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Analytical solution of the steady-state atmospheric fractional diffusion equation in a finite domain

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Abstract. In this work, an analytical solution for the steady-state fractional advection-diffusion equation was investigated to simulate the dispersion of air pollutants in a finite media. The authors propose a method that uses classic integral transform technique (CITT) to solve the transformed problem with a fractional derivative, resulting in a more general solution. We compare the solutions with data from real experiment. Physical consequences are discussed with the connections to generalised diffusion equations. In the wake of these analysis, the results indicate that the present solutions are in good agreement with those obtained in the literature. This report demonstrates that fractional equations have come of age as a decisive tool to describe anomalous transport processes.

Keywords. Anomalous diffusion; fractional diffusion equation; air pollutants; integral transform technique; Mittag–Leffler function.

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1. Introduction

Air quality model in the atmosphere make it possible to assess the substances which exceed concentration threshold. We quote the transport mechanisms of mass transfer that have been broadly used to evaluate the predictive capability of operational dispersion models to reproduce mean concentration of non-reactive air pollutants [1–4]. Also, numerous solitary waves, solitons and elliptic function solutions are constructed in various forms of three-dimensional nonlinear partial differential equations (PDEs) in mathematical physics to emphasise the dispersive long-wave equations utilising the modified extended direct algebraic method [5–8]. In the same vein, the nonlinear evolution equations of the mathematical physical models such as hydrodynamics, the theory of turbulence, shallow water waves theory and other dynamical systems are often used to characterise the dispersion analysis as well as the conservation law of the dynamic system [9–11]. At this stage, it is important to mention that the mechanism of turbulent diffusion is a stake of modern science, for which the conventional framework of differential integer-order equations does not properly characterise the problem of turbulent diffusion. For a long time, the transport of airborne pollutants under the combined effects of advection and diffusion were described by taking into account special forms of eddy diffusivity coefficient and wind speed [12–14]. As an illustration, for depicting the solution of the advection–diffusion equation, the traditional air pollution models assume the wind speed and eddy diffusivity in the atmospheric boundary layer (ABL) [15,16] as constant. In steady state and horizontally homogeneous conditions for functional forms, the wind speed and eddy diffusivity increase with altitude, inversely with pressure [17]. Almost all the dispersion approaches explaining the advection–diffusion equation take into account the eddy diffusivity coefficients specified by K-theory in the ABL [18–22]. In addition, some of these models assumed wind speed

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as a generalised function of height and even eddy diffusivity as a function of downwind distance [23]. In contrast, the above assumption has been stretched out, and most of the crosswind-integrated solutions [24,25] are obtained by considering the wind speed and vertical eddy diffusivity as the power-law functions of vertical height above the ground. But none of these fixes up a sound approach to find the solution with the generalised functional forms of eddy diffusivity and wind speed. A striking consequence of turbulence appears to be the emanation of anomalous diffusion. Anomalous, or non-Fickian dispersion has been widely used since the introduction of continuous time random walks (CTRW) in the predictive potentiality of models relied on the stochastic process of Brownian motion. This predictive ability is the starting point for the so-called conventional advection–dispersion equation [26–29]. In this hypothesis, wind speed has been parametrised as a power-law profile of vertical height above the ground. However, in the atmospheric surface layer, a more realistic profile of wind speed can be obtained by using the local similarity theory. As a consequence, a realistic parametrisation of wind profile can lead to a striking enhancement in the prediction of concentrations of pollutants in the planetary boundary layer (PBL). Albeit, it is admitted that fractional differential equations report the time evolution of anomalous diffusive systems [30] more realistically. In the present work, we shall demonstrate that the fractional differential equation is more suitable than the classical equation to characterise the steady-state regime of anomalous dispersion of pollutants in a turbulent flow. The dispersion model that we introduce in the present work, shows that the models often found in the literature, do not deal with the advection–diffusion equation as disclosed conventionally, however, they reshaped the mathematical structure of this equation to describe more realistically the evolution in space of concentration of air contaminants discharged in a turbulent flow. Many researchers have demonstrated the conformity between these heavy-tailed motions and transport equations that use fractional-order derivatives. Thus, the motions can be heavy-tailed, assuming extremely long-range correlation and fractional derivatives in coordinate [31–33]. To overcome the restrained capability of analytical dispersion, models with particular forms of vertical eddy diffusivity and wind speed to characterise the turbulent dispersion are needed. An integro-differential equation based on the fractional operator properties is introduced. Then, we assume that the exponential decay of the modes are modified so that long-tailed memory effects induce a moderate power-law decay of the modes, in accordance with the Mittag–Leffler pattern [34].

