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A geometric look at the objective gravitational wave function reduction

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Abstract. There is a famous criterion for objective wave function reduction which is derived by using the Shrödinger–Newton equation [L Diosi, *Phys. Lett. A* **105**(**4–5**), 199 (1984)]. In this regard, a critical mass for the transition from quantum world to the classical world is determined for a particle or an object. In this paper, we shall derive that criterion by using the concept of Bohmian trajectories. This study has two consequences. The first is, it provides a geometric framework for the problem of wave function reduction. The second is, it represents the role of quantum and gravitational forces in the reduction process.

Keywords. Gravitational reduction of the wave function; Bohmian quantum potential; Bohmian geodesic deviation equation; Bohmian trajectories.

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1. Introduction

One of the questions that has always been raised is the boundary between quantum and classical mechanics. There is a critical mass for the transition from quantum domain to the classical world [1,2]. Such critical mass determines the macroscopicity or microscopicity of an object. By knowing the density of an object, the macroscopicity is directly determined in terms of the size of the body [2]. We expect that macroscopic bodies obey the rules of classical mechanics, i.e. definite position and momentum, determinism, etc. But, microscopic bodies obey the rules of quantum mechanics, like the uncertainty in position and momentum of the particle. One of the approaches to determine the boundary between quantum mechanics and classical mechanics is the gravitational approach. The outstanding gravitational studies for determining the boundary between the quantum world and the classical world started by Karolyhazy [2]. Diosi's work, based on the Schrödinger–Newton equation, is a remarkable work that was done after that (see refs [1,3,4]). In that equation, there is a term due to the self-gravity of the particle or body. Here, self-gravity is due to the quantum distribution of matter and is definable even for a point-particle. According to the Born rule, mass distribution of a particle or a body is $\rho = |\psi(\mathbf{x}, t)|^2$ which can be used to define self-gravity. In other words, we can consider a particle in different locations simultaneously with the distribution $\rho = |\psi(\mathbf{x}, t)|^2$ (see figure 1). This is a quantum mechanical concept which refers to the uncertainty in the position of the particle and is not obvious in our classical world.

The Schrödinger–Newton equation for a single particle with distribution $\rho = |\psi(\mathbf{x}, t)|^2$ is

$$i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = \left(-\frac{\hbar^2}{2M}\nabla^2 - GM^2 \int \frac{|\psi(\mathbf{x}', t)|^2}{|\mathbf{x}' - \mathbf{x}|} \mathrm{d}^3x' \right) \times \psi(\mathbf{x}, t). \tag{1}$$

By using the stationary state $\psi = \psi(\mathbf{x})\mathrm{e}^{iEt/\hbar}$, for the classical limit, which satisfies the above equation, and using the variational method, a relation between the mass of a particle and the width of its associated stationary wave packet is obtained. That relation, which is

$$\sigma_{(\min)} = \frac{\hbar^2}{Gm^3} \tag{2}$$

provides a criterion for the transition from the quantum world to the classical world or the breakdown of quantum superposition in terms of universal constants and

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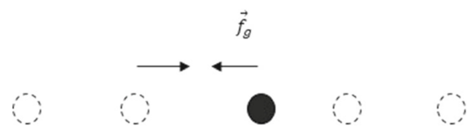

Figure 1. An imagination for the self-gravity of a particle or body in the context of quantum mechanics. The particle has no exact location due to the uncertainty principle.

the mass of the particle (objective property) [1]. For a body with radius R, the minimum wave packet width is $\sigma^{(R)} = (\sigma_{(\min)})^{(1/4)} R^{3/4}$. The critical size of a body for the transition from quantum domain to classical domain is about $R_c = 10^{-5}$ cm. For details, see [2,4].

The works of Karolyhazy and Diosi are conceptually the same. In fact, in the both works, the main concept is the existence of uncertainty in the structure of space–time. But their different results (different characteristic width, collapse time etc.) for extensive objects, are due to the different mathematical approaches [5]. For a point particle, their results are identical. We shall show that our Bohmian analysis is also consistent with their results, i.e., in obtaining the characteristic width similar to the work of Karolyhazy and Diosi for a point particle. We shall not do Bohmian investigation for an extended object in this paper.

