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Dynamical behaviour of fractional-order finance system

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Abstract. In this paper, we developed the fractional-order finance system transmission model. The main objective of this paper is to construct and evaluate a fractional derivative to track the shape of the dynamic chaotic financial system of fractional order. The numerical solution for fractional-order financial system is determined using the Atangana–Baleanu–Caputo (ABC) and Caputo derivatives. Picard–Lindelof's method shows the existence and uniqueness of the solution. Numerical techniques show that ABC derivative strategy can be used effectively to overcome the risk of investment. An active control strategy for controlling chaos is used in this system. The stabilisation of equilibrium is obtained by both theoretical analysis and simulation results.

Keywords. Finance system; fractional derivative; Picard–Lindelof; stability analysis; price index.

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1. Introduction

Among various fields of natural science, researchers are attracted more to nonlinear chaotic systems. Such systems are dynamic and sensitive to initial conditions. Although the chaotic phenomenon first became evident in 1985, the influence of Western science has been affected by the fact that in an economic system the chaotic phenomenon itself makes the macroeconomic feature infinite. The efficiency of the intervention is very small, while the government can take macrocontrol measures like finance policies or monetary policies to intervene. The precise economy is significantly restricted due to the instability and the difficulty and also the rational prediction behaviour. The internal structures of the economy have shown discrepancies and inconsistencies, and in the fields of finance, supply and social economics, extremely difficult phenomena are occurring because of interaction between non-linear variables, increasingly complicated with all kinds of economic problems, and processes developing at a low to a higher level. The monitoring and stabilisation of unstable, regular or stationary solutions for accurate economic forecasts is thus becoming increasingly important [1,2].

The fractional calculus has grown significantly during the last forty years as an excellent modelling technique. This is a strong tool in designing most of the physical processes with a memory effect which cannot be expressed well by integer and differential equations. It has been applied to many fields of science and technology. For an overview of the theory and applications of fractional calculations, we refer to [3–7]. The drawback of the derivative is that it has a non-local property. The current status value is determined through all recent values and historical values for the objective purpose. This excellent property can be used to model several financial variables, primarily because the financial and economic variables always have a time-based memory impact, such as rate of interest, stock price, exchange amount of the future [8–11]. The study of financial and economic activities using fractional derivative model is suggested by Chen in [12]. This includes a variety of macroeconomic factors, such as investment, interest and price indices. In [13,14], the process of sliding and feedback control was respectively studied with the dynamics of chaos and chaos management. The proposed delayed financial system of fractionality and numerical simulations discuss the complex dynamic behaviour of this system. In such a system, there are a wide number of

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interesting dynamic actions, including single, multiple and chaotic motions. There is an overview of transient delays and the fractional impact on disorderly behaviour; a correct time delay can either enhance or remove the appearance of chaos [15].

The complexities of the financial and economic fractional-order structures have been discussed using several mathematical approaches (see [12,15] and references therein). Nevertheless, the physical and human interpretations of these complex phenomena in numerical simulations remain unclear. We have recognised that the occurrence of uncertainty in financial systems in fractions depends on both parameters and fractions. Nonetheless, there is no research on how a specific group of economic data should choose the correct fractional order and parameters. While, for example, we do have a number of methods to handle financial chaos, we do not know the economic understanding of these processes of chaos management. In fact, the financial and economic outcomes are decided by the inner commitment of the real world issues. Therefore, a category should exist which can be used for representing the relevant financial and economic variables, with the corresponding parameters and fractional order.

Researchers have proposed various schemes to solve fractional partial differential equations with Liouville–Caputo and Caputo–Fabrizio fractional operators. Dehghan *et al* [16] applied the HAM to solve linear partial differential equations. In this work, fractional derivatives are described in the Liouville–Caputo sense. Jafari *et al* [17] used the homotopy analysis method (HAM) to obtain the solution of multiorder fractional differential equation [18]. Recently, a few researchers have proposed several approximations and numerical methods for solving variable order of the differential equations [19]. We present a numerical technique to obtain the solution of fractional differential equations containing Atangana–Baleanu derivative [20,21]. Ganji and Jafari solved the variable order integro-differential equations using several polynomials [22]. Jafari *et al* [23] obtained an approximate solution for variable order differential equations based on Bernstein polynomials.

In this paper, we need to use fractional parameters using the Caputo and ABC fractional derivatives to build a model of complex nonlinear differential equations. The primary goal of this article is to use a fresh non-integer order derivative to study the model for finance system and to provide the results of the system with fractional derivative. Picard–Lindelof's method shows the existence and uniqueness of the solution. Numerical simulation are carried out which support and explain the convergence of the results to analysis and control the system. Complex financial system models of complex actions provide a new perspective as a result of patterns and actual behaviour of the financial system's internal structure.

2. preliminaries

The Liouville–Caputo fractional derivative (C) is presented as

$$ {}_{t_0}^{c}D_t^{\xi}\{f(t)\} = \frac{1}{\Gamma(1-\xi)}\int_{t_o}^{t}\frac{\mathrm{d}}{\mathrm{d}t}f(\tau)(t-\tau)^{-\xi}, \quad n-1<\xi\le n, \tag{1} $$

where $\Gamma(\cdot)$ represents the Gamma function [24,25]. Laplace transform to Liouville–Caputo fractional-order derivative is

$$ \mathcal{L}\{{}_{t_0}^{C}D_t^{\xi}\{f(t)\}\}(s) = S^{\xi}F(s) - \sum_{m-1}^{k=0} S^{\xi-k-1}f^{(k)}(0). \tag{2} $$

Recently, Atangana and Baleanu proposed a fractional derivative with Mittag–Leffler function as the kernel of differentiation. This kernel is non-singular and non-local and preserves the benefits of the above Liouville–Caputo derivative [26,27].