This work is organised as follows. In §2, we review some basic definitions and properties of Riemann–Liouville and Caputo fractional derivatives. In §3, the formulation of mathematical problem is presented. In §4, a numerical comparison with our theoretical results against conventional models and experimental data is done, and finally, we conclude in §5.

2. Useful calculus definitions

In this section, we recall some basic definitions:

- The Riemann–Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_\mu, \mu \geq -1$ is defined as [35]

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\mathrm{d}\tau, \quad (\alpha > 0),\ J^0 f(t) = f(t). \tag{1}$$

For the Riemann–Liouville integral we have

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)} t^{\alpha+\gamma}. \tag{2}$$

- The fractional derivative in the Caputo sense is defined as [36]

$$D^\alpha f(t) = J^{n-\alpha} D^n f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}\mathrm{d}\tau, \tag{3}$$

for $n-1 < \alpha \leq n, n \in N, \alpha > 0$.
- The Laplace transform of $f(t)$ is defined as

$$F(s) = \mathscr{L}[f(t)] = \int_0^\infty \mathrm{e}^{-st} f(t)\mathrm{d}t. \tag{4}$$

- The Laplace transform $\mathscr{L}[f(t)]$ of the Riemann–Liouville fractional integral is given by

$$\mathscr{L}\left\{I^\mu f(t)\right\} = s^{-\mu} F(s). \tag{5}$$

- The Laplace transform of the Caputo derivative is given by Caputo [37,38] in the following form:

$$\mathscr{L}\left\{D^\mu f(t)\right\} = S^\mu F(s) - \sum_{k=0}^{n-1} S^{(\mu-k-1)} f^{(k)}, \quad n-1 < \mu \leq n. \tag{6}$$

3. Formulation of the fractional derivative equations

The steady-state of atmospheric dispersion equation, depicting the dynamics of air pollutants, relating to the concentration distribution $c \equiv \bar{c}(x, y, z)$ can be characterised by the following equation [39]:

$$u(x, y, z)\frac{\partial \bar{c}}{\partial x} + v(x, y, z)\frac{\partial \bar{c}}{\partial y} + w(x, y, z)\frac{\partial \bar{c}}{\partial z} = \frac{\partial}{\partial x}\left[K_x(x, y, z)\frac{\partial \bar{c}}{\partial x}\right] + \frac{\partial}{\partial y}\left[K_y(x, y, z)\frac{\partial \bar{c}}{\partial y}\right] + \frac{\partial}{\partial z}\left[K_z(x, y, z)\frac{\partial \bar{c}}{\partial z}\right] + S, \quad (7)$$

where $\bar{c}$ is the average concentration, $u(x, y, z)$, $v(x, y, z)$, $w(x, y, z)$ and $K_x(x, y, z)$, $K_y(x, y, z)$, $K_z(x, y, z)$ are the Cartesian components of wind velocity and eddy diffusivity along the x, y and z directions, respectively. S is the source term. The subsequent assumptions include: the pollutant is considered as inert and there is no extra sinks or sources downwind from the point source, the term including the vertical velocity is much smaller than the horizontal one and the orientation of the x-axis assumes that the velocity is equal to zero, the average horizontal flow is incompressible and horizontally homogeneous, under moderate to strong wind conditions, the diffusion term in the x-direction is smaller than the advection term, i.e.

$$u(x, y, z)\frac{\partial \bar{c}}{\partial x} \gg \frac{\partial}{\partial x}\left[K_x(x, y, z)\frac{\partial \bar{c}}{\partial x}\right]. \quad (8)$$

In addition, the following expression for eddy diffusivity is considered:

$$K_z(x, z) = f_1(x)K(z), \quad (9)$$

where $f_1(x)$ is a dimensionless function of x highlighting a correction $K(z)$ in the vicinity of the source and $K(z)$ is the far-field pattern eddy diffusivity coefficient that is a function of z. The three-dimensional concentration derived by Glênio *et al* (2018) using the method of separation of variables can be deduced from eq. (9) by taking

$$f_1(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\sigma_z^2(x).$$

Thus,

$$s_z(x) = \int_0^x f_1(x')\mathrm{d}x' = \frac{1}{2}\sigma_z^2(x),$$

where σ_z is the dispersion coefficient in the z-direction. The functional form of k_z is characterised to allow a Gaussian distribution for the transverse concentration. According to these considerations, eq. (7) becomes

$$u(z)\frac{\partial \bar{c}}{\partial x} = \frac{\partial}{\partial z}\left[K_z(x, z)\frac{\partial \bar{c}}{\partial z}\right]. \quad (10)$$