The self-gravity of a body or a particle would localise the position distribution of the particle or body. It reduces the uncertainty in the position of the particle. It seems that our classical world is the product of a gravitational reduction [6–8].

Wave function reduction also takes place through a measurement process. In a measurement process, the wave function of a quantum system reduces instantaneously to one of its eigenvectors (wave function collapse). This is one of the postulates of orthodox or standard quantum mechanics, known as collapse postulate. Thus, there is no room for justifying the collapse phenomena in the standard quantum mechanics. In a measurement process, a pure state which is governed by the linear Schrödinger equation, evolves to a mixed state after measurement, through a non-unitary evolution [9–14]. Some physicists modify the Schrödinger equation by adding non-Hamiltonian terms for justifying the non-unitary evolution of the wave function and its reduction process [5,15–18].

In this paper, we do not study what is happening in the measurement processes. Our aim is to study the objective gravitational wave function reduction in the Bohmian context.

If quantum mechanics is universal, our classical world should be in a superposition. But, it seems that our classical world is not in a superposition. For example, a chair in a room is not in its different states of its different degrees of freedom simultaneously. This can be explained by taking into account the self-gravity of the particle (object, body). We shall prove this in the Bohmian context.

After Diosi, the most significant work which has been done, is the gravitational approach of Penrose which is based on two essential concepts in physics: the principle of equivalence in general relativity and the principle of general covariance [6–8]. The overview of Penrose's work is as follows: Consider a body in two different locations with their associated states $|\phi_i\rangle$, $i = 1, 2$. Each state satisfies the Schrödinger equation separately as a stationary state, with a unique Killing vector. The superposed state $|\psi\rangle = \alpha|\phi_1\rangle + \beta|\phi_2\rangle$ is also a stationary state with the unique Killing vector $\mathcal{K} = \partial/\partial t$. When the self-gravity of the body (the curvature of the space–time due to the mass of the object itself) is considered, the quantum state of the gravitational field of the body at different locations, i.e., $|\mathcal{G}_i\rangle$, $i = 1, 2$, must also be taken into account. This changes the state $|\psi\rangle$ to the state $|\psi_{\mathcal{G}}\rangle = \alpha|\phi_1\rangle|\mathcal{G}_1\rangle + \beta|\phi_2\rangle|\mathcal{G}_2\rangle$ which is not a stationary state in the sense that it does not have a unique Killing vector. Thus, the total state decays to one of the states to get a stationary state with definite Killing vector of space–time. In this approach, the decay time for transition from the quantum domain to classical domain is obtained [6].

The gravitational considerations of Penrose which were stated above can be used for justifying the wave function reduction through the measurement process. Usually, in a measurement process we have an apparatus and a microscopic system with their associated wave functions. The quantum state of the apparatus, as a macroscopic body, is entangled with the microscopic or quantum system (an electron for example) during the measurement [6,9,12]. As the apparatus is a macroscopic body, its self-gravity is significant. Then, according to the previous statements, the total entangled state ($|\psi_{\mathcal{G}}\rangle$) is not stationary and it decays to a specific state for having a definite unique Killing vector. Consequently, the microscopic system which is entangled with the apparatus also goes to a specific eigenstate. Here, the role of gravity is obvious [6–8].

Why do we want to study this topic in the Bohmian framework? Because, relativistic Bohmian quantum mechanics can be represented geometrically [19–22]. On the other hand, gravity is described geometrically. Thus, we were persuaded to study the objective gravitational wave function reduction in the Bohmian context.

