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Pricing American put option under fractional Heston model

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Abstract. In this paper, we attempt to provide a solution for the fractional linear complementarity problem related to the evaluation of American put option generated by the fractional Heston stochastic volatility model. Using the Adomian decomposition, a numerical investigation is conducted to confirm the theoretical results.

Keywords. Pricing American option; stochastic volatility; fractional Heston model; Adomian decomposition.

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1. Introduction and formulation of the problem

Pricing American option is one of the most thorny issues in financial mathematics. The early exercise feature inherent in American options was related to a free boundary condition problem [1–4], which is proved to be very complicated. From this perspective, American options have no closed form solutions.

As the dynamics of volatility is intrinsic in terms of the hedging and pricing options and as the assumptions of the constant volatility model [5] are not suitable for realising financial markets, it is important to consider volatility as stochastic, the most famous of which is the Heston Model [6]. The introduction of an additional stochastic volatility factor enormously complicates the pricing of American options.

Several works addressed the problem of American pricing options under a stochastic volatility using multiple methods that are generated by different models. At present, this can only be achieved by means of numerical schemas, which involve solving integral equations [7–9], performing monte Carlo simulations [10,11], using Malliavin calculus [12–14], or discretising a partial differential equation [15,16].

The fractional calculus was invested in several research areas (see [17–21]). Recently, it has been introduced in the mathematical finance field [22–25], mainly to resolve the pricing option problem. For instance, refs [26–28] are devoted to the evaluation of a European option and ref. [29] is devoted to the American option. Many methods are set forward to resolve linear and nonlinear fractional differential equations, see for example [30,31]. In this work, we use the Adomian decomposition method [32–34], which is a successful method in terms of providing analytical solution in linear and nonlinear equations.

From this perspective, we provide a new formula to evaluate the price of an American put option under the fractional Heston model using the Adomian decomposition method. In what follows, we introduce the dynamics of the Heston model. Let S_t and V_t represent two stochastic processes such that S_t is generated by the following process:

$$\mathrm{d}S_t = rS_t\,\mathrm{d}t + S_t\sqrt{V_t}\,\mathrm{d}W_t^S \tag{1}$$

and V_t follows a mean reverting and a square-root diffusion process indicated by

$$\mathrm{d}V_t = k_V(\theta_V - V_t)\,\mathrm{d}t + \sigma_V\sqrt{V_t}\,\mathrm{d}W_t^V, \tag{2}$$

where r is the interest rate which is supposed to be constant, W_t^S and W_t^V are two correlated standard Brownian motions, i.e. $W_t^S = \sqrt{1-\rho^2}B_t^1 + \rho B_t^2$ and $W_t^V = B_t^2$, with B a standard two-dimensional Brownian motion and the coefficient of correlation $\rho \in]-1,1[$. The parameters θ_V, k_V and σ_V are respectively, the long-term mean, the rate of mean reversion and the volatility of the stochastic process V_t. We assume that the volatility process V_t is strictly positive a.s.

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Using the Ito formula, we have the following differential equation:

$$\frac{\partial P}{\partial t}+rS\frac{\partial P}{\partial S}+k(\theta-V)\frac{\partial P}{\partial V}+\frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2}+\rho\sigma VS\frac{\partial^2 P}{\partial S\partial V}+\frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2}-rP=0, \tag{3}$$

where P is the American put price and K is the strike price. The boundary conditions in terms of time can be stated as follows:

$$P(S_t,t)=\max(K-S_t,0)\ \text{ in the exercise case} \tag{4}$$

and

$$P(S_t,t)>\max(K-S_t,0)\ \text{ in the other case.} \tag{5}$$

Therefore, the problem of pricing American put option comes down to a linear complementarity problem under the following system:

$$\left[\frac{\partial P}{\partial t}+rS\frac{\partial P}{\partial S}+k(\theta-V)\frac{\partial P}{\partial V}+\frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2}+\rho\sigma VS\frac{\partial^2 P}{\partial S\partial V}+\frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2}-rP\right]\times(P-(K-S_t))=0$$

$$\left[\frac{\partial P}{\partial t}+rS\frac{\partial P}{\partial S}+k(\theta-V)\frac{\partial P}{\partial V}+\frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2}+\rho\sigma VS\frac{\partial^2 P}{\partial S\partial V}+\frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2}-rP\right]\leq 0$$

$$P-(K-S_t)\geq 0\ \ \forall t.$$

2. Preliminaries

In what follows, we set forward certain definitions related to the fractional calculus which are the basis of our work. For an organic presentation of the fractional theory, please refer to Podlubny's book [35].

