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Dispersion relation of modulational instability for one-dimensional standing solitary waves in hot ultrarelativistic electron–positron plasmas

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Abstract. In this paper, the non-linear dispersion relation of the system for interactions between high intensity laser and hot relativistic electron–positron plasma is derived. We restrict the problem to the standing solitons in ultrarelativistic plasmas and apply the quasineutrality condition. The modulational instability growth rate for different values of plasma temperatures, velocities and wave numbers are illustrated numerically. It is shown that, by increasing the unperturbed plasma enthalpy, the modulational instability growth rate decreases for all the values of plasma fluid velocities. It is also found that the growth rate shows an increasing trend with plasma fluid velocity. Furthermore, the impact of wave amplitude on the modulational instability growth rate is investigated explicitly. The growth rate increases with wave number as well as wave amplitude.

Keywords. Modulational instability; solitary waves; relativistic plasmas.

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1. Introduction

Soliton formation in plasmas [1–13] as well as other mediums [14–16] has recently received increased attention. Due to the non-linear interactions of high-intensity lasers with two-component plasmas, these stationary solitary waves are formed and propagate, provided some conditions are met. In general, these conditions depend on the balance between the non-linearity and dispersion of the system. The former enhances the wave amplitude and narrows the wave width, while the latter has the opposite effect. Regarding electromagnetic solitons, in different studies, depending on the type of plasma and the applied conditions for finding the solitary solutions, there are different eigenvalues and these conditions have influenced the shape and type of solitons. Any change in them, would result in different characteristics for solitary waves. Lehmann and Spatschek [6] studied the existence of stationary electrostatic and electromagnetic waves in ultrarelativistic electron–ion plasmas. Using Poincaré analysis, they investigated the influence of ion mobility on the phase-space structure. The results for electron–ion plasmas in ultrarelativistic case were compared with the phenomena in electron–positron plasmas. It was shown that in this regime the electron–ion plasmas behave qualitatively like electron–positron plasma. Heidari *et al* [7] have studied the propagation of a moving electro-sound wave in fully relativistic two-component plasma for two regions of relative velocities of soliton and plasma fluid. For both mobile and immobile ions, the effect of positron fraction on solitary waves in cold and weakly relativistic electron–positron–ion plasma was investigated by Lu *et al* [8]. Shukla *et al* [9] presented the interaction between intense laser pulses with electron–positron plasma by nonlinear Schrödinger equation. Using the quasineutral approximation, they found the electrostatic potential as a function of the vector potential. They also found the modulational instability growth rate and investigated its variation for different temperatures and intensities. Furthermore, the Sagdeev potential and its variations with different values of frequency and temperature were studied. Tatsuno *et al* [10] studied the nonlinear interaction of an intense electromagnetic wave with hot relativistic electron–positron plasmas using the variational method. They found an effective potential to describe the evolution of the system and showed the possibility of beam self-trapping in the formation of stable two-dimensional soliton structures. Heidari *et al*

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[11] investigated electromagnetic standing solitons with a zero drifting velocity for the case of ultrarelativistic hot electron–positron plasmas. They presented the variations of the standing solitons for different values of plasma temperatures and plasma fluid velocities. It was shown that for higher values of fluid velocity, the non-moving patterns emerge at lower temperatures.

In the present study, we are going to consider the nonlinear propagation of intense light pulses in an electron–positron plasma by taking into account the combined action of the relativistic particle mass increase and the plasma density profile modification. The combined action of these two nonlinear effects produces modulational instability of the intense light pulse. In the next section, it is shown that the system of governing equations can be reduced to a generalised nonlinear Schrödinger equation (NLSE). The NLS equation which describes the nonlinear evolution of the amplitude of relativistic solitary waves in hot *e–p* plasma predicts a modulational instability.

