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Imprecisely defined fractional-order Fokker–Planck equation subjected to fuzzy uncertainty

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Abstract. Fokker–Planck equation with interval and fuzzy uncertainty has been considered in this paper. Also the derivatives involved with respect to time and space are assumed to be fractional in nature. This problem has been solved using variational iteration method (VIM) along with the double parametric form of fuzzy numbers. For the analysis, both triangular and Gaussian normalised fuzzy sets are taken into consideration. Numerical results for different cases have been obtained and those are depicted in terms of plots and are also compared in special cases for the validation. Moreover, using an important method known as successive approximation method, it has also been verified that the obtained solutions are the same as that of VIM as both methods are equivalent.

Keywords. Triangular fuzzy number; Gaussian fuzzy number; r-cut; double parametric form; Fokker–Planck equation; variational iteration method.

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1. Introduction

The Fokker–Planck equation is very important in the fields of natural science, solid-state physics, quantum optics, theoretical biology, chemical physics, circuit theory etc. By using Fokker and Planck equation, Risken [1] described the Brownian motion of particles.

More works related to space- and time-fractional Fokker–Planck equation can be found in [2–12]. Liu *et al* [2] used Riemann–Liouville and Grunwald–Letnikov definitions of fractional derivatives with the lines method to find numerical solution of space-fractional Fokker–Planck equation. Odibat and Momani [3] applied the Adomian decomposition method for solving the said fractional differential equation. Tatari *et al* [4] implemented Adomian decomposition method for solving Fokker–Planck equation. Finite element method has successfully been applied by Deng [5] to obtain the solution of fractional Fokker–Planck equation. This equation has been solved by Chen *et al* [6] using finite difference approximations. Yildirim [7] successfully applied homotopy perturbation method for solving Fokker–Planck equation with space- and time-fractional derivatives. Iterative Laplace transform has been used by Yan [8] to obtain numerical solution of the fractional Fokker–Planck equations. Gajda and Wyłomańska [9] first studied diffusion process driven by fractional Brownian motion delayed by general infinitely divisible subordinator, Then, they showed that the analysed process is the stochastic representation of the Fokker–Planck-type equation which describes the probability density function of an introduced model. Group analysis and exact solution of fractional Fokker–Planck equation has been analysed by Hashemi [10]. Moreover, Baleanu *et al* [11] applied variational iteration method (VIM) for solving parabolic Fokker–Planck equation. Exact solution of the Fokker–Planck equation for isotropic scattering has been studied by Malkov [12]. Very recently, Firoozjaee *et al* [13,14] implemented various important numerical approaches to study the fractional-order Fokker–Planck equation. In [13], Ritz approximation method has been used and Caputo–Fabrizio fractional derivatives are considered for the analysis. However, Firoozjaee *et al* [13] have considered Fokker–Planck equation with space- and time-fractional and non-factional derivatives. There they have transformed the problem into an optimisation problem to obtain the solution. Convergence of the method has also

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been discussed there. Various other significant procedures for solving fractional and non-fractional order differential equations with applications can be found in [14–18] also.

It can be observed from the aforementioned works that the parameters and initial conditions involved are assumed as crisp/precise in nature or defined exactly. However, this assumption is impractical for many real life situations because it is not always possible to get exact or complete information about all the involved parameters as those found by some experiments, observations etc. So in actual scenario one may have only vague, imprecise or incomplete information about the parameters. Hence, those uncertainties can be modelled using fuzzy parameters. So it is very much important to study and solve fractional differential equations defined with fuzzy uncertainty. In this regard, some recent contributions related to the theory of fuzzy differential equation and fuzzy fractional differential equations can be found in [19–26] and [27–30] respectively.

In this paper, space- and time-fractional fuzzy Fokker–Planck equation in the framework of uncertain probability density $\tilde{v}(z,t)$ of the random variable z at time t has been investigated. VIM [31,32] has been applied to convert the equation and necessary conditions into its double parametric form. The main benefit of this scheme is that, the final solution can be expressed as a power series or in a compact form. One can see Abbaoui and Cherruault [33] for the convergence analysis of VIM. Moreover, it can also be seen that some researchers [34–37] have effectively implemented VIM for solving uncertain differential equations. Moreover, for the thorough understanding of fuzzy set theory, fuzzy numbers and fuzzy computations, one can see [38–40].

This paper is organised as follows. In §2 double parametric representations of the considered problem has been addressed. Section 3 presents the general solution using the proposed technique. Followed by this, various cases for different values of the functions involved in the equations are obtained in §4. Section 5 gives numerical results along with discussions. Finally, conclusions are drawn in §6.

2. Double parametric based fuzzy space- and time-fractional Fokker–Planck equation

Fuzzy space- and time-fractional Fokker–Planck equation is first converted to an interval-based equation using r-cut [41] form. Then, the interval-based equation again is converted to a crisp form using double parametric form of fuzzy numbers [41,42]. Then VIM has been applied to achieve the solution.

Let us consider the following fuzzy space- and time-fractional Fokker–Planck equation:

$$\frac{\partial^\alpha \tilde{v}}{\partial t^\alpha} = \left[-\frac{\partial^\gamma}{\partial z^\gamma}A(z) + \frac{\partial^{2\gamma}}{\partial z^{2\gamma}}B(z)\right]\tilde{v}(z,t),$$
$$\text{where } t>0,\ z>0,\ 0<\alpha\le 1 \text{ and } 0<\gamma\le 1. \tag{1}$$

Applying $\frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}$ on both sides of eq. (1) equivalently, the above equation can be expressed as

$$\frac{\partial \tilde{v}}{\partial t} = \left[-\frac{\partial^{\gamma-\alpha+1}}{\partial t^{-\alpha+1}\partial z^\gamma}A(z) + \frac{\partial^{2\gamma-\alpha+1}}{\partial t^{-\alpha+1}\partial z^{2\gamma}}B(z)\right] \times\tilde{v}(z,t), \tag{2}$$

where $t>0,\ z>0,\ 0<\alpha\le 1$ and $0<\gamma\le 1$

with fuzzy initial condition

$$\tilde{v}(z,0) = \tilde{\delta}z, \tag{3}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$, $\frac{\partial^\gamma}{\partial z^\gamma}$ and $\frac{\partial^{2\gamma}}{\partial z^{2\lambda}}$ are the Caputo derivatives (as defined below) of order α and γ. Here $B(z)>0$ and $A(z)$ are the diffusion and the drift coefficient respectively. $B(z)$ and $A(z)$ may also depend on time. The function $\tilde{v}(z,t)$ is supposed to be an uncertain function of time and space, which is vanishing for $t<0$ and $z<0$.