DEFINITION 1

The Riemann–Liouville fractional integral of order $\alpha>0$ is represented as

$$I^{\alpha}_{t_0}x(t)=\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-\tau)^{\alpha-1}x(\tau)\,\mathrm{d}\tau$$

where

$$\Gamma(\alpha)=\int_0^{+\infty}\mathrm{e}^{-t}t^{\alpha-1}\,\mathrm{d}t.$$

DEFINITION 2

The Caputo fractional derivative is expressed as

$$D^{\alpha}_{t_0,t}x(t)=\frac{1}{\Gamma(m-\alpha)}\int_{t_0}^{t}(t-\tau)^{m-\alpha-1}\frac{\mathrm{d}^m}{\mathrm{d}\tau^m}x(\tau)\mathrm{d}\tau,$$

$(m-1<\alpha<m)$.

When $0<\alpha<1$, the Caputo fractional derivative of order α of f reduces to

$$D^{\alpha}_{t_0,t}x(t)=\frac{1}{\Gamma(1-\alpha)}\int_{t_0}^{t}(t-\tau)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}x(\tau)\mathrm{d}\tau. \tag{6}$$

Note that the relation between Riemann–Liouville operator and Caputo fractional differential operator is enacted by the following equality:

$$I^{\alpha}_{t_0}D^{\alpha}_{t_0,t}f(t)=D^{-\alpha}_{t_0,t}D^{\alpha}_{t_0,t}f(t)=f(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}f^k(0),\quad m-1<\alpha\leq m. \tag{7}$$

Like the exponential function used in the solutions of integer-order differential systems, Mittag–Leffler function is frequently used in the solutions of fractional-order differential systems.

DEFINITION 3

The Mittag–Leffler function with two parameters is indicated as

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha>0$, $\beta>0$, $z\in C$.

When $\beta=1$, we have $E_{\alpha}(z)=E_{\alpha,1}(z)$. Besides, $E_{1,1}(z)=\mathrm{e}^z$.

3. Main results

In his research paper [26], Kharrat introduced the fractional Heston model to provide a closed-form solution of a European option. In this work, it is used to tackle pricing American put option. As a matter of fact, we need to resolve the following fractional linear complementarity problem related to our problem:

$$\left[\frac{\partial^{\alpha} P}{\partial t}+rS\frac{\partial P}{\partial S}+k(\theta-V)\frac{\partial P}{\partial V}+\frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2}+\rho\sigma VS\frac{\partial^2 P}{\partial S\partial V}+\frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2}-rP\right]\times(P-(K-S_t))=0$$

Table 1. Pricing American put option for different values of fractional model compared to the classical binomial model (400 time steps), as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/12$).

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
BIN400	20	15	10	4.957	2.2154	1.261	0.1091	0.073	0.002
$P(\alpha = 1)$	20	15	10	4.873	2.2132	1.282	0.1496	0.057	0.003
$P(\alpha = 0.9)$	20	15	10.12	5.269	2.401	1.297	0.151	0.077	0.0033
$P(\alpha = 0.7)$	20	15	10.26	5.471	2.4704	1.311	0.197	0.089	0.0037
$P(\alpha = 0.5)$	20	15	10.53	5.492	2.611	1.377	0.386	0.114	0.0075

Table 2. Pricing American put option for different values of fractional model compared to the classical binomial model (400 time steps), as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/4$).