Let $f\in H^1(a,b), b>a, \xi\in[0,1]$. The new fractional derivative (Atangana–Baleanu derivative in Caputo sense) is defined as

$$ {}_{t_0}^{\mathrm{ABC}}D_t^{\xi}\{f(t)\} = \frac{B(\xi)}{1-\xi}\int_{t_0}^{t}f'(\tau)E_{\xi}\left[-\xi\frac{(t-\tau)^{\xi}}{1-\xi}\right]\mathrm{d}\tau, \quad n-1<\xi(t\le n), \tag{3} $$

where $\xi\epsilon\Re$, $B(\xi)$ represents a normalisation function $B(0)=B(1)=1$ and $E_{\xi}(\cdot)$ represents the Mittag–Leffler function. The equations of Laplace (2) are defined as follows:

$$ \begin{aligned} &\mathcal{L}\{{}_{t_0}^{\mathrm{ABC}}D_t^{\xi}\{f(t)\}\}(s) \\ &\quad = \frac{B(\xi)}{1-\xi}\mathcal{L}\left[\int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}t}f(\tau)E_{\xi}\left[-\xi\frac{(t-\tau)^{\xi}}{1-\xi}\right]\mathrm{d}\tau\right](s) \\ &\quad = \frac{B(\xi)}{1-\xi}\frac{s^{\xi}\mathcal{L}[f(t)](s)-s^{\xi-1}f(0)}{s^{\xi}+\frac{\xi}{1-\xi}}. \end{aligned} \tag{4} $$

3. Laplace homotopy analysis method for systems of FDEs

3.1 *Liouville–Caputo derivative*

The modified homotopy analysis transform method (MHATM) was proposed in [18,27]. The method is an analytical technique based on the combination of HAM and Laplace transform with homotopy polynomial. The main steps of this method are described as follows:

Step 1. Let us consider the following equation:

$$D_t^{\rho}\{g(i,t)\}+\beta[i]g(i,t)+\wedge[i]g(i,t)=\Psi(i,t),\quad i\in\mathbb{N},\quad 0<\rho\leq 1, \tag{5}$$

where $\beta[i]$ is a fixed linear operator in i. The non-linear operator $\wedge(i)$ is the continuing, satisfying Lipschitz $|\wedge(g)-\wedge(g)|\leq\vartheta|g-\phi|$ where the continuous function is $\vartheta>0$, $\Psi(i,t)$. Boundary conditions and initial conditions can be equally treated.

Step 2. We achieve the following equation of mth-order deformation:

$$g_m(i,t)=(X_m+\hbar)g_{m-1}-\hbar(1-X_m)\sum_{i=0}^{j-1}t^i g^{(i-1)}(0)+\hbar\mathcal{L}^{-1}\left(\frac{1}{s^{\rho}}\mathcal{L}\left(B_{m-1}[i]g_{m-1}(i)+\sum_{k=0}^{m-1}P_k(g_0,g_1,\ldots,g_m)-\Psi(i,t)\right)\right). \tag{6}$$

Step 3. The nonlinear definition is $\wedge[i]g(i,t)$ and includes polynomial homotopes.

$$\wedge[g(i,t)]=\wedge\left(\sum_{k=0}^{m-1}g_m(i,t)\right)=\sum_{m=0}^{\infty}P_m g^m.$$

Step 4. A sequence of homotopical polynomials for $m>1$ to extend a nonlinear term in (6). For a description of the solutions in eq. (5), which usually converge quickly to accurate solution.

$$g(i,t)=\sum_{\infty}^{m=0}g_m(i,t).$$

The following time-fractional funding model has been solved by the Caputo fractional-order derivative according to this methodology.

$$\begin{aligned}
{}_0^C D_\tau^{\rho}x(\tau)&=z(\tau)+x(\tau)y(\tau)-ax(\tau)\\
{}_0^C D_\tau^{\rho}y(\tau)&=1-by(\tau)-x(\tau)x(\tau)\\
{}_0^C D_\tau^{\rho}z(\tau)&=-x(\tau)-cz(\tau)
\end{aligned} \tag{7}$$

with initial conditions $x(0)=n_1=0.1$, $y(0)=n_2=4$, $z(0)=n_3=0.5$.

Solution. We use the Laplace transform (1) for the first equation of system (7).

$$s^{\rho}\bar{x}(s)-s^{\rho-1}x(0)=\mathcal{L}\{z(\tau)+x(\tau)y(\tau)-ax(\tau)\}.$$

By using initial conditions in the above equation, we get

$$\bar{x}(s)=\frac{x(0)}{s}+\mathcal{L}\{z(\tau)+x(\tau)y(\tau)-ax(\tau)\}. \tag{8}$$

Applying the Laplace inverse to eq. (8), then

$$x(\tau)=n_1+\mathcal{L}^{-1}\frac{1}{s^{\rho}}\mathcal{L}\{z(\tau)+x(\tau)y(\tau)-ax(\tau)\}.$$

For the other equations in system (7),

$$y(\tau)=n_2+\frac{\tau^{\rho}}{\Gamma\rho+1}+\mathcal{L}^{-1}\frac{1}{s^{\rho}}\mathcal{L}\{-by(\tau)-x(\tau)x(\tau)\}$$

$$z(\tau)=n_3+\mathcal{L}^{-1}\frac{1}{s^{\rho}}\mathcal{L}\{-x(\tau)-cz(\tau)\}.$$

Here we choose a linear form operator.

$$[\phi_j(\tau;q)]=\mathcal{L}[\phi_j(\tau;q)],\ j=1,\ 2,\ 3$$

$(c)=0$ with constants, where c is a constant, we have

$$\begin{aligned}
N[\phi_1(\tau;q)]&=\mathcal{L}[\phi_1(\tau;q)]-n_1-\frac{1}{s^{\rho}}\mathcal{L}\{\phi_3+\phi_1\phi_2-a\phi_1\}\\
N[\phi_2(\tau;q)]&=\mathcal{L}[\phi_2(\tau;q)]-n_2+\frac{1}{s^{\rho}}\mathcal{L}\{-b\phi_2-\phi_1\phi_1\}\\
N[\phi_3(\tau;q)]&=\mathcal{L}[\phi_3(\tau;q)]-n_3+\frac{1}{s^{\rho}}\mathcal{L}\{-\phi_1-c\phi_3\}.
\end{aligned}$$