The process can be governed by the steady-state regime of the advection–diffusion equation with K_z as a constant with a sharp boundary condition

$$\bar{c}_y(0, z) = \lim_{x\to 0^+} \bar{c}_y(x, z) = \delta(z)$$

in the domain $0 \le x < \infty$ and in the domain $-\infty < z < \infty$ the solution is given in terms of the Brownian solution by the Galilei shifted Gaussian

$$\bar{c}_y(x, z) = \frac{1}{\sqrt{4\pi K_z x}} \exp\left(-\frac{z^2}{4K_z x}\right). \quad (11)$$

According to [40,41], we proposed a set of tridimensional solutions, for ground level sources only, taking into account both the wind speed (u) and vertical diffusion coefficient (K_z). Thus, a modified continuous-time random-walk (CTRW) scheme in a velocity field is introduced, and applied to an extended Taylor formula, supposing the initial condition $\bar{c}_y(x, 0) = \delta(x)$.

The corresponding equation in Fourier–Laplace space takes the following form:

$$u(s)\frac{\partial \bar{c}_y(u, k)}{\partial x} = -k^2 K_s(s)\bar{c}_y(u, k), \quad (12)$$

$$= -k^2 \sum_{n=0}^{\infty} \frac{(i)^n}{n!} K_n \bar{c}_y^{(n)}(u, k). \quad (13)$$

Supposing a waiting time distribution, especially for the choice of a long-tailed distribution for ψ, $\psi(u) \sim 1 - (u\tau)^{\gamma}$ for $u \to 0$ and $k \to 0$, following the asymptotic forms $\lambda_c(k) \sim 1 - \sigma^{\mu}|k|^{\mu}$ for the cosine transform, after a tricky development (refer Appendix 1) with regard to the steps previously described by eqs (5) and (6) we find the following fractional generalised diffusion equation:

$$\frac{\partial^{\alpha} \bar{c}_y(x, k)}{\partial x^{\alpha}} - \frac{x^{-\alpha}\delta(x)}{\Gamma(1-\alpha)} = \frac{1}{u(z)}\frac{\partial}{\partial z}\left[K_z \frac{\partial c_y(x, z)}{\partial z}\right]. \quad (14)$$

Equation (10) is subjected to the following boundary conditions:

- The pollutant is located at the point $(0, 0, h_s)$ from a continuous point source with strength Q.

  $u\bar{c}(0, z) = Q\delta(z - h_s)$ at $x = 0$,

  where h_s is the effective stack height and δ is the Dirac-delta generalised function.
- The top and bottom of the PBL are supposed to be impervious. This is supposed to be a perfectly total absorption

  $K_z(x, z)\dfrac{\partial \bar{c}}{\partial z} = 0$ when $z \to 0$ and $z = h$,

  h_s represents the maximum level of the inversion/mixed layer and soil characteristics represents the roughness length on the ground surface.

- The pollutant diffusion is assumed to be completely through the top of the mixed layer positioned at height h

  $\bar{c} = 0 \quad \text{at } z = h.$

- $\bar{c} \to 0$ when x and $z \to \infty$.

The solution is taken by using a separation ansatz [42]

$$\bar{c}_y(x, z) = \sum_{n=0}^{\infty} X_n(x) Z_n(z). \tag{15}$$

This gives the two ordinary differential equations

$$\frac{\partial^\alpha X}{\partial x^\alpha} - \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \lambda^2 X = 0, \tag{16}$$

$$\frac{1}{u(z)} \frac{\partial}{\partial z}\left[K(z)\frac{\partial Z(z)}{\partial z}\right] + \lambda^2 Z(z) = 0. \tag{17}$$

The solution of eq. (17) offers a set of linearly independent eigenfunctions for $Z_k(z)$ and as a result for $\Psi_j(z)$. It is worth noting that for each value of λ_j there is an associated infinite set of real eigenvalues $\lambda_j (j = 0, 1, 2, \ldots, N)$. The integral transform pair technique enables one to consider an unknown function characterised by an eigenfunction expansion series in terms of the eigenfunction $\Psi_j(z)$. We recall the orthogonality properties of the eigenfunctions $Z_k(z)$.

$$\int_{z_0}^{h} u(z) Z_j(z) Z_k(z)\, \mathrm{d}z = N_z(\lambda_j)\delta_{jk}, \tag{18}$$

where δ_{jk} is the Kronecker delta, $N_z(\lambda_j)$ represents the norm of $Z_j(z)$.