Against the collapse theories, there is a different approach known as 'many-words interpretation' or in short 'MWI'. In this approach, the wave function is a real entity and it does not collapse through a measurement [23–26]. Rather, any outcome of the measurement is in a different real word. These different real worlds or branches of our Universe do not interfere with each other. Thus, it seems that the problem of absence of superposition in the classical domain has been resolved in this context. For such an interpretation, there is no need to modify the Schrödinger equation to explain the collapse of the wave function. Hence, the unitary evolution of the Schrödinger equation is preserved in the MWI approach. In MWI, an observer is a part of a quantum mechanical system [27]. Thus, there is no sharp cut-off between quantum and classical worlds. In the opinion of some researchers, one of the drawbacks of the MWI, is the question of what experiment can show the existence of other branches of the many-worlds [5]? But, here, we do not talk about the details of this approach or its abilities and inabilities. We want only to remember that it is an important approach that exists along with the other approaches to explain the measurement processes of a quantum system or its classical limit. Note that in this paper we are not talking about the wave function reduction through the measurement processes. Rather, we discuss about the roles of mass, gravity and quantum potential or quantum force in the objective wave function reduction. We want to explain that the classical world is a world with negligible quantum potential or quantum force. But negligible with respect to what? In fact, with respect to gravity. This is what we are trying to explain in this paper.

Bohmian quantum mechanics is a causal and deterministic theory in which a particle has a definite trajectory with definable physical quantities like in classical mechanics. The probability density of being a particle in the volume $d^3\mathbf{x}$ is equal to $\rho(\mathbf{x}, t) = R^2(\mathbf{x}, t)$ [9,28–30]. In Bohm's own view, the departure from classical mechanics appears in an essential entity known as 'quantum potential'. It is a non-local potential with non-classical features. The primary approach of Bohm was the substitution of the polar form of the wave function,

$$\psi(\mathbf{x}, t) = R(\mathbf{x}, t) \exp\left(i \frac{S(\mathbf{x}, t)}{\hbar}\right),$$

into the Schrödinger equation which leads to a quantum Hamilton–Jacobi equation as

$$\frac{\partial S(\mathbf{x}, t)}{\partial t} + \frac{(\nabla S)^2}{2m} + V(\mathbf{x}) + Q(\mathbf{x}) = 0. \tag{3}$$

The last term in the above equation is the non-relativistic quantum potential which is given by

$$Q = -\frac{\hbar^2 \nabla^2 R(\mathbf{x}, t)}{2m R(\mathbf{x}, t)} = -\frac{\hbar^2 \nabla^2 \sqrt{\rho(\mathbf{x}, t)}}{2m \sqrt{\rho(\mathbf{x}, t)}}. \tag{4}$$

The quantum force exerted on the particle is defined as $\mathbf{f} = -\nabla Q$. The function $S(\mathbf{x}, t)$ is the action of the system. It propagates like a wave front in the configuration space of the system [9]. The momentum field of

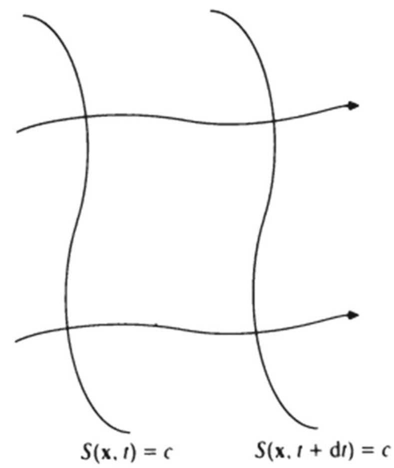

Figure 2. Propagation of function $S(\mathbf{x}, t)$ in the configuration space. Normal to it is the momentum field of the particle.

the particle is a vector field orthogonal to $S(\mathbf{x}, t)$ which is obtained by using the relation $\mathbf{p} = \nabla S(\mathbf{x}, t)$ (see figure 2). The energy of the particle is obtained by using the relation $E = -\partial S/\partial t$ [9].