The so-called deformation equation of zeroth order is given by

$$(1-q)[\phi_j(\tau;q)-u_o(\tau)]=q\hbar[\phi_j(\tau;q)],\quad j=1,\ 2,\ 3.$$

When $q=0$ and $q=1$,

$$\phi_j(\tau;0)=u_0(\tau),\ \phi_j(\tau;1)=u(\tau),\ j=1,\ 2,\ 3,$$

where equations are given for the mth-order deformation

$$\begin{aligned}
\mathcal{L}\{x_m(\tau)-P_m x_{m-1}(\tau)\}&=\hbar R_m(\vec{x}_{m-1},\ \tau)\\
\mathcal{L}\{y_m(\tau)-P_m y_{m-1}(\tau)\}&=\hbar R_m(\vec{y}_{m-1},\ \tau)\\
\mathcal{L}\{z_m(\tau)-P_m z_{m-1}(\tau)\}&=\hbar R_m(\vec{z}_{m-1},\ \tau).
\end{aligned} \tag{9}$$

Using the inverse Laplace transformation in eq. (9), we get

$$R_m(\vec{x}_{m-1},\ \tau)=\mathcal{L}[x_{m-1}(\tau)]-(1-P_m)\times\left(n_1+\frac{1}{s^{\rho}}\mathcal{L}\{z_{m-1}+H_m-a\cdot x_{m-1}\}\right)$$

$$R_m(\vec{y}_{m-1},\ \tau)=\mathcal{L}[y_{m-1}(\tau)]-(1-P_m)\times\left(n_2+\frac{s^{\rho}}{\Gamma\rho+1}+\frac{1}{s^{\rho}}\mathcal{L}\{-b\cdot y_{m-1}-K_m\}\right)$$

$$R_m(\vec{z_{m-1}},\ \tau) = \mathcal{L}[z_{m-1}(\tau)] - (1 - P_m) \times \left(n_3 + \frac{1}{s^\rho}\mathcal{L}\{-x_{m-1} - c \cdot z_{m-1}\}\right)$$

$$x_m(\tau) = (P_m + \hbar)x_{m-1} - \hbar(1 - P_m)(n_1) - \hbar\mathcal{L}^{-1}\left\{\frac{1}{s^\rho}\mathcal{L}\{z_{m-1} + H_m - a \cdot x_{m-1}\}\right\}$$

$$y_m(\tau) = (P_m + \hbar)y_{m-1} - \hbar(1 - P_m) \times \left(n_2 + \frac{t^\rho}{\Gamma\rho + 1}\right) - \hbar\mathcal{L}^{-1}\left\{\frac{1}{s^\rho}\mathcal{L}\{-b \cdot y_{m-1} - K_m\}\right\}$$

$$z_m(\tau) = (P_m + \hbar)z_{m-1} - \hbar(1 - P_m)(n_3) - \hbar\mathcal{L}^{-1}\left\{\frac{1}{s^\eta}\mathcal{L}\{-x_{m-1} - c \cdot z_{m-1}\}\right\}, \tag{10}$$

where

$$H_m = \frac{1}{\Gamma m + 1}\left[\frac{\mathrm{d}^m}{\mathrm{d}q^m}N[(q\phi_1(\tau; q))(q\phi_2(\tau; q))]\right]_{q=0'} \tag{11}$$

$$K_m = \frac{1}{\Gamma m + 1}\left[\frac{\mathrm{d}^m}{\mathrm{d}q^m}N[(q\phi_1(\tau; q))(q\phi_1(\tau; q))]\right]_{q=0'}. \tag{12}$$

Finally, the solutions of eq. (7) are

$$x(\tau) = \sum_{m=0}^{\infty} x_m(\tau), \quad y(\tau) = \sum_{m=0}^{\infty} y_m(\tau),$$
$$z(\tau) = \sum_{m=0}^{\infty} z_m(\tau). \tag{13}$$

Through coupling Laplace transform (1) and its inverse, another type (7) solution can be achieved. The iterative method is

$$x_n(\tau) = n_1 + \mathcal{L}^{-1}\left\{\frac{1}{s^\rho}\mathcal{L}\{z_{n-1}(\tau) + x_{n-1}(\tau)y_{n-1}(\tau) - ax_{n-1}(\tau)\}(s)\right\}(\tau)$$

$$y_n(\tau) = n_2 + \mathcal{L}^{-1}\left\{\frac{1}{s^\rho}\mathcal{L}\{1 - by_{n-1}(\tau) - x_{n-1}(\tau)x_{n-1}(\tau)\}(s)\right\}(\tau)$$

$$z_n(\tau) = n_3 + \mathcal{L}^{-1}\left\{\frac{1}{s^\rho}\mathcal{L}\{-x_{n-1}(\tau) - cz_{n-1}(\tau)\}(s)\right\}(\tau). \tag{14}$$

For the initial conditions of n_1, n_2 and n_3, if n tends to infinity, it is assumed that the solution exists.

Theorem 3.1. *The recursive method given by eq.* (12) *is stable.*

Proof. We take (12) which is given for positive solution. A, B and C can be dealt with in such a way that $0 \le \tau \le \infty$

$$\| x(\tau) \| < A; \quad \| y(\tau) \| < B; \quad \| z(\tau) \| < C.$$

Now we consider a $L2(e,\ f)(0,\ W)$ subset defined by

$$\Xi = \left\{\rho : (e,\ f)(0,\ W) \to \Xi, \frac{1}{\Gamma(\rho)}\int(\tau - \beta)^{(\rho-1)}v(\beta)u(\beta)\mathrm{d}\beta < \infty\right\}.$$

Assuming Θ as the following operator

$$\begin{aligned}\Theta(x,\ y,\ z) &= z(\tau) + x(\tau)y(\tau) - ax(\tau)\\ &= 1 - by(\tau) + x(\tau)x(\tau)\\ &= -x(\tau) - cz(\tau),\end{aligned}$$

we have

$$\begin{aligned}&\langle\Theta(x,\ y,\ z) - \Theta(x_1,\ y_1,\ z_1), (x - x_1,\ y - y_1,\ z - z_1)\rangle\\ &< \left\{\frac{\| z(\tau) - z_1(\tau) \|}{\| x(\tau) - x_1(\tau) \|} + \| y(\tau) - y_1(\tau) \| - a\right\} \times \| x(\tau) - x_1(\tau) \|^2\\ &< \left\{\frac{1}{\| y(\tau) - y_1(\tau) \|} - b - \frac{\| x(\tau) - x_1(\tau) \|^2}{\| y(\tau) - y_1(\tau) \|}\right\} \times \| y(\tau) - y_1(\tau) \|^2\\ &< \left\{-\frac{\| x(\tau) - x_1(\tau) \|}{\| z(\tau) - z_1(\tau) \|} - c\right\} \| z(\tau) - z_1(\tau) \|^2,\end{aligned}$$

where

$$\begin{aligned}&\langle\Theta(x,\ y,\ z) - \Theta(x_1,\ y_1,\ z_1), (x - x_1,\ y - y_1,\ z - z_1)\rangle\\ &< M \| x(\tau) - x_1(\tau) \|^2,\\ &< N \| y(\tau) - y_1(\tau) \|^2,\\ &< O \| z(\tau) - z_1(\tau) \|^2 .\end{aligned} \tag{15}$$