The eigenfunctions Ψ_j of this eigenvalue problem correspond to the ensuing orthogonality property (Ŏzisik, 1985) in the domain Ω defined by the boundaries

$$\int_{\Omega} u(z)\Psi_j(z)\Psi_k(z)\, \mathrm{d}z = \int_{z_0}^{h} u(z)\Psi_j(z)\Psi_k(z)\mathrm{d}z = N_j \delta_{jk}, \tag{19}$$

where $N_j = N_z(\lambda_j)$ is the norm of $\Psi_j(z)$.

According to the above considerations, the following integral transform pair is introduced:

$$\bar{c}_j(x) = \int_{\Omega} u(z)\Psi_j(z) c(x)\, \mathrm{d}z \tag{20}$$

(Transform)

$$c(x) = \sum_{j=0}^{\infty} \frac{\Psi_j(z)}{\bar{N}_j} \bar{c}_j(x) \tag{21}$$

(Inverse)

By introducing these properties

1. If $\mathrm{Re}(\alpha) \geqslant 0$ and $\beta \in \mathbb{C}\,(\mathrm{Re}(\beta) > 0)$, then the equations [43,44]

$$(J^\alpha_{a+} J^\beta_{a+} f)(x) = \left(J^{\alpha+\beta}_{a+} f\right)(x),$$
$$(J^\alpha_{b-} J^\beta_{b-} f)(x) = \left(J^{\alpha+\beta}_{b-} f\right)(x) \tag{22}$$

   are satisfied on almost all points $x \in [a, b]$ for $f \in L_p(a, b)$ $(1 \leqslant p \leqslant \infty)$; if $\alpha + \beta > 1$, then the above relations hold at any point of $[a, b]$.
2. The relation between Caputo fractional derivative and Riemanna–Liouville fractional derivative is given by the following formula [45,46]:

$$D^\alpha_{a+} f(t) = {}^C D^\alpha_{a+} f(t) + \sum_{i=0}^{n-1} \frac{f^{(i)}(a)}{\Gamma(i-\alpha+1)}(t-a)^{i-\alpha}. \tag{23}$$

Equation (14) can be recast in the following form:

$$\frac{{}^C\partial^\alpha \bar{c}(x, y, z)}{\partial x^\alpha} = \frac{1}{u(z)}\frac{\partial}{\partial z}\left[K_z \frac{\partial \bar{c}(x, y, z)}{\partial z}\right] \tag{24}$$

and applying in eq. (24) the inverse formula eq. (21) and taking into account the eigenvalue problem, eq. (14) becomes

$$\sum_{j=0}^{\infty} u(z) \frac{{}^C\partial^\alpha \bar{c}_j(x)}{\partial x^\alpha} \frac{\Psi_j(z)}{\bar{N}_j} = -\sum_{j=0}^{\infty} u(z) \frac{\lambda_j^2 \Psi_j(z)}{\bar{N}_j} \bar{c}_j(x) \tag{25}$$

and multiplying both sides of eq. (25) by $\Psi_i(z)$ one gets

$$\sum_{j=0}^{\infty} \frac{{}^C\partial^\alpha}{\partial x^\alpha} \frac{\bar{c}_j(x)}{\bar{N}_j} \int_C u(z)\Psi_i(z)\Psi_j(z)\, \mathrm{d}z = -\sum_{j=0}^{\infty} \frac{\bar{c}_j(x)}{\bar{N}_j} \lambda_j^2(x) \int_C u(z)\Psi_i(z)\Psi_j(z)\, \mathrm{d}z. \tag{26}$$

Then, considering the orthogonality expression (eqs. (19) in (26)) the following ordinary differential system is obtained:

$$\frac{{}^C\partial^\alpha \bar{c}_j(x)}{\partial x^\alpha} + \lambda_j^2 \bar{c}_j(x) = 0. \tag{27}$$

We obtain a non-trivial solution

$$\bar{c}_j(x^\alpha) = \bar{c}_j(0) E_\alpha[-\lambda^2 x^\alpha], \tag{28}$$

where E_α is the Mittag–Leffler function [47].