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
BIN400	20	15	10.1706	5.027	3.485	1.995	0.775	0.193	0.052
$P(\alpha = 1)$	20	15.009	10.177	5.005	3.482	1.878	0.768	0.198	0.059
$P(\alpha = 0.9)$	20	15.019	10.192	5.103	3.523	2.182	0.788	0.225	0.081
$P(\alpha = 0.7)$	20	15.093	10.201	5.254	3.779	2.294	0.922	0.274	0.096
$P(\alpha = 0.5)$	20	15.128	10.2175	5.670	4.041	2.503	1.156	0.346	0.117

$$\left[\frac{\partial^\alpha P}{\partial t} + rS\frac{\partial P}{\partial S} + k(\theta - V)\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \rho\sigma VS\frac{\partial^2 P}{\partial S\partial V} + \frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2} - rP\right] \le 0$$

$$P - (K - S_t) \ge 0 \ \ \forall t.$$

where $0 < \alpha \le 1$.

Under stochastic volatility, to compute the value of American put price $P(S_t, V_t)$, we need to resolve the following nonlinear fractional differential equation:

$$D_t^\alpha P(S_t, V_t) + A[P](S_t, V_t) = 0, \quad 0 < \alpha \le 1 \tag{8}$$

in the unbounded domain $\{(S_t, V_t) | S_t \ge 0, V_t \ge 0$ and $t \in [0, T]\}$ with the initial value

$$P(S_0, V_0). \tag{9}$$

For boundary conditions, in case of a put option, at maturity T with an exercise price K, the payoff function is

$$\max(K - S_T, 0), \tag{10}$$

where

$$D_t^\alpha = \frac{\partial^\alpha}{\partial t}$$

and

$$A[P] = rS\frac{\partial P}{\partial S} + k(\theta - V)\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \rho\sigma VS\frac{\partial^2 P}{\partial S\partial V} + \frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2} - rP.$$

Theorem. *Let $(P_t)_{t\ge 0}$ be the American option price at time t. Under the hypotheses of the Heston model, at time l with $l < t$, the American put option price, which corresponds to the solution of the previous fractional linear complementarity problem, is equal to*

$$P(S_l, V_l) = \max[\max(K - S_l, 0); e^{-r(t-l)}E_\alpha(-(t-l)^\alpha \times A[P(S_t, V_t)])], \tag{11}$$

where $0 < \alpha \le 1$, E_α is the Mittag–Leffler function and

$$A[P] = rS\frac{\partial P}{\partial S} + k(\theta - V)\frac{\partial P}{\partial V} + \frac{1}{2}VS^2\frac{\partial^2 P}{\partial S^2} + \rho\sigma VS\frac{\partial^2 P}{\partial S\partial V} + \frac{1}{2}\sigma V\frac{\partial^2 P}{\partial V^2} + rP.$$

Proof. Multiplying eq. (8) by the operator $D_t^{-\alpha}$ and taking into account (4), we get

$$P(S_t, V_t) = P(S_l, V_l) + D_t^{-\alpha}(-A[P](S_t, V_t)). \tag{12}$$

Therefore, using the Adomian decomposition method in the domain $[l, t]$, the solution is indicated as

$$P(S_t, V_t) = P(S_l, V_l) + \sum_{k=1}^{\infty} P_k(S_t, V_t). \tag{13}$$

By substituting (13) into (8), we have

$$P_{n+1}(S_t, V_t) = D_t^{-\alpha}(-A[P_n](S_t, V_t)) = -A[P(S_l, V_l)]^n D_t^{-\alpha}\left(\frac{(t-l)^{n\alpha}}{\Gamma(1+n\alpha)}\right). \tag{14}$$

Thus, we get

$$P(S_l, V_l) = \sum_{k=0}^{\infty}(-1)^k\frac{(t-l)^{k\alpha}}{\Gamma(1+k\alpha)}A[P(S_t, V_t)]^k = E_\alpha(-(t-l)^\alpha A[P(S_t, V_t)]). \tag{15}$$

Table 3. Pricing American put option for different values of fractional model compared to the classical binomial model (400 time steps), as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/2$).