Dispersion relation provides a relationship between the wave vector and the frequency of a wave and describes under which conditions the wave can propagate and under which conditions it cannot propagate [17–21]. In [17] a generalised linear dispersion relation for drift waves was obtained to represent both the collisional loss of momentum to the ions and the thermal relaxation with themselves. In this study, normalised drift wave growth rate and normalised frequency were investigated as a function of normalised parallel wave number for various parameters. The variations of normalised frequency with the normalised wave number for different cases are studied in [18]. In this study, when an electromagnetic wave propagates in a strongly magnetised cold plasma, a dispersion relation is derived considering the quantum electrodynamical phenomenon of elastic photon–photon scattering as well as the influence of an electron–positron pair plasma. Hongsit *et al* [19] derived the growth rate of long-wavelength transverse instabilities of the wave number using the small-k method.

Here, we proceed to find the one-dimensional dispersion relation of the standing solitary wave propagating in a hot ultrarelativistic electron–positron plasma in the quasineutral approximation. In §2, starting from the Maxwell and electron–positron fluid equations, we present a wave equation governing the dynamics of the electromagnetic wave propagating in the pair plasma. We show that it can be reduced to a generalised nonlinear Schrödinger equation. The latter leads to the nonlinear dispersion relation. The dispersion relation can be used to analyse the modulational instability due to nonlinear coupling between high frequency and low frequency motions. Section 3 is devoted to the numerical analysis for investigating the variations of modulational instability growth rate with plasma temperature, wave intensity, wave number and plasma fluid velocity. We finally conclude the paper with a summary of our findings in §4.

2. Theoretical model and nonlinear dispersion relation

We consider a relativistic plasma consisting of two-particle species, hot electrons and positrons, and carry out all our calculations in the stationary one-dimensional frame

$$\xi = \gamma_V\,(x - Vt) \text{ and } \tau = \gamma_V\left[t - \left(V/c^2\right)x\right]$$

so that all quantities are assumed to depend only on the variable ξ. Introducing the dimensionless quantities

$$\tilde{\xi} = \left(\omega_p/c\right)\xi,\quad \tilde{t} = \omega_p t,\quad \tilde{w} = w/w_0,\quad \tilde{a} = (e/w_0)\,a,$$
$$\tilde{\phi} = (e/w_0)\,\phi,\quad \tilde{V} = V/c \quad\text{and}\quad \tilde{n} = n/n_0,$$

we arrive at the dimensionless equation [21],

$$a_{\xi\xi} + \Big(\omega^2 - \gamma_0\,(\beta_0 - V) \times \sum_\alpha \Big\{\Big[\gamma_V^{-2}\sigma_\alpha\phi + \gamma_0 w_0\,(\beta_0 V - 1)\Big]^2 - \gamma_V^{-2}\left(w_\alpha^2 + a^2\right)\Big\}^{-1/2}\Big)\,a = 0 \tag{1}$$

for the transverse component of the electromagnetic equation and the plasma species density as

$$n_\alpha = \pm\gamma_0\frac{(\beta_0 - V)}{R_\alpha}w_\alpha \tag{2}$$

in which

$$R_\alpha = \sqrt{[\gamma_V^{-2}\sigma_\alpha\phi + \gamma_0 w_0\,(\beta_0 V - 1)]^2 - \gamma_V^{-2}\left(w_\alpha^2 + a^2\right)}, \tag{3}$$

where subscript 0 represents quantities at infinity. $\omega_p = \left(4\pi n_0\, e^2/m_e\right)^{1/2}$ is the electron plasma frequency, α represents the two species of the plasma, a is the vector potential, ϕ is the scalar potential, $\sigma_\alpha = q_\alpha/e$, e is the magnitude of the electron charge, c is the speed of light in vacuum, ω is the electromagnetic wave frequency, V is the group velocity of the soliton, υ is the plasma species velocity and $\gamma = \left(1 - \upsilon^2/c^2\right)^{-1/2}$ is the Lorentz relativistic factor. Also,

$$w = mc^2 + \frac{\upsilon}{\upsilon - 1}\frac{p}{n}$$

is the enthalpy per fluid particle [11], where p is the pressure and m is the rest mass of electrons and positrons.

The adiabatic index υ is equal to 5/3 and 4/3 for the non-relativistic and ultrarelativistic cases respectively. In ultrarelativistic regime, the second term in the definition of the enthalpy is dominated and we shall obtain $w_\alpha = 4k_BT_\alpha$ for the enthalpy of the species.