If a non-null vector (x_1, y_1, z_1) is also taken into consideration, using a certain routine like above, we get

$$\begin{aligned}&\langle\Theta(x,\ y,\ z) - \Theta(x_1,\ y_1,\ z_1), (x - x_1,\ y - y_1,\ z - z_1)\rangle\\ &< M \| x(\tau) - x_1(\tau) \| \| x(\tau) \|\\ &< N \| y(\tau) - y_1(\tau) \| \| y(\tau) \|\\ &< O \| z(\tau) - z_1(\tau) \| \| z(\tau) \| .\end{aligned} \tag{16}$$

From the results obtained in eqs (15) and (16), we conclude that the iterative method used is stable. □

3.2 *Atangana–Baleanu–Caputo derivative*

Following the methodology described in [18,19] we solve the following time-fractional finance system via ABC fractional-order derivative

$$
\begin{aligned}
{}^{\mathrm{ABC}}_{0}D^{\rho}_{\tau}x(\tau) &= z + xy - ax\\
{}^{\mathrm{ABC}}_{0}D^{\rho}_{\tau}y(\tau) &= 1 - by - xx\\
{}^{\mathrm{ABC}}_{0}D^{\rho}_{\tau}z(\tau) &= -x - cz
\end{aligned}
\tag{17}
$$

with initial conditions $x(0) = n_1 = 0.1$, $y(0) = n_2 = 4$, $z(0) = n_3 = 0.5$.

Solution. The Laplace transform (3) is applied to the first equation of system (16), and we have

$$
\frac{B(\rho)}{1-\rho}\frac{s^{\rho}\tilde{x}(s) - s^{\rho_1}x(0)}{s^{\rho}+\frac{\rho}{1-\rho}} = \mathcal{L}\{z(\tau) + x(\tau)y(\tau) - ax(\tau)\}.
$$

We are able to take initial conditions and simplify the equation above

$$
\tilde{x}(s) = \frac{x(0)}{s} + \frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}} \times\mathcal{L}\{z(\tau) + x(\tau)y(\tau) + ax(\tau)\}.
\tag{18}
$$

We get the inverse transformation of the Laplace into eq. (17).

$$
x(\tau) = n_1 + \mathcal{L}^{-1}\left\{\frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}\{z(\tau) + x(\tau)y(\tau) + ax(\tau)\}\right\}.
$$

For the other equations shown in (16), we have

$$
\begin{aligned}
y(\tau) &= n_2 + \frac{(1-\rho)}{B(\rho)s} + \frac{\rho}{B(\rho)s(\rho)} + \mathcal{L}^{-1}\left\{\frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}\{-by(\tau) - x(\tau)x(\tau)\}\right\},\\
z(\tau) &= n_3 + \mathcal{L}^{-1}\left\{\frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}\{-x(\tau) - cz(\tau)\}\right\}.
\end{aligned}
$$

We select a linear operator of the kind in this case.

$$
[\phi_{\mathrm{j}}(\tau;\mathrm{q})] = \mathcal{L}[\phi_j(\tau;q)],\quad j = 1,\ 2,\ 3.
\tag{19}
$$

$(\mathfrak{c}) = 0$, with c being constant. The following system is defined next.

$$
\begin{aligned}
N[\phi_1(\tau;q)] &= \mathcal{L}[\phi_1(\tau;q)] - n_1 - \frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}\{\phi_3 - \phi_1\phi_2 + a\phi_1\}\\
N[\phi_2(\tau;q)] &= \mathcal{L}[\phi_2(\tau;q)] - n_2 - \frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}\{b\phi_2 + \phi_1^2\}\\
N[\phi_3(\tau;q)] &= \mathcal{L}[\phi_3(\tau;q)] - n_3 - \frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\frac{1}{s^{\rho}}\mathcal{L}\{\phi_1 + c\phi_3\}.
\end{aligned}
$$

The deformation equations of the mth-order are given by

$$
\begin{aligned}
\mathcal{L}\{x_m(\tau) - P_m x_{m-1}(\tau)\} &= \hbar R_m(\vec{x_{m-1}},\ \tau)\\
\mathcal{L}\{y_m(\tau) - P_m y_{m-1}(\tau)\} &= \hbar R_m(\vec{y_{m-1}},\ \tau)\\
\mathcal{L}\{z_m(\tau) - P_m z_{m-1}(\tau)\} &= \hbar R_m(\vec{z_{m-1}},\ \tau).
\end{aligned}
\tag{20}
$$

When we use the inverse Laplace transformation to eq. (18), we have

$$
\begin{aligned}
R_m(\vec{x_{m-1}},\ \tau) &= \mathcal{L}[x_{m-1}(\tau)] - (1-P_m)n_1 + \frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}\{z_{m-1} + H_{m-1} - ax_{m-1}\}\\
R_m(\vec{y_{m-1}},\ \tau) &= \mathcal{L}[y_{m-1}(\tau)] - (1-P_m) \times\left(n_2 + \frac{(1-\rho)}{B(\rho)} + \frac{\rho s^{\rho}}{\Gamma(\rho+1)}\right) - \frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}\{by_{m-1} + K_{m-1}\}\\
R_m(\vec{z_{m-1}},\ \tau) &= \mathcal{L}[z_{m-1}(\tau)] - (1-P_m)n_3 - \frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}\{x_{m-1} + cz_{m-1}\}.
\end{aligned}
$$

