\omega = 0$ shown in figure 2 is a neutral curve. It is obvious from figure 2 that, as magnetic parameter Ha increases, Re_c rapidly increases. Our results are comparable with Takashima [23]. In this figure, the neutral curve for each value of Ha is known as the parabola. For any wave number, the flow is stable against perturbations when Re decreases from the critical number. Different values of Re are resembling in the sequence at critical wave number k. As the Hartmann value increases, the unstable area decreases rapidly. The force resulting from the magnetic field has a damped impact on the flow of fluid. The flow of electrically conducting fluid stabilises in the presence of transversal magnetic field. Instability vanishes as $Re \to \infty$ and is more prevalent at low values of the Reynolds number. The results shown in figure 2 for small values of Ha are similar to those of Lock [1].

The temporal growth rate ω as a function of Re for the range $Ha = 0.1$–3.3 are sketched in figure 3. Each curve has a global maximum for different values of magnetic fields. Since the magnetic field can reduce the disturbances, a weak magnetic field strength gives a larger maximum Re. By increasing the value of Ha the growth rate approaches the neutral point of eigenvalue.

The temporal growth with Hartmann value for $Re =$ 4805–16805 is shown in figure 4. When the magnetic field is weaker, larger Reynold number corresponds to small growth rate. The flow would be very rapid when Ha increases and for large values of Re. Figure 4 shows that larger Ha will correspond to a smaller growth rate. The magnetic field suppresses stability.

For $Re =$ 4000–12000, temporal growth curves with wave number k is shown in figure 5. We can see that two maxima are observed in the temporal growth rate

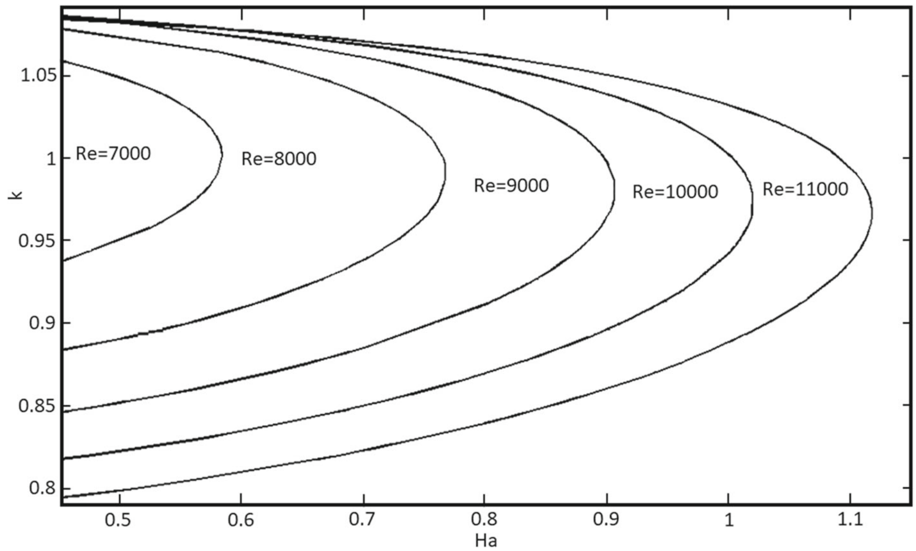

Figure 6. Neutral stability curves for various values of *Re*.

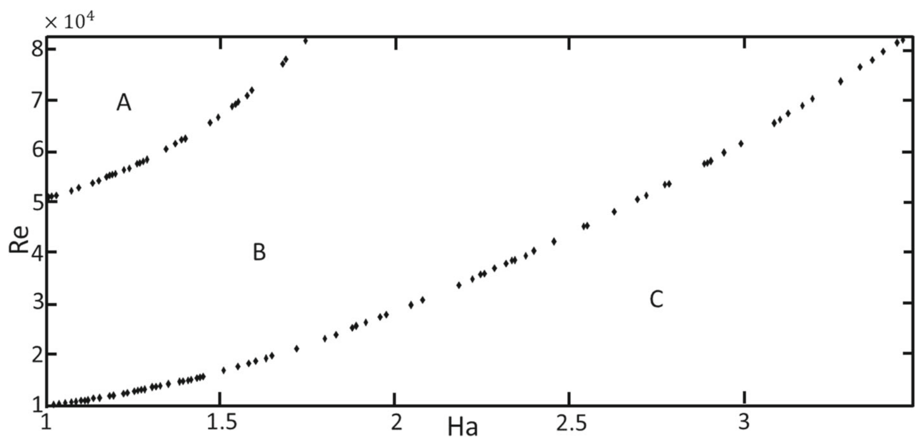

Figure 7. Streamwise instability curves for $k = 1$.

curves which change positions for various values of *Re*. Maximum in long wave area near zero wave number, with a neutral zero growth is observed when $Re = 4000$, while the maximum determines a critical Reynold value at $Re_c = 12000$ which lies at wave number at $k = 0.97$ with least growth rate. From the figure it is also observed that higher Reynolds number leads to greater growth rate.

Stability curve for the stabilising flow is observed in figure 6. The area surrounded by curves is the unstable region whereas stable area represents the outer side of the curve. The curves in this figure indicate again that, greater values of *Ha* correspond to greater stable region with a specific flow rate and magnetic field force decreases the growth rate of perturbation. There are two competitive effects on perturbations in the basic flow, the inertial energy transfer from the mean flow to disturbances and the suppression by the magnetic field. As shown in figure 6, the former effect will destabilise the flow when *Re* increases: but the increase of *Ha* enhances the latter effect and makes the flow more stable. Therefore, it is possible to stabilise the flow of the electrically conductive fluid between two parallel planes by increasing the strength of the applied magnetic field. Thus, this method can be used for flow control.

In figure 7 for stream-wise perturbations of temporal instability, neutral curves are presented in Re–Ha parameter space for $k = 1$. The two curves in the figure separate the unstable and stable areas. The flow is stable in the outer side of area B and unstable area lies between the two lines A and C. It is observed that there is a coupling relationship between *Ha*, *Re* and wave number *k* of perturbations which will result in the instability of the flow. Magnetic field has an important role in the critical values of wave number, Reynold number and Hartmann number.

5. Conclusions and remarks

Considering the MHD impact, linear stability of Poiseuille flow exposed to the vertical constant magnetic field is analysed by using collocation technique for the low *Re* range. Coupling effects of *Ha*, wave number and *Re* are explored in the present work. Analysis of linear stability is provided in detail. From the results we can arrive at the following contributions:

- The unstable region gets smaller for large magnetic field.
- As *Re* increases, the growth rate increases, which makes the flow unstable.
- For various values of wave number, magnetic field reduces the growth of perturbation.

For different *Ha*, there is little effect of perturbations in span-wise direction that increase the critical values of *Re*.

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