For the solution of eq. (17), we assume the polynomial description for the velocity and the turbulent diffusivity in a finite medium. In this case, the velocity and the eddy diffusivity profiles are specified as

$$u(z) = u_r z^{\alpha} \tag{29a}$$

$$k(z) = k_r z^{\beta}. \tag{29b}$$

The above equations are the so-called Berliand's profile [48]. Equation (17) can be recast as

$$\frac{\partial}{\partial z} z^{\beta} \frac{\partial Z}{\partial z} + \lambda^2 \left(\frac{u_r}{k_r}\right) z^{\alpha} Z = 0. \tag{30}$$

The scheme to solve the analytical new form with the Neumann boundary condition is showed in the present work, defining a non-zero value of the eigenvalue, thus introducing the new set of variables $\eta = z^{(\alpha-\beta+2)/2}$ and $Z(z) = z^{(1-\beta)/2} f(\eta)$. The initial eigenvalue problem eq. (17) becomes

$$\eta^2 f''(\eta) + \eta f'(\eta) + [\kappa^2 \eta^2 - \mu^2] f(\eta) = 0. \tag{31}$$

Equation (31) is the Bessel equation, and the solution can be given by the following equation:

$$Z(z) = z^{(1-\beta)/2}[a_1 J_{\mu}(\omega_j z^{q/2}) + a_2 J_{-\mu}(\omega_j z^{q/2})]. \tag{32}$$

$$\mu = \frac{1-\beta}{q}, \tag{33a}$$

$$\omega_j = \frac{\lambda_j (u_r/k_r)}{q/2}, \tag{33b}$$

$$q = \alpha - \beta + 2, \tag{33c}$$

using the second boundary condition term $\partial Z/\partial z = 0$ at $z = 0$ and this yields $a_1 = 0$. Similarly, introducing the condition $Z = 0$, at $z = h$

$$J_{-\mu+1}(\omega_j h^{q/2}) = 0. \tag{34}$$

Equation (34) leads to the Sturm–Liouville eigenvalue problem which has the corresponding norm:

$$\bar{N}_j = \begin{cases} \dfrac{u_r}{(\alpha+1)} h^{\alpha+1}, & j = 0 \\ \dfrac{u_r}{q} h^q J_{-\mu}(\omega_j h^{q/2}), & j = 1, 2, 3 \ldots \end{cases} \tag{35}$$

Substituting eqs (28) and (35) in eq. (21), the final solution is expressed in the following form:

$$\bar{c}_y(x, z) = \left[a_0 + z^{(1-\beta)/2} \sum_{j=1}^{\infty} a_n J_{-\mu}(\omega_j z^{q/2}) E_{\alpha}(\lambda^2 x^{\alpha}) \right]. \tag{36}$$

We quote that the ordinary and generalised Mittag–Leffler functions interpolate between an entirely exponential law and power-law like behaviour of the phenomena guided by ordinary kinetic equations and their fractional counterparts [49,50]. Assume that special values for integer $\alpha = n$ are

$$E_0(x) = \frac{1}{1-x}, \tag{37}$$

$$E_1(x) = \exp^x, \tag{38}$$

$$E_2(x) = \cosh(\sqrt{x}). \tag{39}$$

Using the first boundary condition [eq. (36)] and source $x = 0$

$$u_r z^{\alpha} \left[a_0 + z^{(1-\beta)/2} \sum_{j=1}^{\infty} a_n J_{-\mu}(\omega_j z^{q/2}) E_{\alpha}(\lambda^2 x^{\alpha}) \right] = Q\delta(z - h_s), \tag{40}$$

introducing the orthogonality property of Bessel function $z^{(1-\beta)/2} J_{-\mu}(\omega_i z^{q/2})$, $i \geq 1$ and integrating with respect to z with limits 0 to h, one gets

$$a_0 = \frac{Q}{N_j}, \quad j = 0, \tag{41}$$

$$a_n = \frac{Q h_s^{(1-\beta)/2}}{N_j} \frac{J_{-\mu}(\omega_j h_s^{q/2})}{J_{-\mu}^2(\omega_j h^{q/2})}. \tag{42}$$

The solution of the concentration distribution can be obtained as follows:

$$\bar{c}_y(x, z) = Q\left[\frac{(\alpha+1)}{u_r h^{\alpha+1}} + \frac{(z h_s)^{(1-\beta)/2} q}{u_r h^q} \times \sum_{j=1}^{\infty} \frac{J_{-\mu}(\omega_j z^{q/2}) J_{-\mu}(\omega_j h_s^{q/2})}{J_{-\mu}^2(\omega_j h^{q/2})} E_{\alpha}(-\lambda_j^2 x^{\alpha})\right]. \tag{43}$$