With the initial position $\mathbf{x}_0$ and the initial wave function $\psi(\mathbf{x}_0, t_0)$, which gives the initial velocity, the position of the particle is obtained using the guidance equation

$$\frac{d\mathbf{x}(t)}{dt} = \left(\frac{\nabla S(\mathbf{x}, t)}{m}\right)_{\mathbf{X}=\mathbf{x}(t)}. \tag{5}$$

The initial wave function gives the initial phase S_0 and consequently initial velocity. With the initial velocity

$$\mathbf{v}_0 = \frac{\nabla S_0}{m} = \frac{\mathbf{p}_0}{m},$$

and the initial position $\mathbf{x}_0$, the evolution of the system is obtained. But we are always faced with a distribution of the initial positions and velocities practically. Thus, there is always an uncertainty in the prediction of evolution of the system practically. The dynamics of the system is obtained using [9]

$$m \frac{d^2\mathbf{x}}{dt^2} = \nabla\left(Q(\mathbf{x}) + \mathbf{V}(\mathbf{x})\right).$$

The expression $\mathbf{X} = \mathbf{x}(t)$ in relation (5) means that among all the possible trajectories, one of them is chosen. Due to our ignorance with respect to all the initial data we are faced with an ensemble of trajectories. The possibility of definition of trajectories in Bohmian quantum physics provides the capability of a geometric visualisation.

In §2, we shall argue how an objective explanation for the classical limit of a free particle based on Bohmian trajectories is possible. In §3, we shall derive a Poisson-like relation for the Bohmian quantum potential for the gravitational reduction of wave function based on deviation of Bohmian trajectories. For the possibility of developing and generalising the subject in future, we start with the relativistic calculations, then we shall consider non-relativistic limit of deviation equation for deriving Diosi's formula. The effect of gravitational field of the particle is considered as a curved space–time with a fixed metric tensor $g_{\mu\nu}$. The relativistic generalisation of the concept of self-gravity for a particle in the Schrödinger–Newton equation is that the particle is affected by the space–time curvature due to its mass–energy. It is natural that if we did not consider the uncertainty in position of the particle in the framework of quantum mechanics, such interpretation for self-gravity either in the relativistic or in the non-relativistic domain was not possible.

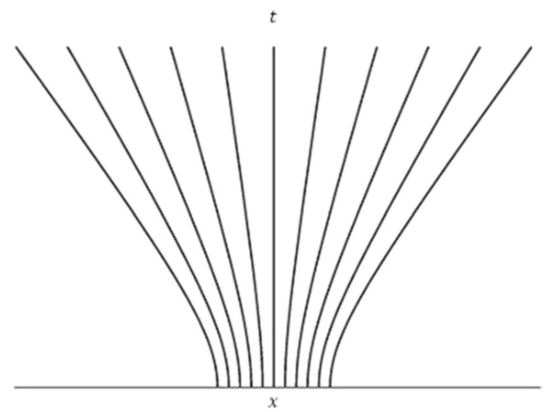

Figure 3. The distribution of trajectories of the particle which is guided by a wave packet in (1 + 1)-dimensional space–time. Due to uncertainty in position of the particle, we are faced with an ensemble of trajectories.

2. Towards an objective reduction based on Bohmian trajectories

In Bohmian quantum mechanics, the usual condition for getting classical limit is the vanishing of Bohmian quantum potential. Vanishing of quantum force is needed in some situations [9]. These conditions do not give an objective criterion for the reduction of wave function or the classical limit of a quantum system. In other words, by vanishing the quantum potential or quantum force, one cannot estimate the needed mass or objective properties of an object for transition from quantum world to classical world. The reason is that the mass of the particle or body does not have any active role in its dynamical evolution. This may be achieved by considering the effects of gravity of the particle on its dynamics.

Let us first study the trajectories of a free particle which is guided by a Gaussian wave packet. The amplitude of the wave packet of a free particle with zero initial group velocity is [9]

$$R = (2\pi\sigma^2)^{-3/4} \mathrm{e}^{-\mathbf{x}^2/4\sigma^2}, \tag{6}$$

where σ is the random mean square width of the packet at time t which represents the spreading of the wave packet [9]. The quantum potential for this system and the quantum force exerted on the particle are obtained using the relations:

$$Q = -\frac{\hbar^2 \nabla^2 R}{2mR} = \frac{\hbar^2}{4m\sigma^2}\left(3 - \frac{\mathbf{x}^2}{2\sigma^2}\right) \tag{7}$$

and

$$f = -\nabla Q = \frac{\hbar^2}{4m\sigma^2}\mathbf{x}. \tag{8}$$

In Bohmian quantum mechanics, the quantum force is responsible for the spreading of wave packet [9]. But, in the standard quantum mechanics, the dispersion of a wave packet is explained by using the Heisenberg uncertainty principle. Now, we want to clarify how the concept of Bohmian trajectories help us to get a criterion for the wave function reduction of a free particle.