DEFINITION 2.1

Caputo derivative [43]

The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha}D^m f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^m(\tau)\mathrm{d}\tau}{(t-\tau)^{\alpha+1-m}}, \\ m-1<\alpha<m,\ m\in N \\ \frac{\mathrm{d}^m}{\mathrm{d}t^m}f(t),\ \alpha=m,\ m\in N, \end{cases}$$

where the parameter α is the order of the derivative and is allowed to be real or complex. In this paper, only real and positive α has been considered. For the Caputo's derivative we have

$$D^\alpha C = 0, \text{ where } C \text{ is a constant},$$

$$D^\alpha t^\beta = \begin{cases} 0,\ (\beta\leqslant\alpha-1) \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}\ (\beta>\alpha-1). \end{cases}$$

Here $J^{m-\alpha}$ is known as the Riemann–Liouville integral operator [43] of order $m-\alpha\ge 0$.

Next the above fuzzy fractional Fokker–Planck equation (2) can be written in r-cut form as

$$\left[\frac{\partial \underline{v}(z,t;r)}{\partial t}, \frac{\partial \bar{v}(z,t;r)}{\partial t}\right]$$
$$= -\left[\frac{\partial^{\gamma-\alpha+1}\underline{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{\gamma}}A(z), \frac{\partial^{\gamma-\alpha+1}\bar{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{\gamma}}A(z)\right]$$
$$+\left[\frac{\partial^{2\gamma-\alpha+1}\underline{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{2\gamma}}B(z), \frac{\leq\partial^{2\gamma-\alpha+1}\bar{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{2\gamma}}B(z)\right], \tag{4}$$

subjected to the r-cut form of fuzzy initial condition as

$$\left[\underline{v}(z,0;r), \bar{v}(z,0;r)\right] = \left[\underline{\delta}(r), \bar{\delta}(r)\right] z.$$

Using the double parametric form [41,42], eq. (4) can be expressed as

$$\left\{\beta\left(\frac{\partial \bar{v}(z,t;r)}{\partial t} - \frac{\partial \underline{v}(z,t;r)}{\partial t}\right) + \frac{\partial \underline{v}(z,t;r)}{\partial t}\right\}$$
$$= -A(z)\left\{\beta\left(\frac{\partial^{\gamma-\alpha+1}\bar{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{\gamma}} - \frac{\partial^{\gamma-\alpha+1}\underline{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{\gamma}}\right) + \frac{\partial^{\gamma-\alpha+1}\underline{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{\gamma}}\right\}$$
$$+B(z)\left\{\beta\left(\frac{\partial^{2\gamma-\alpha+1}\bar{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{2\gamma}} - \frac{\partial^{2\gamma-\alpha+1}\underline{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{2\gamma}}\right) + \frac{\partial^{2\gamma-\alpha+1}\underline{v}(z,t;r)}{\partial t^{-\alpha+1}\partial z^{2\gamma}}\right\} \tag{5}$$

subject to the fuzzy initial condition

$$\left\{\beta\left(\bar{v}(z,0;r) - \underline{v}(z,0;r)\right) + \underline{v}(z,0;r)\right\}$$
$$= \left\{\beta\left(\bar{\delta}(r) - \underline{\delta}(r)\right) + \underline{\delta}(r)\right\} z \text{ where } r, \beta \in [0,1].$$

Let us now denote

$$\left\{\beta\left(\frac{\partial \bar{v}(z,t;r)}{\partial t} - \frac{\partial \underline{v}(z,t;r)}{\partial t}\right) + \frac{\partial \underline{v}(z,t;r)}{\partial t}\right\} = \frac{\partial \tilde{v}(z,t;r,\beta)}{\partial t},$$

$$\left\{\beta\left(\frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{\gamma}\bar{v}(z,t;r)}{\partial z^{\gamma}} - \frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{\gamma}\underline{v}(z,t;r)}{\partial z^{\gamma}}\right) + \frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{\gamma}\underline{v}(z,t;r)}{\partial z^{\gamma}}\right\}$$
$$= \frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{\gamma}\tilde{v}(z,t;r,\beta)}{\partial z^{\gamma}},$$

$$\left\{\beta\left(\frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{2\gamma}\bar{v}(z,t;r)}{\partial z^{2\gamma}} - \frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{2\gamma}\underline{v}(z,t;r)}{\partial z^{2\gamma}}\right) + \frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{2\gamma}\underline{v}(z,t;r)}{\partial z^{2\gamma}}\right\}$$
$$= \frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{2\gamma}\tilde{v}(z,t;r,\beta)}{\partial z^{2\gamma}},$$

$$\left\{\beta\left(\bar{v}(z,0;r) - \underline{v}(z,0;r)\right) + \underline{v}(z,0;r)\right\} = \tilde{v}(z,0;r,\beta)$$

and

$$\left\{\beta\left(\bar{\delta}(r) - \underline{\delta}(r)\right) + \underline{\delta}(r)\right\} = \tilde{\delta}(r,\beta).$$

Substituting the above terms in eq. (5) gives

$$\frac{\partial \tilde{v}(z,t;r,\beta)}{\partial t} = -A(z)\frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{\gamma}\tilde{v}(z,t;r,\beta)}{\partial z^{\gamma}} + B(z)\frac{\partial^{-\alpha+1}}{\partial t^{-\alpha+1}}\frac{\partial^{2\gamma}\tilde{v}(z,t;r,\beta)}{\partial z^{2\gamma}} \tag{6}$$

with initial condition

$$\tilde{v}(z,0;r,\beta) = \tilde{\delta}(r,\beta)z.$$

Now eq. (6) can be solved using VIM to obtain $\tilde{v}(z,t;r,\beta)$. Substituting $\beta = 0$ and 1 the lower and upper bounds of the solution respectively are obtained as

$$\tilde{v}(z,t;r,0) = \underline{v}(z,t;r) \text{ and } \tilde{v}(z,t;r,1) = \bar{v}(z,t;r).$$

In the next section VIM has been applied to solve eq. (6).