For $\alpha = 1$ we get the required solution similar to that obtained by Sharan and Kumar [18]:

$$\bar{c}_y(x, z) = Q\left[\frac{(\alpha+1)}{u_r h^{\alpha+1}} + \frac{(z h_s)^{(1-\beta)/2} q}{u_r h^q} \times \sum_{j=1}^{\infty} \frac{J_{-\mu}(\omega_j z^{q/2}) J_{-\mu}(\omega_j h_s^{q/2})}{J_{-\mu}^2(\omega_j h^{q/2})} \exp(-\lambda_j^2 x)\right]. \tag{44}$$

4. Results and discussion

The performance of the developed model has been evaluated against experimental dataset from the experiments carried out in Prairie Grass (M L Barad, 1958). The Prairie Grass campaign was conducted in the summer of 1956 in O'Neill, Nebraska. The Prairie grass tracer experiments were released from a continuous point source at a height of 0.46 m in which the release was 1.5 m above the ground surface. The micro-meteorological parameters were also measured. Fortunately, we can have a typical and widely used dataset of dispersion observations used during the evaluation of dispersion models for ground-level releases of continuous plume over flat terrain. In all, 68 attempts of sulphur dioxide was made at a height of 0.46 m, without taking into account runs series 65, 66, 67 and 68 within a release height at 1.5 m. For the occasion, five sampling arcs, 50, 100, 200, 400, 800 m downwind from the source were needed. The relevant dataset is available at http://www.dmu.dk/International/Air/Models/Background/ExcelPrairie.htm. The condition attached in this dataset for normalised crosswind-integrated concentrations and meteorological parameters (L, u^*, w^* and h) are detailed. The surface roughness length $z_0 = 0.006$ m [12]. In our study, we assume $z_r = 1$ m. The statistical indices of table 1 point out that a good agreement is obtained between the experimental data and the model. In the wake of the statistical indices (Hanna, 1989) these parameters focus on the agreement between model predictions and observations [18]. Usually, to evaluate the performance of the dispersion models, the Environmental Protection Agency (EPA) recommends a well-known set of statistical indices (Cox and Tikvart, 1990) defined as follows:

Normalised mean square error (NMSE):

$$\text{NMSE} = \overline{(c_p - c_o)^2}/\bar{c}_o\bar{c}_p.$$

Fractional bias (FB):

$$\text{FB} = (\bar{c}_o - \bar{c}_p)/0.5(\bar{c}_o + \bar{c}_p).$$

Correlation coefficient (R):

$$R = \overline{(c_o - \bar{c}_o)(c_p - \bar{c}_p)}/(\sigma_o\sigma_p).$$

Fractional variance:

$$\text{FS} = (\sigma_o - \sigma_p)/0.5(\sigma_o + \sigma_p).$$

Factor of two (FA2):

$$\text{FA2} = 0.5 \le c_p/c_o \le 2,$$

where σ_p and σ_o represent the standard deviations of c_p and c_o respectively, the over bars indicate the average over all measurement terms (N). These parameters focus on the agreement between model predictions and observations [18]. A model appears to be perfect for ideal values

$$\text{NMSE} \le 0.4, -0.3 \le \text{FB} \le 0.3$$

and

$$\text{Correlation coefficient (COR)} = \text{FA2} = 1.$$

Similarly, the distribution of data in table 2 reinforces annotations previously developed, and hence the model reproduced the observed concentrations fairly well. The statistical indices used to assess the performance of these models, The Sharan-PL model (44), the CITT-MM model and α-CITT model (43), are summarised in table 1. Table 3 reveals the average wind speed in the longitudinal direction (u) and it indicates that the coefficient

Table 1. Comparison between the three models according to the standard statistical performance measures.

Model	NMSE	FA2	FB	COR	FS
Ideal	0	1	0	1	0
Sharan-PL	0.06	0.95	−0.16	0.97	0.12
CITT-MM	0.08	0.79	−0.09	0.96	0.13
α-CITT	0.06	0.98	−0.03	0.96	0.28

Table 2. Values of normalised crosswind-integrated concentration (CPMW)10^{-4} s m^{-2} and (CPL)10^{-4} s m^{-2} computed with Mooney and Wilson (1993) and linear functions respectively. Co 10^{-4} s m^{-2} is the observed concentration in Prairie Grass experiment.

Distance (m)	50	100	200	400	800	Run
CPMW	934	803.58	610.92	418.85	258.36	1
CPL	733	440.35	185.65	31.87	11.3	
Co	886.16	276.59	69.38	21.35	8.01	
CPMW	957.12	890.06	738.87	565.83	412.5	17
CPL	897.72	680.99	425.39	230.57	110.87	
Co	1101.1	678.02	364.38	191.88	110.03	

Table 3. Domains of the values of meteorological parameters recorded in the Prairie Grass experiment.