We restrict the problem to the standing solitons from now on and proceed to derive the nonlinear dispersion relation of the system. To this end, we apply the quasineutrality condition, $n_{e^-} = n_{e^+}$ in eq. (2). We reach

$$w_p^2\left(\phi^2 + 2\gamma_0 w_0\phi + \gamma_0^2 w_0^2\right) - w_e^2\left(\phi^2 - 2\gamma_0 w_0\phi + \gamma_0^2 w_0^2\right) = (w_p^2 - w_e^2)a^2. \tag{4}$$

Note that, for electron–positron plasmas, where the masses and the moduli of the charges of the two species are equal, the inertia of charge particles constituting the plasma is the same. So, no charge separation is expected to occur and applying the quasineutrality approximation is allowed. The quasineutrality assumption, in fact, gives rise to a single equation only for the vector potential and absence of the scaler potential. In other words, we try to find the scalar potential in terms of the vector potential amplitude. By neglecting terms of ϕ^2 and higher orders, the electrostatic potential can be written as

$$\phi = \frac{\gamma_0^2 w_0^2 - a^2}{2\gamma_0 w_0}\frac{w_e^2 - w_p^2}{w_e^2 + w_p^2}. \tag{5}$$

Inserting eq. (5) into eq. (3), after some straightforward algebraic simplifications yields

$$R_e = R_p = \sqrt{2/(w_e^2 + w_p^2)} \times\sqrt{\gamma_0^2 w_0^2 - (w_e^2 + w_p^2)/2 - a^2} \tag{6}$$

and

$$\nabla^2 a - \mathrm{i}\frac{\partial a}{\partial t} - \frac{\gamma_0\beta_0\sum_\alpha w_\alpha\sqrt{\sum_\alpha w_\alpha^2}}{\sqrt{2}w_e w_p\sqrt{\gamma_0^2 w_0^2 - \sum_\alpha w_\alpha^2/2 - a^2}}a = 0 \tag{7}$$

which is a generalised NLSE. Now, we make the ansatz $a = (a_0 + a_1)\,\mathrm{e}^{\mathrm{i}\delta t}$ where the perturbation a_1 is defined as

$$a_1 = (X + \mathrm{i}Y)\exp\mathrm{i}\,(kx - \Omega t)\,. \tag{8}$$

The wave amplitude a_0 is real ($a_0 \gg a_1$) and δ is a constant corresponding to a nonlinear frequency shift. Also, X and Y are real constants and $\Omega\,(k)$ denotes the frequency (wave number) of the low-frequency modulations. Finding δ for the lowest order, one obtains $\delta = M\left(\Lambda - a_0^2\right)^{-1/2}$. Linearising eq. (7) with respect to the perturbation a_1, the nonlinear dispersion relation reads as

$$[\Omega - \delta]^2 = k^2 + M\left(\Lambda^{-3/2}a_0 + \Lambda^{-1/2}\right) \tag{9}$$

in which

$$M = \frac{\gamma_0\beta_0\sum_\alpha w_\alpha\sqrt{\sum_\alpha w_\alpha^2}}{\sqrt{2}w_e w_p},$$

$$\Lambda = \sqrt{\gamma_0^2 w_0^2 - \sum\nolimits_\alpha w_\alpha^2/2 - a_0^2}. \tag{10}$$

3. Numerical results in ultrarelativistic regime

It is worth noting that the general form of the enthalpy per fluid particle is defined as [11]

$$w_\alpha = K_3\left(Z_\alpha\right)/K_2\left(Z_\alpha\right) \tag{11}$$

in which K_2 and K_3 are the MacDonald functions of the second and third order, respectively. The argument in K_2 and K_3 is the ratio of the rest energy to the thermal energy and is defined as $Z_\alpha = 1/k_BT_\alpha$, where k_B and T_α are the Boltzmann constant and the temperature of the particles, respectively. For non-relativistic temperatures, $k_BT_\alpha \ll 1\,(Z_\alpha \gg 1)$, the general form of the enthalpy reduces to

$$w_\alpha = 1 + 5/2Z_\alpha \tag{12}$$

while for ultrarelativistic high temperatures the condition $k_BT_\alpha \gg 1\,(Z_\alpha \ll 1)$ is satisfied and we have