It has been demonstrated in ref. [9] that the trajectory of the particle which is guided by a Gaussian wave packet is

$$\mathbf{x}(t) = \mathbf{x}_0 \left(1 + \left(\frac{\hbar t}{2m\sigma_0^2}\right)^2\right)^{1/2}, \tag{9}$$

where $\mathbf{x}_0$ denotes the initial position of the particle. If the initial position of the particle is at $\mathbf{x}_0 = 0$ (in the middle of the wave packet), the trajectory is a straight line. The quantum force $\mathbf{f} = -\nabla Q$ for such a trajectory vanishes and its trajectory is classical. Figure 3 shows that for initial position $\mathbf{x}_0 \neq 0$, the trajectories are not straight lines and the particle is affected by a quantum force.

So far, we found that the curved trajectories are the results of the quantum force. For returning them to the classical straight trajectories, we need an agent to keep the deviation between trajectories constant. The best candidate is the self-gravity of the particle. In other words, gravity prevents from more dispersion and consequently more uncertainty in the position of the particle. When the mass of the particle tends to a specific value, its self-gravity increases. Then, the quantum force and gravitational force can be equal and deviation remains constant. This is achieved when the mass of the particle affects its dynamics. Here, the mass of the particle has an active role, not a passive one. In the following, we explain this idea further.

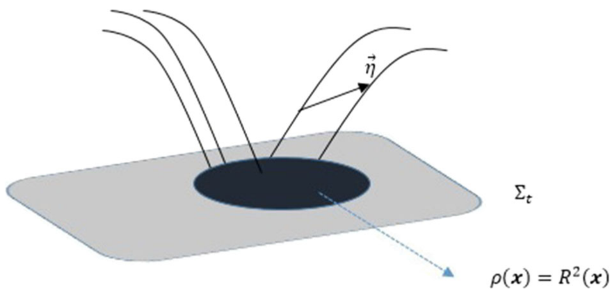

Figure 4. The distribution of time-like trajectories associated with a wave packet on a space-like hybersurface Σ_t. Vector η represents the deviation vector between two neighbour trajectories.

Constant deviation implies that the width of the wave packet remains constant. In other words, we shall face with a wave packet in the form

$$\psi = R(\mathbf{x}) \exp\left(-\frac{iEt}{\hbar}\right),$$

where $R(\mathbf{x})$ is the amplitude in relation (6). An obvious difference between these two types of forces is that gravitational force is defined in a real space, while the quantum force is defined in the configuration space of the particle or system. This type of view brings up this picture in our mind that in the reduction process, quantum effects do not vanish absolutely; rather the self-gravity of the particle as an attractive force, does not allow the quantum force to appear. While the wave packet evolves according to the Schrödinger equation, the particle can be at different locations in the space, with different probability densities $\rho(\mathbf{x}) = R^2(\mathbf{x})$ which is determined by the wave function. By spreading the wave packet, the deviation between trajectories varies with time. On the other hand, the self-gravity of the particle would localise the trajectories. This leads to the equality of the two forces. Figure 4 schematically represents the distribution of an ensemble of time-like trajectories associated with a wave packet.

Fortunately, it can be proved that Bohmian trajectories do not cross each other. Thus, these trajectories can be considered as a congruence in space. In fact, the single-valuedness of the wave function for a closed loop leads to

$$\Delta S = \oint_c \mathrm{d}S = \oint_c \nabla S \cdot \mathrm{d}\mathbf{x} = \oint_c \mathbf{p} \cdot \mathrm{d}\mathbf{x} = nh. \qquad (10)$$

Then, the net phase difference around a closed loop is constant and for every instant at the position $\mathbf{x}$, the momentum $\mathbf{p} = \nabla S$ is single-valued [9].