Stability	u_* (m s^{-1})	w_* (m s^{-1})	h (m)	L (m)
Stable	$0.014 \leq u_* \leq 50$	$-0.15 \leq w_* \leq -0.43$	$0 \leq h \leq 200$	$0.40 \leq L \leq 328$
Unstable	$0.14 \leq u_* \leq 0.67$	$0.70 \leq w_* \leq 2.33$	$96 \leq h \leq 2100$	$-329 \leq L \leq -6$

K is chosen, in order of comparison, as being the average value of the diffusion coefficient over the longest x distance of each experiment. Tables 1 and 2 show good agreement concerning the performance between the proposed fractional models. In particular, in table 1 from the statistical indices obtained according to the Prairie Grass dispersion experiment the α-CITT model (43) works much better than the other models, mainly, looking at the NMSE and the FB. Thus, NMSE $= 0.06$ for the α-CITT model against 0.08 and 0.06 for the CITT-MM and Sharan-PL respectively, while FB corresponds to -0.03 for α-CITT and -0.09 and -0.16 for CITT-MM and Sharan-PL models respectively. From the point of view of statistical physics, normal diffusion is the basis of the well-known Brownian motion of particles. The advantage of fractional models over standard integer derivative models is that the former can very well describe the inherent processes of unusually exponential or heavy tail decay. As an illustration, compared to the traditional Gaussian models, fractional models display direct consequence of the anomalous diffusion present in the turbulent diffusion that characterises a distribution with a power-law profile of average quadratic displacement.

The solutions obtained by this new method are different from those obtained by other researchers because of the following reasons.

- By selecting particular values of $0 < \alpha < 1$, eq. (43) has various types of special solutions in the form of analytic, trigonometric, hyperbolic and special functions method with specific types of boundary conditions for a bounded domain.

  But still some of our solutions have similarities with the following.
- Solutions eq. (43) for $\alpha = 1$ has similarity with the analytic solutions of the atmospheric advection–diffusion equation with the stratification of the boundary condition [18,51]. The solution has been obtained by applying the separation of variable method and Bessel's equation.
- Solutions eq. (43) for $\alpha = 1$ has similarities with the solutions obtained by [52,53] using analytical solution of the advection–diffusion equation with the Neumann (total reflection) boundary conditions and variable separable method, orthogonality condition of Bessel function and the special functions method.

  But the differences are the large scale downwind as constant and vertical eddy diffusivities considered as an explicit function of downwind distance and vertical height.

Our remaining solutions are recent and have not been simulated before.

In the power-law profiles (29a) and (29b) we have considered the parameters β varying from 0.4 to 0.9. For $\alpha = 1$, we obtain from eq. (26), a set of first-order ordinary differential equation (ODEs). Thus, the accuracy of results is in the same line with those established previously by some comprehensive analytical models for the distribution of air pollutants, using variable separable method, orthogonality condition of Bessel function and the special function method [51,52]. We assume that the modified form of $K_z(x, z)$ is given as described by Mooney and Wilson 1993, and $f_1(x)$ is the correction of $K(z)$ near the source dispersion. Regarding the analysis of Prairie Grass concerning the atmospheric dispersion, as stated by Irwin *et al* (2007) the crosswind integrated concentration of the plume dispersion is observed as having a Gaussian shape, that is the basic characteristic in all atmospherical transport and diffusion models. Finally, in order to justify the best approximated value for the α-CITT model, the analysis is conducted on the basis of the values $\alpha = 0.8$, $\alpha = 0.99$, and $\beta = 0.7$, $\beta = 0.8$ in steps of 10^{-2}. The best performance occurs when it is close to 0.90. The variations noted in the choice of values proceed from the importance of the characterisation of the different diffusion coefficients used throughout the process. In fact, the Gaussian solution obtained from Brownian motion results from the central limit theorem. Of course, diffusion under the influence of power-law function can be modelled by the advection–dispersion equation, the fractional case may be different for the advection and the transport. The motivation of this work is the good characterisation of the non-Fickian anomalous transport, it involves random walk in continuous time, with probability distributions different from those of the Brownian motion. Indeed, many works show that Fick's law does not always describe the dispersion well. Fick's law says that from a localised injection, the concentration takes the Gaussian shape with the standard deviation proportional to the square root of coordinate axis. Experimental analysis has shown evidence of qualitatively different behaviours, while replacing the Gaussian forms with Levy's stable laws in a velocity field. This effect seems to be the origin of this phenomena [18,54,55] and for describing what is going on it is necessary to go beyond the framework of the classical models of diffusion applied in small-scale Markov process. In fact, anomalous diffusion seems well suited to take into account the memory effects that are observed in many complex systems. Anomalous diffusion appears more effective in describing particle motion in turbulent flow [56] and dispersive property as stated [57]. As result, the fractional derivative models have an obvious physical meaning and are easy to obtain by adjusting experimental or field measurement data. This discussion shows the effectiveness and power of the new method in the application of 'orthogonality condition of Bessel function'. This method will help the researchers to apply the special functions method to many fractional differential equations.