$$w_\alpha = 4/Z_\alpha. \tag{13}$$

Though we have been using normalised quantities, we present a few general examples to obtain a better estimate on some of the main quantities discussed in this paper. From equating the thermal and electron rest mass energies (i.e. $k_BT_\alpha = m_\alpha c^2$) a minimal temperature $T_{0\,\mathrm{min}} = 6\times 10^9$K for the plasma can be determined. The condition for having relativistic plasma could be stated as $T_0 \gg 6\times 10^9$ K. For $w_0 = 4k_BT_0/mc^2 = 100$ assumed in the manuscript and using the Boltzmann constant $k_B = 1.38\times 10^{-23}$J/K, one obtains $T_0 = 1.49\times 10^{11}$K for the temperature. This temperature for plasma with one degree of freedom indicates an energy of 12.8 MeV for particles if we recall the conversion factor: 1 eV $= 11600k$. For a non-relativistic plasma, $k_BT_\alpha \ll mc^2$ and $w_0 = 10^{-3}$ for instance, the particle energy in the plasma amounts to 128 eV. Figure 1 shows the impact of particle temperatures explicitly. In this figure, the modulational instability growth rate $\Gamma = -\mathrm{i}\Omega$ vs. the equilibrium temperature of plasma for different values of fluid velocity is plotted. It is clear that, as w_0 increases, Γ decreases for all values of β_0. In order to

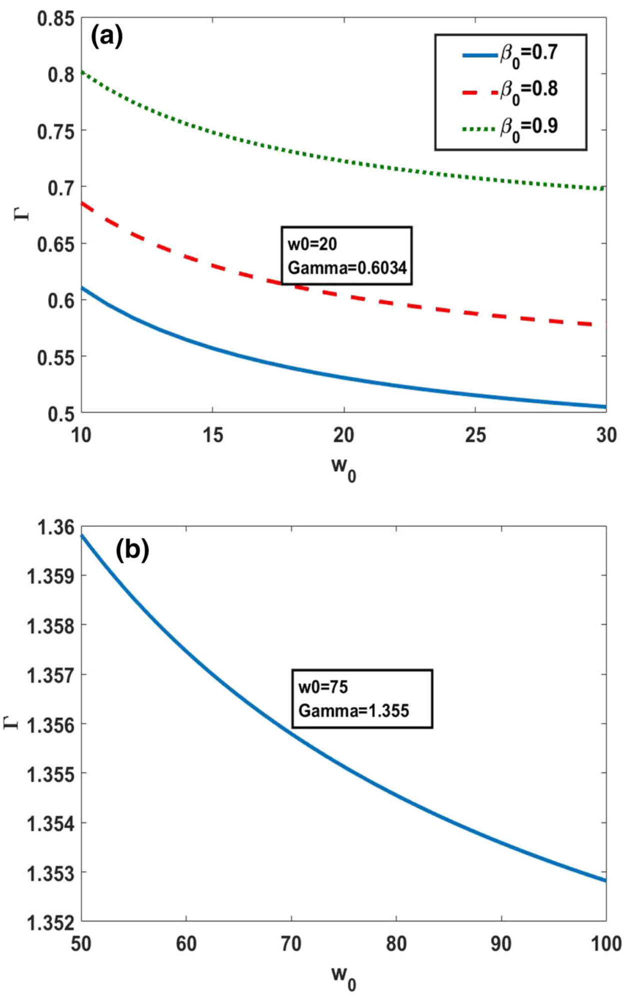

Figure 1. Variation of the modulational instability growth rate with the plasma temperature for (**a**) $k = 0.1$, $a_0 = 0.1$ and (**b**) $k = 0.4$, $a_0 = 0.1$, $\beta_0 = 0.9$.