In the next section, we demonstrate that Diosi's formula is obtained through the investigation of the deviation η^μ between Bohmian trajectories.

3. Bohmian trajectories and wave function reduction

For the possibility of developing and generalising the subject in future, we obtain the deviation between trajectories for a relativistic particle. On the other hand, the Bohmian dynamics of relativistic matter can be described as a conformal transformation of the background metric [19–21]. Hence, for the richness of the topic, we start from the relativistic case. Karolyhazy, in his famous paper [2], started from a relativistic action, because he was searching for answer in a more fundamental way, such as the properties of space–time. Penrose, too, relates the gravitational reduction to the fundamental principles of the space–time dynamics, such as general covariance and equivalence principle [6–8].

The generalisation of self-gravity to the relativistic case is as follows: The quantum distribution of the particle $\rho = |\psi|^2$ makes the space–time curved. Then the dynamics of the particle is affected by this curvature. In other words, the particle moves in its own gravity. This can be justified in the quantum mechanical framework where there is uncertainty in the position of the particle. We represent the metric of the space–time with the symbol $g_{\mu\nu}$. By this imagination, we are going to calculate the deviation between trajectories.

Here, in addition to gravity, the quantum force is present. Thus, we first derive Bohmian acceleration for a spinless relativistic particle. The Hamilton–Jacobi equation in a curved space is in the form [31]:

$$\nabla_\mu S \nabla^\mu S = p_\mu p^\mu = \mathcal{M}^2, \qquad (11)$$

where

$$\mathcal{M}^2 = m^2(1 + \mathcal{Q}). \qquad (12)$$

Also,

$$\mathcal{Q} = \frac{\hbar^2}{m^2} \frac{\nabla_\mu \nabla^\mu R}{R} \qquad (13)$$

is the relativistic Bohmian quantum potential of a spinless particle in a curved background [32]. Here, R stands for the amplitude of the relativistic wave function and S is the action of the particle or the phase of its associated wave function [9,19]. The explicit form of $\mathcal{Q}$ is not needed, because we need the non-relativistic form of the Bohmian quantum potential (4) for obtaining Diosi's formula. Four-momentum of the particle is defined as follows:

$$p^\mu = \mathcal{M} u^\mu. \qquad (14)$$

In the presence of quantum force, the geodesic equation $u^\nu \nabla_\nu u^\mu = 0$, with four-velocity u^μ, is not valid anymore. It will be replaced by the equation

$$u^\nu \nabla_\nu u^\mu = a^\mu_{(B)}, \tag{15}$$

where $a^\mu_{(B)}$ refers to the Bohmian acceleration of the particle due to the quantum force. For deriving a relation for the Bohmian acceleration, we start from relation (11). By differentiation from both sides of relation (11) with respect to the parameter τ which parametrises the trajectories of the particle, we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(p_\mu p^\mu) = \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{M}^2. \tag{16}$$

Then,

$$2p_\mu \frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = 2\mathcal{M}\frac{\mathrm{d}\mathcal{M}}{\mathrm{d}\tau}. \tag{17}$$

On the other hand, we have

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\nabla_\nu p^\mu = u^\nu \nabla_\nu p^\mu, \tag{18}$$

where we have used the replacement:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \to \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\nabla_\nu = u^\nu \nabla_\nu. \tag{19}$$

Now, by substituting (18) and (19) into (17), we get

$$p_\mu(u^\nu \nabla_\nu p^\mu) = \mathcal{M}u_\mu \nabla^\mu \mathcal{M} = \mathcal{M}u^\mu \nabla_\mu \mathcal{M}. \tag{20}$$

Substitution of relation (14) into (20), leads to

$$\mathcal{M}u_\mu(u^\nu \nabla_\nu(\mathcal{M}u^\mu)) = \mathcal{M}u_\mu \nabla^\mu \mathcal{M} \tag{21}$$

or

$$u^\nu \nabla_\nu(\mathcal{M}u^\mu) = \nabla^\mu \mathcal{M} \tag{22}$$

which gives

$$u^\mu u^\nu \nabla_\nu \mathcal{M} + \mathcal{M}u^\nu \nabla_\nu u^\mu = \nabla^\mu \mathcal{M}. \tag{23}$$