The mth-order deformation equation (18) solution is specified as

$$
\begin{aligned}
x_m(\tau) &= (P_m + \hbar)x_{m-1} - \hbar(1-P_m)n_1 + \hbar\mathcal{L}^{-1}\left\{\frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}[z_{m-1} + H_m - ax_{m-1}]\right\}\\
y_m(\tau) &= (P_m + \hbar)y_{m-1} - \hbar(1-P_m) \times\left(n_2 + \frac{(1-\rho)}{B(\rho)} + \frac{w^{\rho}}{B(\rho)\Gamma\rho+1}\right) - \hbar\mathcal{L}^{-1}\left\{\frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}[by_{m-1} + K_m]\right\}\\
z_m(\tau) &= (P_m + \hbar)z_{m-1} - \hbar(1-P_m)n_3 + \hbar\mathcal{L}^{-1}\left\{\frac{(1-\rho)s^{\rho}+\rho}{B(\rho)s^{\rho}}\mathcal{L}[-x_{m-1} - cz_{m-1}]\right\},
\end{aligned}
\tag{21}
$$

where

$$
H_m = \frac{1}{\Gamma(m+1)}\left[\frac{\mathrm{d}^m}{\mathrm{d}q^m}N[(q\phi_1(\tau;q))(q\phi_2(\tau;q))]\right]_{q=0'}
\tag{22}
$$

$$
K_m = \frac{1}{\Gamma(m+1)}\left[\frac{\mathrm{d}^m}{\mathrm{d}q^m}N[(q\phi_1(\tau;q))(q\phi_1(\tau;q))]\right]_{q=0'}.
\tag{23}
$$

We have the followings convergence solutions:

$$x_{n+1}(\tau) = \frac{1-\rho}{B(\rho)}\{z_n(\tau) + x_n(\tau)y_n(\tau) - ax_n(\tau)\} + \frac{\rho}{B(\rho)\Gamma(\rho)}\int_0^\tau (\tau - w)^{\rho-1}\{z_n(w) + x_n(w)y_n(w) - ax_n(w)\}\mathrm{d}w,$$

$$y_{n+1}(\tau) = \frac{1-\rho}{B(\rho)}\{1 - by_n(\tau) - x_n(\tau)x_n(\tau)\} + \frac{\rho}{B(\rho)\Gamma(\rho)}\int_0^\tau (\tau - w)^{\rho-1}\{1 - by_n(w) - x_n(w)x_n(w)\}\mathrm{d}w,$$

$$z_{n+1}(\tau) = \frac{1-\rho}{B(\rho)}\{-x_n(\tau) - cz_n(\tau)\} + \frac{\rho}{B(\rho)\Gamma(\rho)} \times \int_0^\tau (\tau - w)^{\rho-1}\{-x_n(w) - cz_n(w)\}\mathrm{d}w.$$

Theorem 3.2. *The Picard–Lindelof method shows the existence of the solution.*

Proof. The following operators are considered:

$$\begin{aligned} &\Xi_1(\tau,\ \varsigma) = z(\tau) + x(\tau)y(\tau) - ax(\tau) \\ &\Xi_2(\tau,\ \varsigma) = 1 - by(\tau) - x(\tau)x(\tau) \\ &\Xi_3(\tau,\ \varsigma) = -x(\tau) - cz(\tau). \end{aligned} \tag{24}$$

Let

$$\Omega_1 = \sup \| \gamma_{\epsilon,k_1}\Xi_1(t,\varsigma)\|;$$
$$\Omega_2 = \sup \| \gamma_{\epsilon,k_2}\Xi_2(t,\varsigma)\|;$$
$$\Omega_3 = \sup \| \gamma_{\epsilon,k_3}\Xi_3(t,\varsigma)\|;$$

where

$$\gamma_{\epsilon,k_1} = |\tau - a, \tau + a| \times [\theta - k_1, \theta + k_1] = \epsilon_1 \times k_1$$
$$\gamma_{\epsilon,k_2} = |\tau - a, \tau + a| \times [\theta - k_2, \theta + k_2] = \epsilon_1 \times k_2$$
$$\gamma_{\epsilon,k_3} = |\tau - a, \tau + a| \times [\theta - k_3, \theta + k_3] = \epsilon_1 \times k_3.$$

Considering the Picard's operator, we have

$$\Theta : \gamma(\epsilon_1, k_1, k_2, k_3) \to \gamma(\epsilon_1, k_1, k_2, k_3) \tag{25}$$

which is defined as follows:

$$\Theta\Omega(\tau) = \Omega_0(\tau)_\Delta(\tau, \Omega(\tau))\frac{1-\rho}{B(\rho)} + \frac{\rho}{B(\rho)\Gamma(\rho)}\int_0^\tau (\tau - w)^{\rho-1}\Delta(w, \Omega(w))\mathrm{d}w,$$

$$\| \Omega(\tau)\|_\infty \le \max\{k_1, k_2, k_3\},$$

$$\begin{aligned} \| \Omega(\tau) - \Omega_0(\tau)\| &= \left\|\Delta(\tau,\ \Omega(\tau))\frac{1-\rho}{B(\rho)} + \frac{\rho}{B(\rho)\Gamma(\rho)}\int_0^\tau (\tau - w)^{\rho-1}\Delta(w), \Omega(w)\mathrm{d}w\right\| \\ &\le \frac{1-\rho}{B(\rho)} \| \Delta(\tau,\ \Omega(\tau))\| + \frac{\rho}{B(\rho)\Gamma(\rho)}\int_0^\tau (\tau - w)^{\rho-1} \| \Delta(\tau), \Omega(\tau)\| \,\mathrm{d}w \\ &\le \frac{1-\rho}{B(\rho)}X = \max\{k_1, k_2, k_3\} + \frac{\rho}{B(\rho)}\xi\vartheta^\rho \le \vartheta\xi \le k = \max\{k_1, k_2, k_3\}. \end{aligned}$$