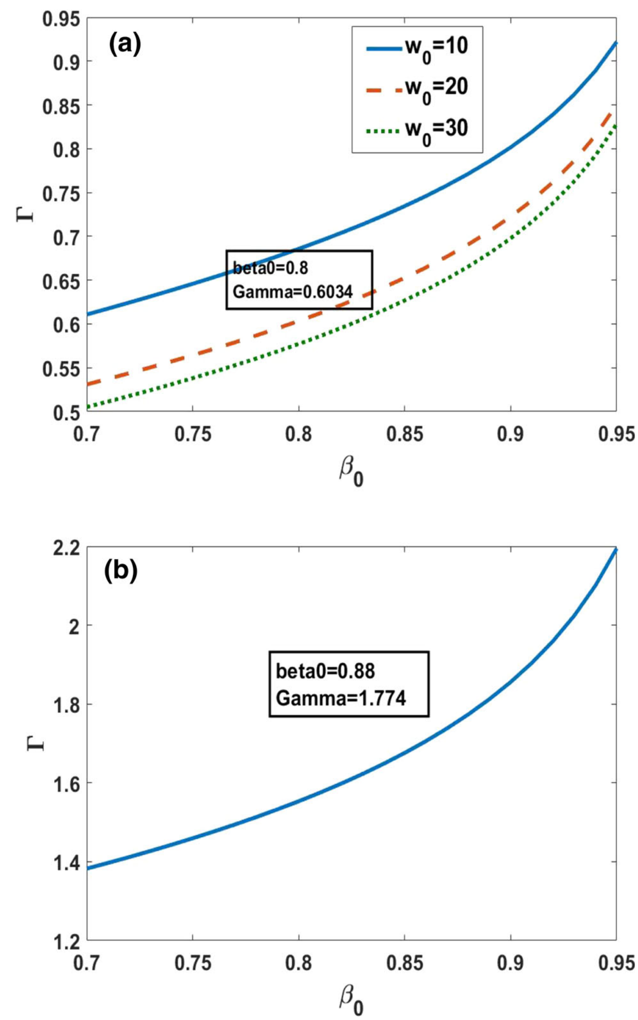

Figure 2. Variation of the modulational instability growth rate with the fluid velocity for (**a**) $k = 0.1$, $a_0 = 0.1$ and (**b**) $k = 0.7$, $a_0 = 0.1$, $w_0 = 30$.

show the consistency of the results, data cursor is used to import some relating points on figures 1–6.

Variation of Γ with the plasma fluid velocity at infinity β_0, is depicted in figure 2. The top panel is plotted for different values of plasma temperature and $k = 0.1$ while the down panel is illustrated for $w_0 = 30$ and $k = 0.7$. It can be seen that by increasing the plasma fluid velocity, the growth rate shows a growing trend for all given values of w_0 and k.

Figure 3 demonstrates the variations of the growth rate as a function of enthalpy for various values of wave numbers. We note that for a certain value of the enthalpy, the growth rate decreases with wave number. Also, for all values of k, as the enthalpy increases, the growth rate decreases.

To demonstrate the impact of wave number, we present four plots in figure 4. The plots are done for $a_0 = 0.1$, $w_0 = 30$ and different values of $k = (0.1$, 0.4, 0.7, and 1). It is notable that, for each arbitrary value of plasma fluid velocity, the growth rate is larger for graphs with larger wave numbers.

The variations of the modulational growth rate with wave amplitude is illustrated explicitly in figure 5. By increasing the wave amplitude, the modulational growth rate increases for both the values of wave numbers ($k = 0.5$ and $k = 1$). Figure 6 is devoted to investigation of simultaneous changes in the modulational growth rate with the enthalpy and the plasma fluid velocity.

In our calculations, the role of plasma enthalpy on the modulational instability growth rate is investigated. The plasma is considered ultrarelativistic consisting of hot electrons and positrons. After the rapid development of laser technology in the last ten years, it is foreseen that sometimes in the future intensities on the order of 10^{26-28} W/cm^2 could be available, entering the range of laser fields where an effective production of electron–positron pairs is expected. So, the present results should