S/K	0.8	0.85	0.9	0.95	1	1.05	1.1	1.15	1.2
BIN400	20	15.002	10.174	5.271	3.495	2.019	0.776	0.204	0.0703
$P(\alpha = 1)$	20	15.019	10.18	5.294	3.491	2.073	0.7901	0.217	0.0731
$P(\alpha = 0.9)$	20	15.105	10.216	5.346	3.581	2.194	0.7985	0.249	0.114
$P(\alpha = 0.7)$	20	15.113	10.223	5.497	3.737	2.317	0.9593	0.283	0.172
$P(\alpha = 0.5)$	20	15.159	10.391	5.729	3.963	2.641	1.0656	0.377	0.249

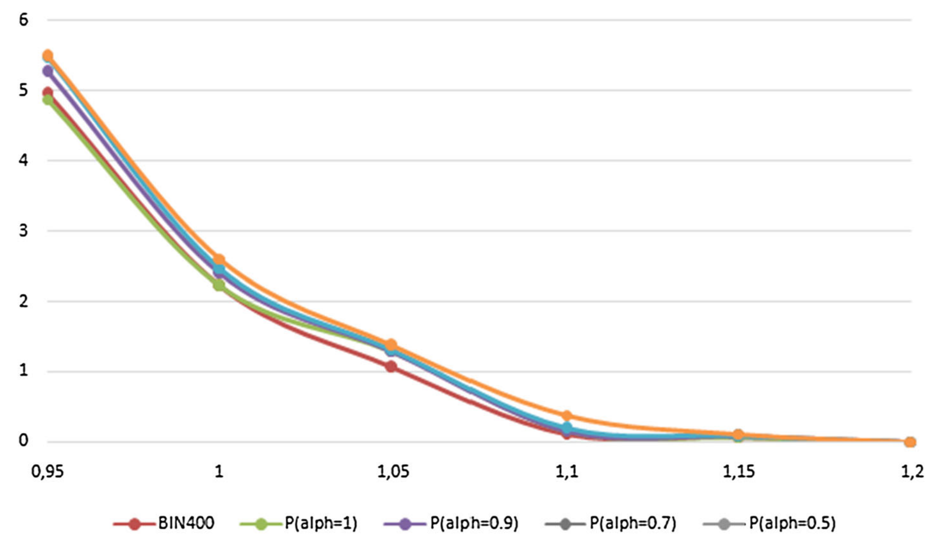

Figure 1. Pricing American put option for different values of fractional model compared to the classical binomial model (400 time steps), as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/12$).

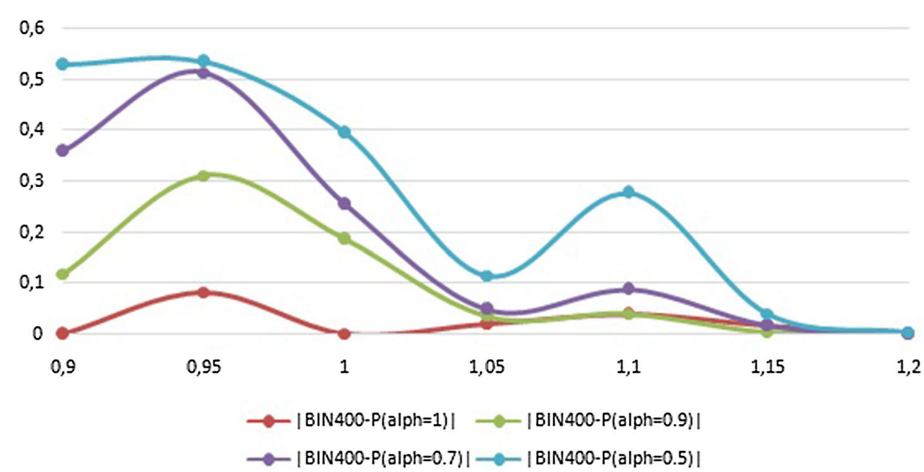

Figure 2. The error between the value of American put option under the fractional Heston model for different values of α and the classical binomial model (400 time steps) under stochastic volatility, as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/12$).