3. Solution to the fuzzy fractional Fokker–Planck equation using VIM

According to VIM one may first construct a correction functional as

$$\tilde{v}_{n+1}(z,t;r,\beta) = \tilde{v}_n(z,t;r,\beta) + \int_0^t \lambda(\tau) \times \left\{\begin{array}{l} \frac{\partial}{\partial \tau}\tilde{v}_n(z,\tau;r,\beta) + A(z)\frac{\partial^{-\alpha+1}}{\partial \tau^{-\alpha+1}} \\ \times \frac{\partial^{\gamma}}{\partial z^{\gamma}}\hat{\tilde{v}}_n(z,\tau;r,\beta) \\ -B(z)\frac{\partial^{-\alpha+1}}{\partial \tau^{-\alpha+1}}\frac{\partial^{2\gamma}}{\partial z^{2\gamma}}\hat{\tilde{v}}_n(z,\tau;r,\beta) \end{array}\right\} d\tau. \tag{7}$$

Making the above expression stationary, and observing $\delta\hat{\tilde{v}}_n = 0$, we have

$$\delta\tilde{v}_{n+1}(z,t;r,\beta) = \delta\tilde{v}_n(z,t;r,\beta) + \delta\int_0^t \lambda(\tau) \left\{\begin{array}{l} \frac{\partial}{\partial \tau}\tilde{v}_n(z,\tau;r,\beta) + A(z)\frac{\partial^{-\alpha+1}}{\partial \tau^{-\alpha+1}}\frac{\partial^{\gamma}}{\partial z^{\gamma}}\hat{\tilde{v}}_n(z,\tau;r,\beta) \\ -B(z)\frac{\partial^{-\alpha+1}}{\partial \tau^{-\alpha+1}}\frac{\partial^{2\gamma}}{\partial z^{2\gamma}}\hat{\tilde{v}}_n(z,\tau;r,\beta) \end{array}\right\} d\tau$$
$$= \delta\tilde{v}_n(z,t;r,\beta) + \lambda(\tau)\delta\tilde{v}(z,t;r,\beta) - \int_0^t \lambda'(\tau)\delta\tilde{v}_n(z,\tau;r,\beta)d\tau = 0$$
$$= (1+\lambda(\tau))\delta\tilde{v}_n(z,t;r,\beta)$$

$$-\int_0^t \lambda'(\tau)\delta\tilde{v}_n(z,\tau;r,\beta)\mathrm{d}\tau = 0.$$

Thus, we obtain the Euler–Lagrange equation

$$\lambda'(\tau) = 0 \tag{8}$$

with natural boundary

$$1 + \lambda(\tau) = 0. \tag{9}$$

So, the Lagrange multiplier can be easily identified as follows:

$$\lambda = -1.$$

Replacing the recognised Lagrange multiplier into eq. (6), the following variational iteration formula can be achieved:

$$\tilde{v}_{n+1}(z,t;r,\beta)=\tilde{v}_n(z,t;r,\beta) - \int_0^t \left\{ \begin{array}{l} \dfrac{\partial}{\partial\tau}\tilde{v}_n(z,\tau;r,\beta)+A(z) \\ \times\dfrac{\partial^{-\alpha+1}}{\partial\tau^{-\alpha+1}}\dfrac{\partial^{\gamma}}{\partial z^{\gamma}}\tilde{v}_n(z,\tau;r,\beta) \\ -B(z)\dfrac{\partial^{-\alpha+1}}{\partial\tau^{-\alpha+1}}\dfrac{\partial^{2\gamma}}{\partial z^{2\gamma}}\tilde{v}_n(z,\tau;r,\beta) \end{array} \right\}\mathrm{d}\tau. \tag{10}$$

It is interesting to mention that using successive approximation method (SAM) [18] we have obtained the successive iteration formula which is the same as the above variational iteration formula obtained by VIM. So one can conclude that both the methods are equivalent. Accordingly, it has been verified for different cases as below to show that the solutions obtained by both the methods are the same.

4. Case studies

Case A: Let us consider

$$\alpha = 1,\ A(z) = z \text{ and } B(z) = \frac{z^2}{2}$$

with triangular fuzzy initial condition in parametric form as $\tilde{v}(z,\ 0; r) = [0.1r + 0.9,\ 1.1 - 0.1r]z$. Hence eq. (2) will become

$$\frac{\partial\tilde{v}}{\partial t} = \left[-\frac{\partial^{\gamma}}{\partial z^{\gamma}}z + \frac{\partial^{2\gamma}}{\partial z^{2\gamma}}\frac{z^2}{2}\right]\tilde{v}(z,t), \tag{11}$$

where $t > 0,\ x > 0,\ 0 < \alpha \leqslant 1$ and $0 < \gamma \leqslant 1$.

Using double parametric form, eq. (6) and the corresponding fuzzy initial condition will become

$$\frac{\partial\tilde{v}(z,t;r,\beta)}{\partial t} = \left[-\frac{\partial^{\gamma}}{\partial z^{\gamma}}z + \frac{\partial^{2\gamma}}{\partial z^{2\gamma}}\frac{z^2}{2}\right]\tilde{v}(z,t;r,\beta)$$

with initial condition

$$\tilde{v}(z,\ 0; r,\beta) = \beta(0.2 - 0.2r) + (0.1r + 0.9)z.$$

Applying VIM, initial approximation can be written as

$$\tilde{v}_0(z,t;r,\beta) = \{\beta(0.2 - 0.2r) + (0.1r + 0.9)\}\,z,$$

and using the variational iteration formula (10) for this case, we have

$$\tilde{v}_1(z,t;r,\beta) = \{\beta(0.2 - 0.2r) + (0.1r + 0.9)\} \times \left(z + t\left[\frac{3z^{3-2\gamma}}{\Gamma(4-2\gamma)} - \frac{2z^{2-\gamma}}{\Gamma(3-\gamma)}\right]\right), \tag{12}$$