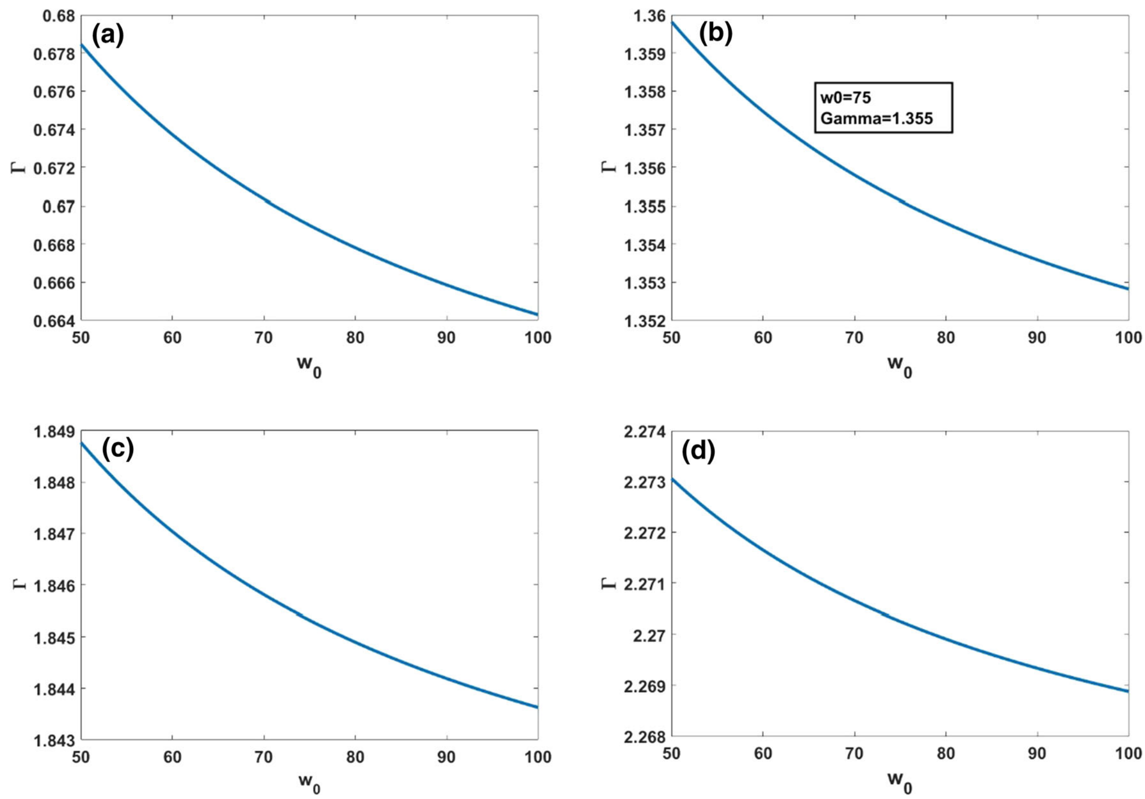

Figure 3. The growth rate as a function of w_0 for $a_0 = 0.1$, $\beta_0 = 0.9$ and (**a**) $k = 0.1$, (**b**) $k = 0.4$, (**c**) $k = 0.7$ and (**d**) $k = 1$.

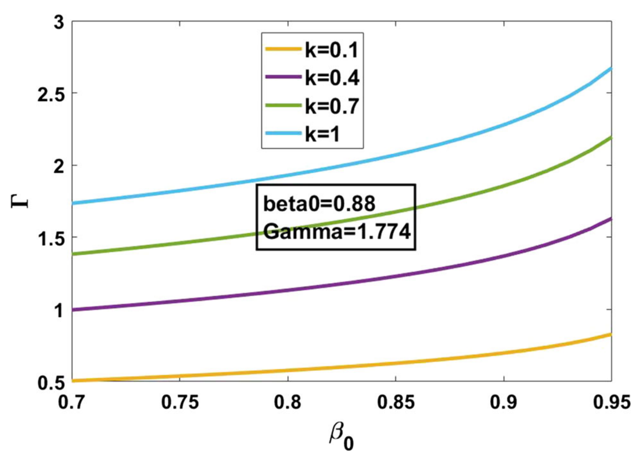

Figure 4. The growth rate as a function of β_0 for $a_0 = 0.1$, $w_0 = 30$.

be useful in understanding the dynamics of intense light pulses in pair plasmas which ought to be created by high-energy laser beams in the forthcoming years. They may also help to understand the origin of localized γ-ray bursts in the fireballs which are composed of electron–positron pairs and radiation [9]. In order to have an appropriate sight about the range of instability growth rate, one can compare the range of the drift wave growth rate which is one of the main conclusions of ref. [17], with the results appeared here. Interested reader can also refer to ref. [9].