5. Conclusion

In this work, we present an analytical solution of the steady-state atmospheric fractional diffusion equation in a finite domain. The present work reinforces the author's previous researches in the field of air pollutants dispersion. In this model, we use fractional calculus tools to determine the approximated solution, by introducing the classic integral transform technique (CITT) to solve the transformed problem. Thus, an integro-differential equation based on the fractional operator properties is introduced. Then, we assume the exponential decay of the modes that are modified so that long tailed memory effects can induce a moderate power-law decay of the modes, in accordance with the Mittag–Leffler pattern. The transport models highlight the probabilistic processes representing the movement of air particles. Their probability densities also govern conventional partial differential equation (PDE) with non-local operators, which completely characterises the process if we disregard all boundary conditions. The studied models have the advantages that the probabilistic process makes it possible to understand observable theories which are inaccessible using the conventional PDE associated with a single instant. Thus, fractional differential equations are essential for modelling natural phenomena with realistic condition limits. Each of these points of view makes it possible to process data from experiments. Hence, we conclude that the fractional derivative-type proposed behaves well with respect to the classical properties of the integer calculation. Thus, fractional calculus is a decisive tool, in modelling turbulent diffusion and non-conservative phenomena.

Appendix

To see a possible generalisation of this problem, in the same spirit as in Weiss treatise [58], in Fourier–Laplace space

$$u\bar{c}_y(k,u) - \bar{c}_y\,(k,0) = u\psi(u)\phi\,(k)\,\bar{c}_y\,(k,u) \\ -\psi(u)\bar{c}_y\,(k,0)\,, \tag{45}$$

the Fourier transform of $\phi\,(k)$ is an operator in k. Thus

$$\phi\,(k)\,\bar{c}_y\,(k,x) \equiv \lambda_c(k)\bar{c}_y\,(k,x)\,. \tag{46}$$

Assume that subdiffusion is characterised by a finite transfer variance Σ^2 associated with a diverging characteristic waiting time. Considering the expansions for small k and the usual long-time limit [59], we reach the relation

$$\bar{c}_y(k,u) - \frac{\bar{c}_y(k,0)}{u} = u^{-\alpha}L(z)\bar{c}_y(k,u). \tag{47}$$

Using the definition of the Riemann–Liouville fractional differentiation of order $1-\alpha$, $0<\alpha<1$ [60]

$${}_0D_x^{1-\alpha}\bar{c}_y(k,u) = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial x}\int_0^x \mathrm{d}x'\frac{\bar{c}_y(k,x')}{(x-x')^{1-\alpha}}. \tag{48}$$

The integration of the corresponding theorem of Laplace transformation can be shown to hold for fractional integration

$$\mathscr{L}\{{}_0D_x^{-\alpha}c_y(k,x)\} = u^{-\alpha}c_y(k,u). \tag{49}$$

We obtain the fractional partial equation

$$\frac{\partial\bar{c}_y(k,u)}{\partial x} = {}_0D_x^{1-\alpha}L(z)\bar{c}_y(k,u), \tag{50}$$

by considering the operator

$$L(z) = \frac{\partial}{\partial z}\left[K_z(x,z)\frac{\partial}{\partial z}\right].$$

Let us consider the fractional diffusion-type eq. (50), in combination with the separation ansatz

$$\bar{c}_y(x,z) = \sum_{n=0}^{\infty} X_n(x)Z_n(z). \tag{51}$$

The resulting equation

$$\frac{\mathrm{d}X(x)}{\mathrm{d}x}({}_0D_x^{1-\alpha}X)^{-1} = \frac{L(z)Z}{Z} = -\lambda \tag{52}$$

is then separated into the pair of eigenequations [42,61]

$${}_0D_x^{\alpha}X(x) - \frac{x^{-\alpha}\delta(0)}{\Gamma(1-\alpha)} = \lambda_{n,\alpha}X(x) \tag{53}$$

$$L(z)Z(z) = -\lambda_{n,\alpha}Z \tag{54}$$

for an eigenvalue $\lambda_{n,\alpha}$ of $L(z)$.

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