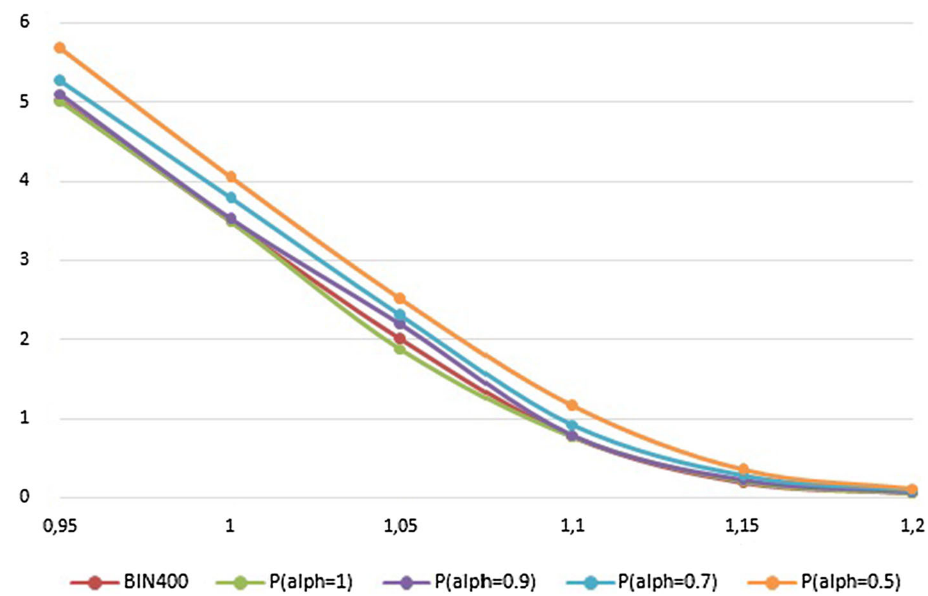

Figure 3. Pricing American put option for different values of fractional model compared to the classical binomial model (400 time steps), as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/4$).

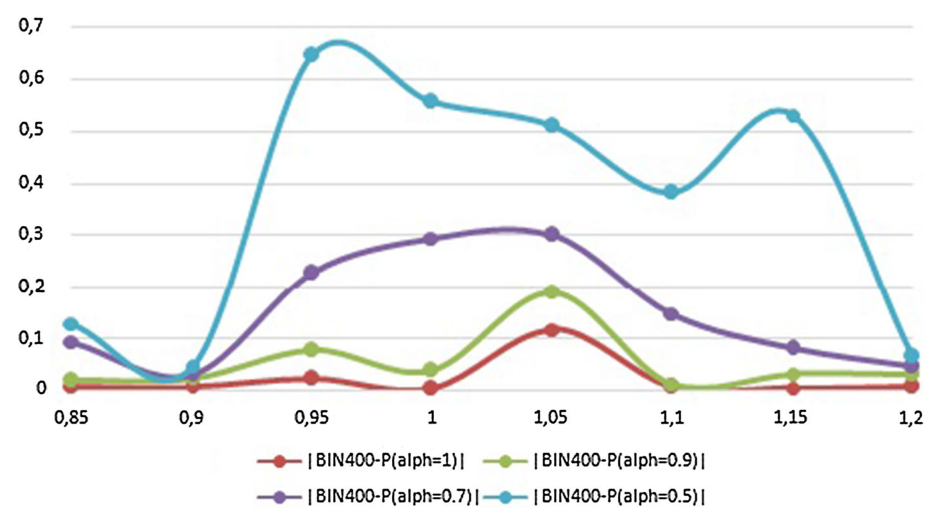

Figure 4. The error between the value of American put option under the fractional Heston model for different values of α and the classical binomial model (400 time steps) under stochastic volatility, as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/4$).

The convergence of power series of the fractional Heston model is guaranteed for a real and positive α. □

4. Simulations and numerical results

In the following, we shall display the numerical study of our proposed model for different orders of fractional derivative. We shall exhibit and plot the price of the American put option under the fractional Heston model and its difference with the one provided by the binomial model under stochastic volatility (see tables 1–3, figures 1–6).