$$\tilde{v}_2(z,t;r,\beta) = \{\beta(0.2 - 0.2r) + (0.1r + 0.9)\} \times \left[\begin{array}{l} z + \left(\dfrac{3z^{3-2\gamma}}{\Gamma(4-2\gamma)} - \dfrac{2z^{2-\gamma}}{\Gamma(3-\gamma)}\right)t \\ + \left\{ \begin{array}{l} \dfrac{2\Gamma(4-\gamma)z^{3-2\gamma}}{\Gamma(3-\gamma)\Gamma(4-2\gamma)} \\ -\left(\dfrac{3\Gamma(5-2\gamma)}{\Gamma(4-2\gamma)} + \dfrac{\Gamma(5-\gamma)}{\Gamma(3-\gamma)}\right) \\ \times\dfrac{z^{4-3\gamma}}{\Gamma(5-3\gamma)} \\ +\dfrac{3}{2}\dfrac{\Gamma(6-2\gamma)z^{5-4\gamma}}{\Gamma(4-2\gamma)\Gamma(6-4\gamma)} \end{array} \right\}\dfrac{t^2}{2} \end{array} \right] \tag{13}$$

and so on.

Similarly, higher-order approximation can be obtained using eq. (10). Here the approximate solution may be taken as

$$\tilde{v}(z,t;r,\beta)=\tilde{v}_2(z,t;r,\beta) = \{\beta(0.2-0.2r)+(0.1r+0.9)\} \times \left[\begin{array}{l} z + \left(\dfrac{3z^{3-2\gamma}}{\Gamma(4-2\gamma)} - \dfrac{2z^{2-\gamma}}{\Gamma(3-\gamma)}\right)t \\ + \left\{ \begin{array}{l} \dfrac{2\Gamma(4-\gamma)z^{3-2\gamma}}{\Gamma(3-\gamma)\Gamma(4-2\gamma)} \\ -\left(\dfrac{3\Gamma(5-2\gamma)}{\Gamma(4-2\gamma)} + \dfrac{\Gamma(5-\gamma)}{\Gamma(3-\gamma)}\right) \\ \times\dfrac{z^{4-3\gamma}}{\Gamma(5-3\gamma)} \\ +\dfrac{3}{2}\dfrac{\Gamma(6-2\gamma)z^{5-4\gamma}}{\Gamma(4-2\gamma)\Gamma(6-4\gamma)} \end{array} \right\}\dfrac{t^2}{2} \end{array} \right]. \tag{14}$$

Substituting $\beta = 0$ and 1, the lower and upper bounds of the solution can be expressed respectively as

$$\underline{v}(z,t;r,0) = \underline{v}_2(z,t;r,0) = (0.1r+0.9) \times \left[z + \left(\frac{3z^{3-2\gamma}}{\Gamma(4-2\gamma)} - \frac{2z^{2-\gamma}}{\Gamma(3-\gamma)} \right) t \left\{ + \frac{2\Gamma(4-\gamma)z^{3-2\gamma}}{\Gamma(3-\gamma)\Gamma(4-2\gamma)} - \left(\frac{3\Gamma(5-2\gamma)}{\Gamma(4-2\gamma)} + \frac{\Gamma(5-\gamma)}{\Gamma(3-\gamma)} \right) \times \frac{z^{4-3\gamma}}{\Gamma(5-3\gamma)} + \frac{3}{2} \frac{\Gamma(6-2\gamma)z^{5-4\gamma}}{\Gamma(4-2\gamma)\Gamma(6-4\gamma)} \right\} \frac{t^2}{2} \right] \tag{15}$$

and

$$\bar{v}(z,t;r,1) = \bar{v}_2(z,t;r,1) = (1.1-0.1r) \left[z + \left(\frac{3z^{3-2\gamma}}{\Gamma(4-2\gamma)} - \frac{2z^{2-\gamma}}{\Gamma(3-\gamma)} \right) t \times + \left\{ \frac{2\Gamma(4-\gamma)z^{3-2\gamma}}{\Gamma(3-\gamma)\Gamma(4-2\gamma)} - \left(\frac{3\Gamma(5-2\gamma)}{\Gamma(4-2\gamma)} + \frac{\Gamma(5-\gamma)}{\Gamma(3-\gamma)} \right) \times \frac{z^{4-3\gamma}}{\Gamma(5-3\gamma)} + \frac{3}{2} \frac{\Gamma(6-2\gamma)z^{5-4\gamma}}{\Gamma(4-2\gamma)\Gamma(6-4\gamma)} \right\} \frac{t^2}{2} \right]. \tag{16}$$

As discussed already, this problem has also been solved by SAM [18] and as per this, applying $L^{-1}[\cdot]$ on both sides of eq. (11) gives

$$\tilde{v}(z,t;r,\beta) = \tilde{v}(z,0;r,\beta) + \int_0^t \left(-\frac{\partial^\gamma}{\partial z^\gamma} z + \frac{\partial^{2\gamma}}{\partial z^{2\gamma}} \frac{z^2}{2} \right) \times \tilde{v}(z,t;r,\beta) \mathrm{d}\tau.$$

For the zeroth approximation we have

$$\tilde{v}_0(z,t;r,\beta) = \{\beta(0.2-0.2r) + (0.1r+0.9)\} z.$$

The SAM gives successive iteration formula for this case as

$$\tilde{v}_{n+1}(z,t;r,\beta) = \tilde{v}_0(z,t;r,\beta) z + \int_0^t \left(-\frac{\partial^\gamma}{\partial z^\gamma} z + \frac{\partial^{2\gamma}}{\partial z^{2\gamma}} \frac{z^2}{2} \right) \times \tilde{v}_n(z,t;r,\beta) \mathrm{d}\tau. \tag{17}$$