Here we request that

$$\vartheta < \frac{k}{\xi}.$$

We have together with the metric, the fixed point theorem for the Banach space

$$\| \Theta\Omega_1 - \Theta\Omega_2\|_\infty = \sup \|_{\tau\epsilon\varepsilon} |\Omega_1 - \Omega_2|,$$

$$\begin{aligned} \| \Theta\Omega_1 - \Theta\Omega_2\| &= \Big\|\{\Delta(\tau,\ \Omega_1(\tau)) - \Delta(\tau,\ \Omega_2(\tau))\}\frac{1-\rho}{B(\rho)} \\ &\quad + \frac{\rho}{B(\rho)\Gamma(\rho)}\int_0^\tau (\tau - w)^{\rho-1}\{\Delta(w, \Omega_1(\tau)) - \Delta(w, \Omega_2(\tau))\}\mathrm{d}w\Big\| \\ &\le \frac{1-\rho}{B(\rho)} \| \Delta(w, \Omega_1(\tau) - \Delta(w, \Omega_2(\tau)\| \\ &\quad + \frac{\rho}{B(\rho)\Gamma(\rho)}\int_0^\tau (\tau - w)^{\rho-1}\{\Delta(w, \Omega_1(\tau)) - \| \Delta(w, \Omega_2(\tau))\}\| \,\mathrm{d}w \\ &\le \frac{1-\rho}{B(\rho)}\omega \| \Omega_1(\tau) - \Omega_2(\tau)\| + \frac{\rho\omega}{B(\rho)\Gamma(\rho)} \times \int_0^\tau (\tau - w)^{\rho-1} \| \Omega_1(\tau) - \Omega_2(\tau)))\| \,\mathrm{d}w \\ &\le \left\{\frac{1-\rho}{B(\rho)}\omega + \frac{\rho\omega\vartheta^\rho}{B(\rho)\Gamma(\rho)}\right\} \| \Omega_1(\tau) - \Omega_2(\tau)\| \,\mathrm{d}w \\ &\le \vartheta\omega \| \Omega_1(\tau) - \Omega_2(\tau)\|, \end{aligned}$$

$\omega < 1$, and because Ω is a contraction, the defined operator $[\Theta]$ is also a contraction. We come to the conclusion that system (22) is the only solution. The Atangana–Baleanu fractional integral numerical approximation is given using the Adams–Moulton rule

$$w_w^\rho[f(\tau_{n+1})] = \frac{1-\rho}{B(\rho)}\frac{f(\tau_{n+1} - f(\tau_n))}{2} + \frac{\rho}{\Gamma(\rho)}\sum_{k=0}^{\infty}\left[\frac{f(\tau_{k+1} - f(\tau_k))}{2}\right]b_k^\rho,$$

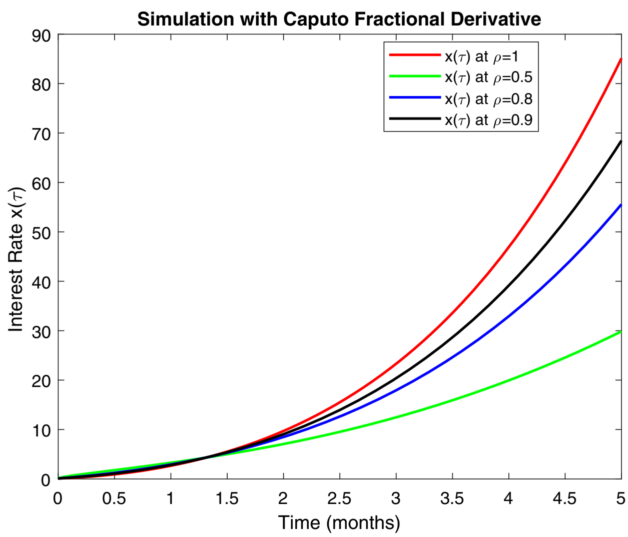

Figure 1. Interest rate $x(\tau)$ with Caputo derivative.

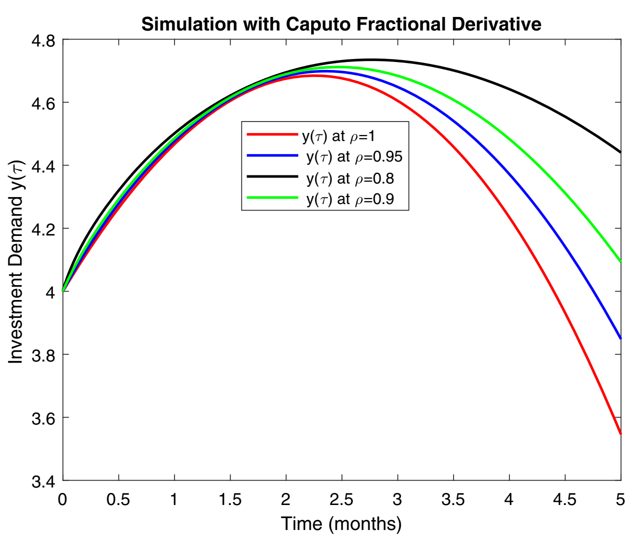

Figure 2. Investment demand $y(\tau)$ with Caputo derivative.

where $b_k^\rho = (k+1)^{1-\rho} - (k)^{1-\rho}$.

By applying the above results, we get the required results which are bounded and converged to steady state point. Hence solution exists. Using the above numerical scheme, we have

$$\begin{aligned} x_{(n+1)}(t) - x_{(n)}(t) &= x_0^n(t) \\ &+ \left\{ \frac{1-\rho}{B(\rho)} \left[\left(\frac{z_{(n+1)}(t) - z_{(n)}(t)}{2} \right) \right.\right. \\ &+ \left(\frac{x_{(n+1)}(t) - x_{(n)}(t)}{2} \right) \left(\frac{y_{(n+1)}(t) - y_{(n)}(t)}{2} \right) \\ &\left.\left. -a \left(\frac{x_{(n+1)}(t) - x_{(n)}(t)}{2} \right) \right] \right\} \\ &+ \frac{\rho}{B(\rho)} \sum_{k=0}^{\infty} (k+1)^{1-\rho} \left[\left(\frac{z_{(k+1)}(t) - z_{(k)}(t)}{2} \right) \right. \\ &+ \left(\frac{x_{(k+1)}(t) - x_{(k)}(t)}{2} \right) \left(\frac{y_{(n+1)}(t) - y_{(n)}(t)}{2} \right) \\ &\left. -a \left(\frac{x_{(n+1)}(t) - x_{(n)}(t)}{2} \right) \right] \end{aligned} \quad (26)$$