At this stage of the analysis,we have investigated the put price as a function of moneyness. As data, we have considered $K = 100$, $V_0 = 0.2$, $r = 0.05$. For the time of maturity, we have considered three cases: the first one is equal to $1/12$, the second equals $1/4$ and finally the third equals $1/2$.

Departing from the obtained results (see figures 1–6), all curves have the same profiles as the one related to the binomial model under stochastic volatility, which is in good accordance with the option's theory.

Referring to figures 2, 4 and 6, and for different times of maturity, we notice that, when moneyness is located between 0.95 and 1.15, there is a slight difference

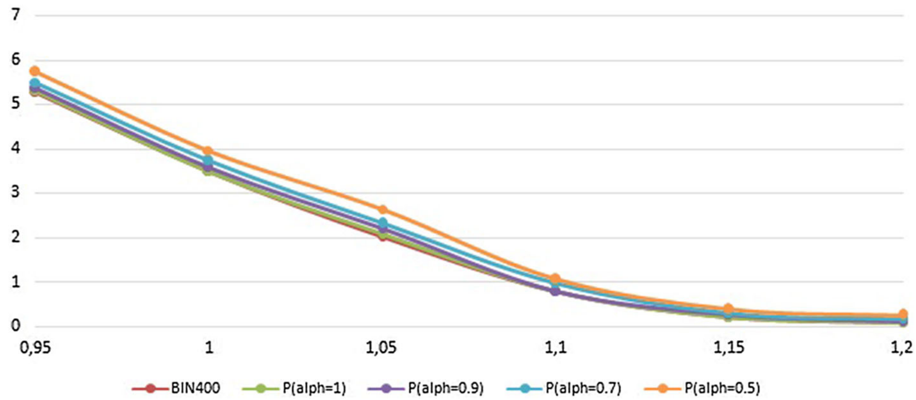

Figure 5. Pricing American put option for different values of the fractional model compared to the classical binomial model (400 time steps), as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/2$).

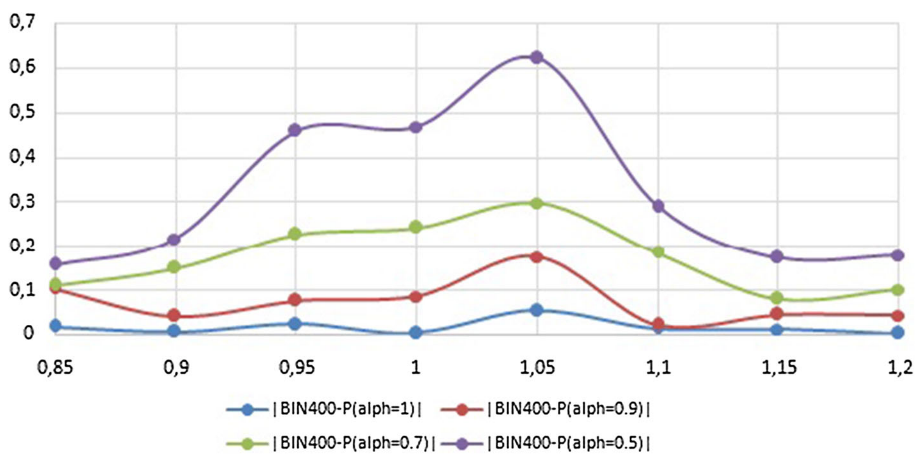

Figure 6. The error between the value of American put option under the fractional Heston model for different values of α and the classical binomial model (400 time steps) under stochastic volatility, as a function of moneyness ($K = 100$, $V_0 = 0.2$, $r = 0.05$, $T = 1/2$).

between the obtained results and the binomial model. On the other hand, when moneyness is not located between 0.95 and 1.15, the difference between the premium of the American put option is almost useless, which holds in terms of accuracy, for every value of the fractional order.

5. Conclusion

Using the Adomian decomposition, we provide and prove the convergence of power series related to the price of the American put option under the fractional Heston model. In order to calculate the theoretical results, we set forward numerical solutions for different values of fractional order. All the results are correlated with the American option theory.

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