Substituting

$$\tilde{v}_0(z,t;r,\beta) = \{\beta(0.2-0.2r) + (0.1r+0.9)\} z$$

into the above equation we obtain

$$\tilde{v}_1(z,t;r,\beta) = \{\beta(0.2-0.2r) + (0.1r+0.9)\} \times \left(z + t \left[\frac{3z^{3-2\gamma}}{\Gamma(4-2\gamma)} - \frac{2z^{2-\gamma}}{\Gamma(3-\gamma)} \right] \right),$$

$$\tilde{v}_2(z,t;r,\beta) = \{\beta(0.2-0.2r) + (0.1r+0.9)\} \times \left[z + \left(\frac{3z^{3-2\gamma}}{\Gamma(4-2\gamma)} - \frac{2z^{2-\gamma}}{\Gamma(3-\gamma)} \right) t + \left\{ \frac{2\Gamma(4-\gamma)z^{3-2\gamma}}{\Gamma(3-\gamma)\Gamma(4-2\gamma)} - \left(\frac{3\Gamma(5-2\gamma)}{\Gamma(4-2\gamma)} + \frac{\Gamma(5-\gamma)}{\Gamma(3-\gamma)} \right) \times \frac{z^{4-3\gamma}}{\Gamma(5-3\gamma)} + \frac{3}{2} \frac{\Gamma(6-2\gamma)z^{5-4\gamma}}{\Gamma(4-2\gamma)\Gamma(6-4\gamma)} \right\} \frac{t^2}{2} \right]$$

and so on. Hence we may write the approximate solution as

$$\tilde{v}(z,t;r,\beta) = \tilde{v}_2(z,t;r,\beta).$$

It can be seen that this solution is the same as that of the solution obtained by VIM. This is because successive iteration formula obtained in eq. (17) is the same as the variational iteration formula obtained by VIM in this case. And it can be verified by substituting the values

$$A(z) = z \quad \text{and} \quad B(z) = \frac{z^2}{2}$$

in the generalised variational iteration formula, i.e. eq. (10), and after simplification one can get

$$\tilde{v}_{n+1}(z,t;r,\beta) = \tilde{v}_n(z,t;r,\beta) - \tilde{v}_n(z,t;r,\beta) + \tilde{v}_n(z,0;r,\beta) + \int_0^t \left\{ -\frac{\partial^\gamma}{\partial z^\gamma} z + \frac{\partial^{2\gamma}}{\partial z^{2\gamma}} \frac{z^2}{2} \right\} \times \tilde{v}_n(z,\tau;r,\beta) \mathrm{d}\tau.$$

Now using the double parametric form of fuzzy initial condition as represented above

$$\tilde{v}(z,\ 0;r,\beta) = \beta(0.2-0.2r) + (0.1r+0.9) z$$

we have

$$\tilde{v}_n(z,0;r,\beta) = \{\beta(0.2-0.2r) + (0.1r+0.9)\} z.$$

Thus, the above equation leads to

$$\tilde{v}_{n+1}(z,t;r,\beta) = \{\beta(0.2-0.2r) + (0.1r+0.9)\} z + \int_0^t \left\{ -\frac{\partial^\gamma}{\partial z^\gamma} z + \frac{\partial^{2\gamma}}{\partial z^{2\gamma}} \frac{z^2}{2} \right\} \times \hat{\tilde{v}}_n(z,\tau;r,\beta) \mathrm{d}\tau. \tag{18}$$

Now it can clearly be seen that eqs (17) and (18) are exactly the same.

Moreover, it can be observed that crisp results obtained by the present analysis (for $r = 1$) exactly matches with Yildirim [7]. Setting $r = 1$ and $\gamma = 1$ in eqs (15) and (16), one may get the solution of the above problem as

$$v(z, t;) = z\left[1 + t + \frac{t^2}{2} + \cdots\right]. \tag{19}$$

The above solution can be written in closed form as

$$v(z, t;) = z\mathrm{e}^t. \tag{20}$$

Case B: Next, we consider

$$A(z) = \frac{z}{6} \quad \text{and} \quad B(x) = \frac{z^2}{12}$$

with Gaussian fuzzy initial condition viz.

$$\tilde{v}(z, 0; r) = z^2\left[\beta\left(0.2\sqrt{-2\log_e r}\right) + (1 - 0.1\sqrt{-2\log_e r})\right].$$

Again, using the above procedure we have

$$\tilde{v}_0(z, t; r, \beta) = z^2\left\{\beta\left(0.2\sqrt{-2\log_e r}\right) + (1-0.1\sqrt{-2\log_e r})\right\}, \tag{21}$$

$$\tilde{v}_1(z, t; r, \beta) = \left\{\beta\left(0.2\sqrt{-2\log_e r}\right) + (1-0.1\sqrt{-2\log_e r})\right\} \times\left(z^2 + \frac{2x^{4-2\gamma}}{\Gamma(5-2\gamma)} - \frac{z^{3-\gamma}}{\Gamma(4-\gamma)}\right)\frac{t^\alpha}{\Gamma(\alpha+1)} \tag{22}$$

$$\tilde{v}_2(z, t; r, \beta) = \left\{\beta\left(0.2\sqrt{-2\log_e r}\right) + (1 - 0.1\sqrt{-2\log_e r})\right\} \times\left[\begin{array}{l} z^2 + \left(\dfrac{2z^{4-2\gamma}}{\Gamma(5-2\gamma)} - \dfrac{z^{3-\gamma}}{\Gamma(4-\gamma)}\right)\dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ + \left(\begin{array}{l} \dfrac{\Gamma(5-\gamma)}{6\Gamma(4-\gamma)}\dfrac{z^{4-2\gamma}}{\Gamma(5-2\gamma)} \\ -\left(\begin{array}{l}\dfrac{\Gamma(6-2\gamma)}{3\Gamma(5-2\gamma)} \\ +\dfrac{1}{12}\dfrac{\Gamma(6-\gamma)}{\Gamma(4-\gamma)}\end{array}\right)\dfrac{z^{5-3\gamma}}{\Gamma(6-3\gamma)} \\ +\dfrac{1}{6}\dfrac{\Gamma(7-2\gamma)z^{6-4\gamma}}{\Gamma(5-2\gamma)\Gamma(7-4\gamma)} \end{array}\right)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{array}\right] \tag{23}$$

and so on.