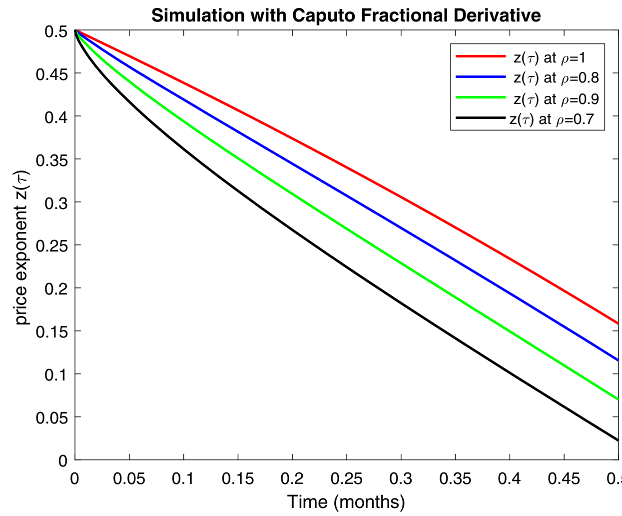

Figure 3. Price exponent $z(\tau)$ with Caputo derivative.

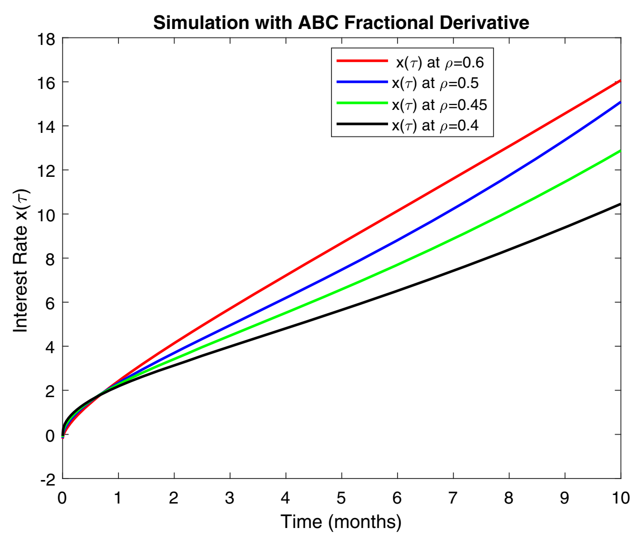

Figure 4. Interest rate $x(\tau)$ with ABC derivative.

$$\begin{aligned} y_{(n+1)}(t) - y_{(n)}(t) &= y_0^n(t) \\ &+ \left\{ \frac{1-\rho}{B(\rho)} \left[1 - b \left(\frac{y_{(n+1)}(t) - y_{(n)}(t)}{2} \right) \right.\right. \\ &\left.\left. - \left(\frac{x_{(n+1)}(t) - x_{(n)}(t)}{2} \right) \left(\frac{x_{(n+1)}(t) - x_{(n)}(t)}{2} \right) \right] \right\} \\ &+ \frac{\rho}{B(\rho)} \sum_{k=0}^{\infty} (k+1)^{1-\rho} \left[1 - b \left(\frac{y_{(k+1)}(t) - y_{(k)}(t)}{2} \right) \right. \end{aligned}$$

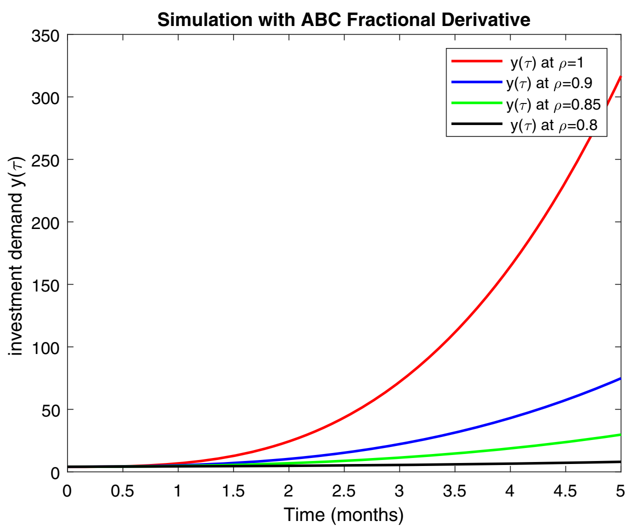

Figure 5. Investment demand $y(\tau)$ with ABC derivative.

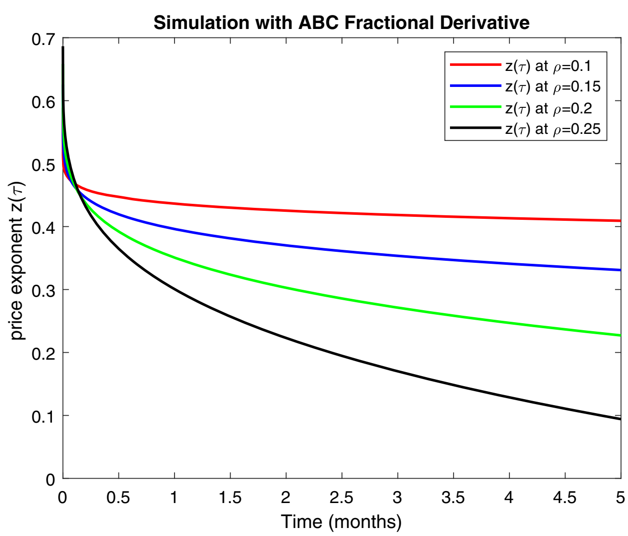

Figure 6. Price exponent $z(\tau)$ with ABC derivative.