4. Summary and conclusion

Using the hydrodynamics equations in a fully relativistic plasma, we derived the transverse component of electromagnetic equation and the plasma density as a function of the scalar and vector potentials, plasma fluid velocity, enthalpy and the solitary wave velocity. we used the quasineutral approximation to obtain the nonlinear dispersion relation of the standing solitary waves in hot ultrarelativistic electron–positron plasma. The variations of the modulational instability growth rate for different values of plasma temperatures, plasma fluid velocities, wave amplitude and wave numbers are investigated. It is found that, the modulational instability growth rate shows a growing trend with enthalpy, wave number and wave amplitude. On the contrary, by increasing the plasma temperature, we observed a decrease in the growth rate.

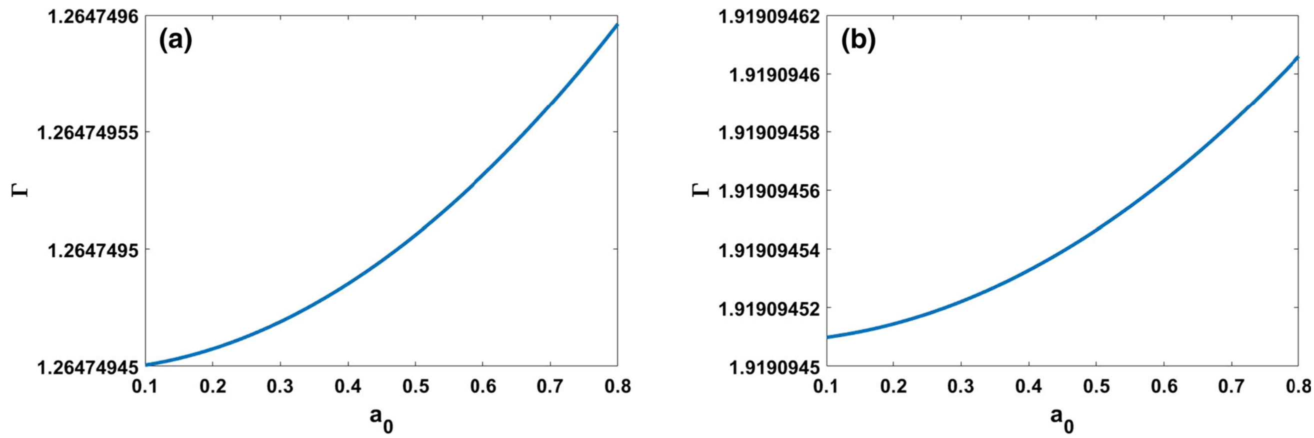

Figure 5. The growth rate as a function of a_0 for fixed values of $\beta_0 = 0.8$, $w_0 = 100$ and (**a**) $k = 0.5$ and (**b**) $k = 1$.

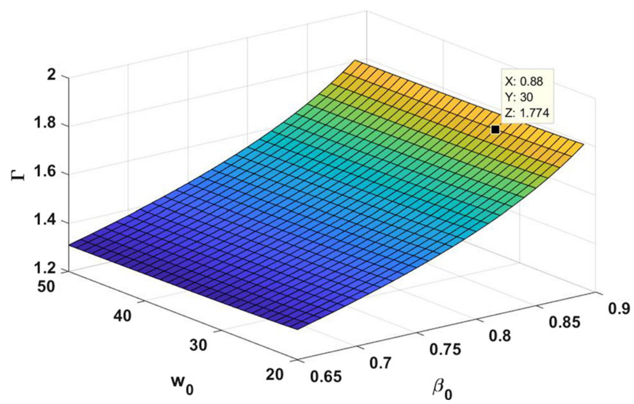

Figure 6. The growth rate as a function of w_0 and β_0 for $a_0 = 0.1$ and $k = 0.7$.

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