As such, the generalised form can be written as

$$\tilde{v}(z, t; r, \beta) = \tilde{v}_2(z, t; r, \beta) = \left\{\beta\left(0.2\sqrt{-2\log_e r}\right) + (1 - 0.1\sqrt{-2\log_e r})\right\} \times\left[\begin{array}{l} z^2 + \left(\dfrac{2z^{4-2\gamma}}{\Gamma(5-2\gamma)} - \dfrac{z^{3-\gamma}}{\Gamma(4-\gamma)}\right)\dfrac{t^\alpha}{\Gamma(\alpha+1)} \\ + \left(\begin{array}{l} \dfrac{\Gamma(5-\gamma)}{6\Gamma(4-\gamma)}\dfrac{z^{4-2\gamma}}{\Gamma(5-2\gamma)} \\ -\left(\begin{array}{l}\dfrac{\Gamma(6-2\gamma)}{3\Gamma(5-2\gamma)} \\ +\dfrac{1}{12}\dfrac{\Gamma(6-\gamma)}{\Gamma(4-\gamma)}\end{array}\right)\dfrac{z^{5-3\gamma}}{\Gamma(6-3\gamma)} \\ +\dfrac{1}{6}\dfrac{\Gamma(7-2\gamma)z^{6-4\gamma}}{\Gamma(5-2\gamma)\Gamma(7-4\gamma)} \end{array}\right)\dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{array}\right]. \tag{24}$$

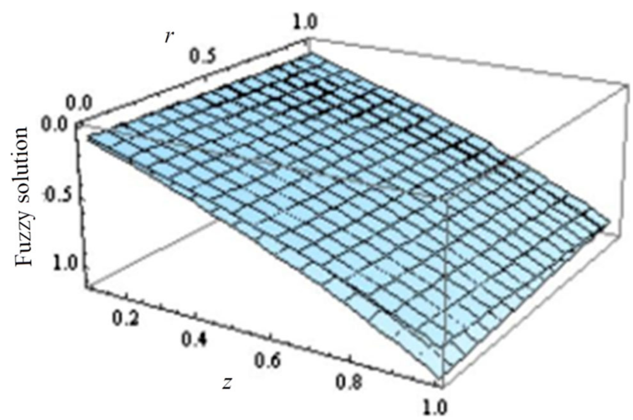

Figure 1. Fuzzy solution for Case A by varying z from 0.1 to 1 when $\gamma = 0.5$, $\alpha = 1$ and $t = 0.5$.

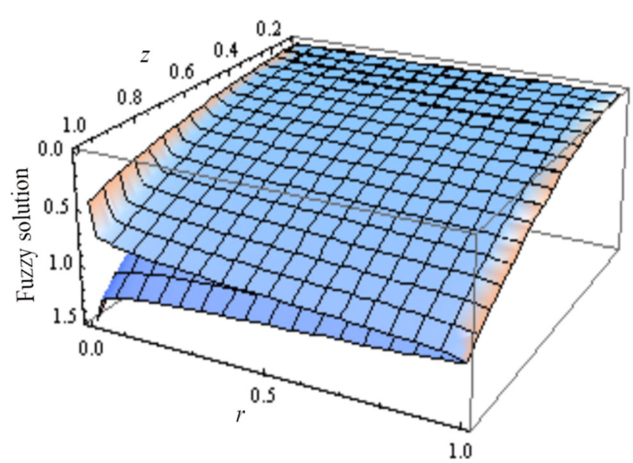

Figure 2. Fuzzy solution for Case B by varying z from 0.1 to 1 when $\gamma = 0.5$, $\alpha = 1$ and $t = 0.5$.

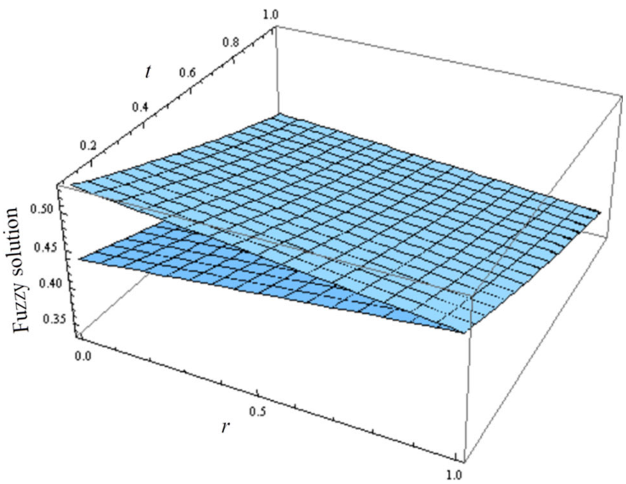

Figure 3. Fuzzy solution for Case A by varying t from 0.1 to 1 when $\gamma = 0.5$, $\alpha = 1$ and $z = 0.5$.

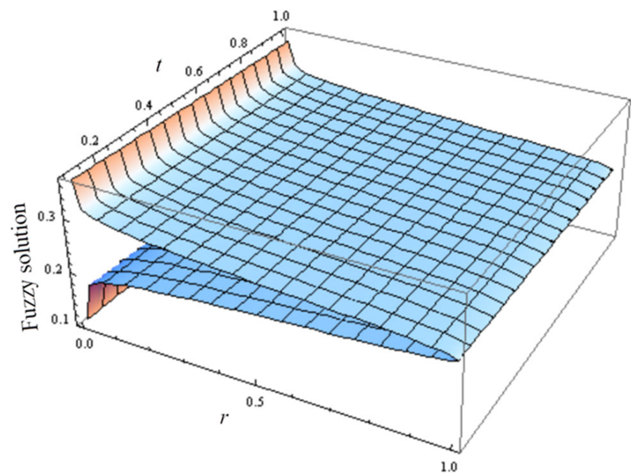

Figure 4. Fuzzy solution for Case B by varying t from 0.1 to 1 when $\gamma = 0.5$, $\alpha = 1$ and $z = 0.5$.