$$-\left(\frac{x_{(k+1)}(t)-x_{(k)}(t)}{2}\right)\left(\frac{x_{(k+1)}(t)-x_{(k)}(t)}{2}\right)\Bigg] \tag{27}$$

$$\begin{aligned} z_{(n+1)}(t)-z_{(n)}(t) &= z_0^n(t) \\ &+\left\{\frac{1-\rho}{B(\rho)}\left[-\left(\frac{x_{(n+1)}(t)-x_{(n)}(t)}{2}\right)\right.\right. \\ &\left.\left.-c\left(\frac{z_{(n+1)}(t)-z_{(n)}(t)}{2}\right)\right]\right\} \\ &+\frac{\rho}{B(\rho)}\sum_{k=0}^{\infty}(k+1)^{1-\rho}\left[-\frac{x_{(k+1)}(t)-x_{(k)}(t)}{2}\right. \\ &\left.-c\frac{z_{(k+1)}(t)-z_{(k)}(t)}{2}\right]. \end{aligned} \tag{28}$$

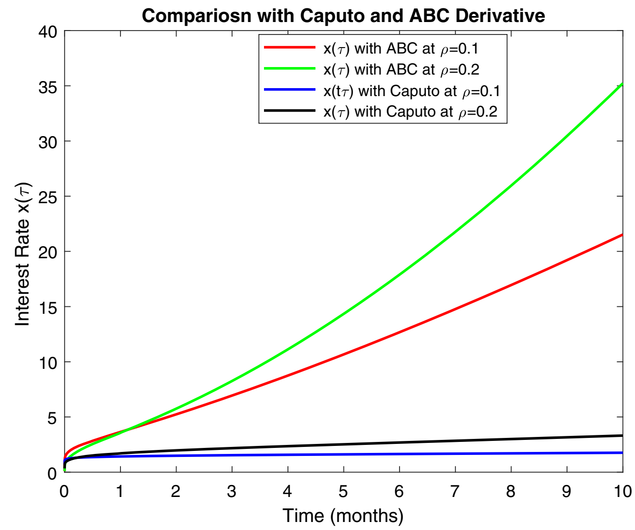

Figure 7. Comparison of $x(\tau)$ with ABC and Caputo derivatives.

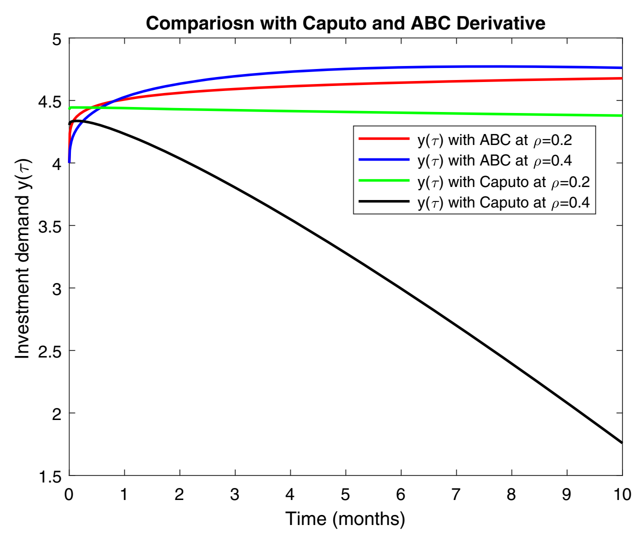

Figure 8. Comparison of $y(\tau)$ with ABC and Caputo derivatives.

4. Results and discussion

A nonlinear system of fractional differential equation was presented using the ABC derivative as the analytical solution for the fractional-order model. By utilising Caputo and ABC partial derivative, the numerical results of interest rate, investment demand and price exponent for various fractional estimations of α are obtained. Figures 1–3 refer to the graphical solution of the finance system with Caputo derivative of the finance system. In these figures, we noticed that interest rate, investment demand and the price exponent have more level of freedom as contrasted with ordinary derivatives. In figures 4–6, we use ABC fractional-order derivative of the financial system and we effectively saw that interest

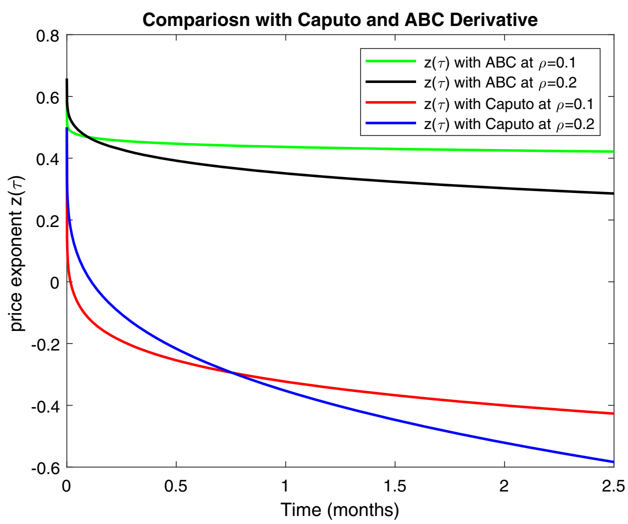

Figure 9. Comparison of $z(\tau)$ with ABC and Caputo derivatives.

rate, investment demand and price exponent rate have better estimation than ordinary derivatives. The comparison is also done for a fractional-order chaotic system with Caputo and ABC derivative in figures 7–9. It should be observed that the behaviour of the finance system is almost the same but ABC derivative gives more appropriate and comfortable behaviour in system for close loop design. Interest rate begins to rise according to the initial conditions as investment demand and price exponent begin to fall, which shows the financial system's actual macroeconomic behaviour. It is observed here that the result obtained using the technique for chaotic fractional system is more appropriate and reliable to control and maintain the financial risk management at fractional parameters. The simulation time is given in month in figures 1–9 and the step size used in evaluating the approximate solutions is $h = 0.0054$.

5. Conclusions

In this paper, dynamical chaotic fractional-order model with ABC derivative is used to analyse the economic system. The basis of this fractional model consists of exponentially decreasing non-singular kernels that appear in the derivation of the ABC. The fractional-order financial model is presented with theoretical and numerical investigation to control the economic system. Model offers the effect of evaluating numerical results on critical minimum interest rate which can be seen in the graphical representation of the variables by using the Caputo and Caputo–Fabrizio derivative. We observe that the ABC non-integer order derivative is revealing more absorbing characteristics than classical derivative by using numerical simulation. This research provides important results for the finance system having interest rate, investment demand and price exponent as state variables. This method is simple, appropriate and helpful to understand the complex financial system and show the actual behaviour of the finance system as fractional values cover the overall situation occurred while the classical derivative of the system represents the behaviour at integer values.

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