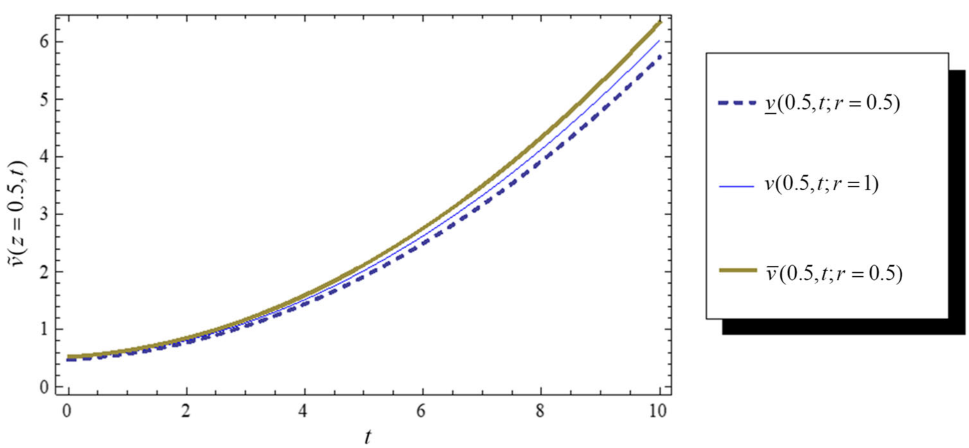

Figure 5. Interval result for Case A when $z = 0.5$, $\gamma = 0.75$, $\alpha = 1$ and t varying from 0 to 10.

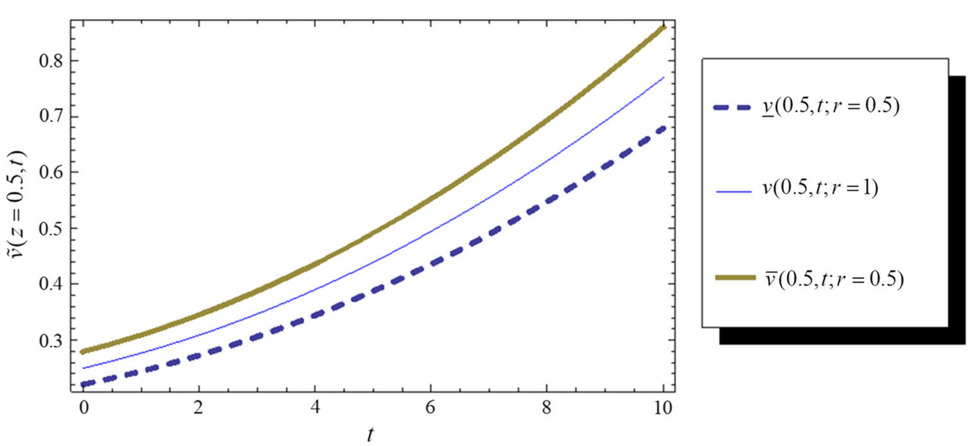

Figure 6. Interval result for Case B when $z = 0.5$, $\gamma = 0.75$, $\alpha = 1$ and t varying from 0 to 10.

For $\beta = 0$ and 1 in $\tilde{v}(z, t; r, \beta)$ we get the lower and upper bounds of the fuzzy solutions respectively as

$$\underline{v}(z, t; r, 0) = \underline{v}_2(z, t; r, 0) = (1 - 0.1\sqrt{-2\log_e r}) \times \left[z^2 + \left(\frac{2z^{4-2\gamma}}{\Gamma(5-2\gamma)} - \frac{z^{3-\gamma}}{\Gamma(4-\gamma)} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} + \left(\frac{\Gamma(5-\gamma)}{6\Gamma(4-\gamma)} \frac{z^{4-2\gamma}}{\Gamma(5-2\gamma)} - \left(\frac{\Gamma(6-2\gamma)}{3\Gamma(5-2\gamma)} + \frac{1}{12} \frac{\Gamma(6-\gamma)}{\Gamma(4-\gamma)} \right) \frac{z^{5-3\gamma}}{\Gamma(6-3\gamma)} + \frac{1}{6} \frac{\Gamma(7-2\gamma) z^{6-4\gamma}}{\Gamma(5-2\gamma)\Gamma(7-4\gamma)} \right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right] \tag{25}$$

and

$$\bar{v}(z, t; r, 1) = \bar{v}_2(z, t; r, 1) = (1 + 0.1\sqrt{-2\log_e r}) \times \left[z^2 + \left(\frac{2z^{4-2\gamma}}{\Gamma(5-2\gamma)} - \frac{z^{3-\gamma}}{\Gamma(4-\gamma)} \right) \frac{t^\alpha}{\Gamma(\alpha+1)} + \left(\frac{\Gamma(5-\gamma)}{6\Gamma(4-\gamma)} \frac{z^{4-2\gamma}}{\Gamma(5-2\gamma)} - \left(\frac{\Gamma(6-2\gamma)}{3\Gamma(5-2\gamma)} + \frac{1}{12} \frac{\Gamma(6-\gamma)}{\Gamma(4-\gamma)} \right) \frac{z^{5-3\gamma}}{\Gamma(6-3\gamma)} + \frac{1}{6} \frac{\Gamma(7-2\gamma) z^{6-4\gamma}}{\Gamma(5-2\gamma)\Gamma(7-4\gamma)} \right) \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right]. \tag{26}$$

In this case also for $r = 1$ (crisp result), the present result is again found to be exactly the same as that of Yildirim [7]. In this case also SAM [18] gives the same solution as VIM.

5. Results and discussions

Numerical results for fuzzy fractional Fokker–Planck equation with different $A(z)$, $B(z)$ and fuzzy initial conditions are computed by VIM. The results obtained by the present analysis are compared with the existing solution of Yildirim [7] in special cases to show the validation of the proposed analysis.

Triangular fuzzy solutions for Case A and Gaussian fuzzy solutions for Case B are depicted in figures 1 and 2 respectively by varying z from 0.1 to 1 for $\gamma = 0.5$, $\alpha = 1$ and $t = 0.5$. Similarly, fuzzy solutions for both the cases obtained by varying the time from 0.1 to 1 for $\gamma = 0.5$, $\alpha = 1$ and $z = 0.5$ are given in figures 3 and 4. It may be worth mentioning that for all the cases, the present results exactly agree with the solution of Yildirim [7] in special case of $r = 1$. Interval solution plots (where $r = 0.5$ and 1) for both Cases A and B are given in figures 5 and 6 respectively by varying t from 0 to 10 for $z = 0.5$, $\gamma = 0.75$ and $\alpha = 1$. It is interesting to note that in both the cases, the bounds of the uncertain velocity of a particle under the influence of drag forces and random forces gradually increases with the increase in time.

6. Conclusions

In this paper, VIM-based fuzzy solution has successfully been obtained for fuzzy fractional-order Fokker–Planck equation. The double parametric form approach along with VIM is found to be easy and straightforward. This does not split the original fuzzy differential equations into sets of equations. This reduces the computational cost. Here, performance of the present method has been shown by considering the initial condition in terms of triangular as well as Gaussian fuzzy numbers. It is interesting to note for $r = 1$ in both the cases that the lower and upper solutions are equal and found to be in good agreement with the existing work of [7]. Moreover, it has also been verified that using SAM [18] the obtained results are the same as that of the solutions obtained by VIM [31,32] for different cases as both the methods are similar.

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References

[1] H Risken, *The Fokker–Planck equation: Method of solution and applications* (Springer, Berlin, 1989)
[2] F Liu, V Anh and I Turner, *J. Comput. Appl. Math.* **166**, 209 (2004)
[3] Z Odibat and S Momani, *Phys. Lett. A* **369**, 349 (2007)
[4] M Tatari, M Dehghan and M Razzaghi, *Math. Comput. Model.* **45**, 639 (2007)
[5] W Deng, *SIAM J. Numer. Anal.* **47**, 204 (2008)
[6] S Chen, F Liu, P Zhuang and V Anh, *Appl. Math. Model.* **33**, 256 (2009)
[7] A Yildirim, *J. King Saud Univ.* **22**, 257 (2010)
[8] L Yan, *Abstr. Appl. Anal.* **2013**, 1 (2013)
[9] J Gajda and A Wyłomańska, *Phys. A Stat. Mech. Appl.* **405**, 104 (2014)
[10] M S Hashemi, *Physica A* **417**, 141 (2015)
[11] D Baleanu, H M Srivastava and X J Yang, *Prog. Fract. Differ. Appl.* **1**, 1 (2015)

[12] M A Malkov, *Phys. Rev. D* **95**, 023007-1 (2017)
[13] M A Firoozjaee, H Jafari, A Lia and D Baleanu, *J. Comput. Appl. Math.* **339**, 367 (2018)
[14] M A Firoozjaee, S A Yousefia and H Jafari, *MATCH Commun. Math. Comput. Chem.* **74**, 449 (2015)
[15] H Jafari and H Tajadodi, *Therm. Sci.* **22**, 277 (2018)
[16] S S Roshan, H Jafari and D Baleanu, *Math. Methods Appl. Sci.* **41**, 9134 (2018)
[17] A Singh, S Das, S H Ong and H Jafari, *J. Comput. Nonlinear Dyn.* **14**(**4**), 041003-1 (2019)
[18] H Jafari, *Appl. Math. Lett.* **32**, 1 (2014)
[19] S L Chang and L A Zadeh, *IEEE Trans. Systems Man Cybernet.* **2**, 30 (1972)
[20] D Dubois and H Prade, *Fuzzy Sets Syst.* **8**, 225 (1982)
[21] N Mikaeilvand and S Khakrangin, *Neural Comput. Appl.* **21**, S307 (2012)
[22] A Khastan, J J Nieto and R R López, *Inf. Sci.* **222**, 544 (2013)
[23] D Qiu, W Zhang and C Lu, *Fuzzy Sets Syst.* **295**, 72 (2016)
[24] S Tapaswini, S Chakraverty and T Allahviranloo, *Comput. Math. Model.* **28**, 278 (2017)
[25] S Tapaswini, S Chakraverty and J J Nieto, *Sadhana* **42**, 45 (2017)
[26] K Nematollah, S S Roushan and H Jafari, *Int. J. Appl. Comput. Math.* **4**, 33 (2018)
[27] R P Agarwal, V Lakshmikantham and J J Nieto, *Nonlinear Anal. Theory Methods Appl.* **72**, 2859 (2010)
[28] E Khodadadi and E Celik, *Fixed Point Theory Appl.* **2013**, 1 (2013)
[29] S Chakraverty and S Tapaswini, *Comput. Model. Eng. Sci.* **103**, 71 (2014)
[30] A Rivaz, O S Fard and T A Bidgoli, *SeMA J.* **73**, 149 (2016)
[31] J H He, *Int. J. Nonlinear Mech.* **34**, 699 (1999)
[32] J H He, *Appl. Math. Comput.* **114**, 115 (2000)
[33] K Abbaoui and Y Cherruault, *Comput. Math. Appl.* **29**, 103 (1995)
[34] H Jafari, M Saeidy and D Baleanu, *Cent. Eur. J. Phys.* **10**, 76 (2012)
[35] A F Jameel, *Approx. Sci. Comput.* **2014**, 1 (2014)
[36] S Abbasbandy, T Allahviranloo, P Darabi and O Sedaghatfar, *Math. Comput. Appl.* **16**, 819 (2011)
[37] M M Hosseini, F Saberirad and B Davvaz, *Int. J. Fuzzy Syst.* **18**, 875 (2016)
[38] H J Zimmermann, *Fuzzy set theory and its application* (Kluwer Academic Publishers, Boston/Dordrecht/London, 2001)
[39] L Jaulin, M Kieffer, O Didrit and E Walter, *Applied interval analysis* (Springer, Paris, 2001)
[40] M Hanss, *Applied fuzzy arithmetic: An introduction with engineering applications* (Springer, Berlin, 2005)
[41] S Chakraverty, S Tapaswini and D Behera, *Fuzzy arbitrary order systems: Fuzzy fractional differential equations and applications* (Wiley, New Jersey, 2016)
[42] S Chakraverty, S Tapaswini and D Behera, *Fuzzy differential equations and applications for engineers and scientists* (CRC Press, Florida, 2016)
[43] I Podlubny, *Fractional differential equations* (Academic Press, New York, 1999)