By multiplying both sides of the above equation by $1/\mathcal{M}$, we obtain

$$u^\nu \nabla_\nu u^\mu = -u^\mu u^\nu \frac{\nabla_\nu \mathcal{M}}{\mathcal{M}} + \frac{\nabla^\mu \mathcal{M}}{\mathcal{M}}. \tag{24}$$

Now, we should express the right-hand side of the above equation in terms of quantum potential. For this purpose, we start from eq. (12) to get

$$2\mathcal{M}\nabla_\nu \mathcal{M} = m^2 \nabla_\nu \mathcal{Q} \Rightarrow \frac{\nabla_\nu \mathcal{M}}{\mathcal{M}} = \frac{1}{2}\frac{m^2 \nabla_\nu \mathcal{Q}}{\mathcal{M}^2}. \tag{25}$$

By using relation (12) again in the above relation we get

$$\frac{\nabla_\nu \mathcal{M}}{\mathcal{M}} = \frac{1}{2}\frac{m^2 \nabla_\nu \mathcal{Q}}{m^2(1+\mathcal{Q})} = \frac{1}{2}\frac{\nabla_\nu \mathcal{Q}}{1+\mathcal{Q}} = \frac{1}{2}\nabla_\nu \ln(1+\mathcal{Q}). \tag{26}$$

Now, we substitute this result into relation (24) to get

$$u^\nu \nabla_\nu u^\mu = -\frac{1}{2}u^\mu u^\nu \nabla_\nu \ln(1+\mathcal{Q}) + \frac{1}{2}\nabla^\mu \ln(1+\mathcal{Q}). \tag{27}$$

The left-hand side of this equation is the Bohmian acceleration (15). Thus, we get Bohmian acceleration in the following form:

$$a^\mu_{(B)} = -\frac{1}{2}u^\mu u^\nu \nabla_\nu \ln(1+\mathcal{Q}) + \frac{1}{2}\nabla^\mu \ln(1+\mathcal{Q}). \tag{28}$$

As the quantum potential $\mathcal{Q}$ is very small with respect to the classical energies, ($(\hbar^2/m^2) \ll 1$ in relation (13), specially for a particle with significant mass and self-gravity), we can assume that $\mathcal{Q} \ll 1$ and $\ln(1+\mathcal{Q}) \simeq \mathcal{Q}$. Then, the last relation reduces to

$$a^\mu_{(B)} = -\frac{1}{2}u^\mu u^\nu \nabla_\nu \mathcal{Q} + \frac{1}{2}\nabla^\mu \mathcal{Q}. \tag{29}$$

This problem is studied in a fixed background metric with the signature $(+1, -1, -1, -1)$, i.e. we do not consider the back-reaction effects of matter on space–time.

Four-vector field $u^\mu = \partial x^\mu/\partial\tau$ is tangent to the trajectories. The deviation vector between two neighbouring trajectories is defined as $\eta^\mu = \partial x^\mu/\partial s$, in which the parameter s parametrises the curves between neighbouring trajectories so that η^μ is tangent to them. Also, η^μ and u^μ are orthogonal. The velocity field for the deviation vector between two neighbourhood trajectories is defined as

$$v^\mu = \frac{\mathrm{d}\eta^\mu}{\mathrm{d}\tau} = u^\nu \nabla_\nu \eta^\mu, \tag{30}$$

where we have used relation (19). The acceleration of the deviation vector is

$$\frac{\mathrm{d}v^\mu}{\mathrm{d}\tau} = \frac{\mathrm{d}^2\eta^\mu}{\mathrm{d}\tau^2} = \frac{\mathrm{d}}{\mathrm{d}\tau}(u^\nu \nabla_\nu \eta^\mu) = u^\lambda \nabla_\lambda(u^\nu \nabla_\nu \eta^\mu), \tag{31}$$

where we have used relation (19) again. According to the definitions

$$\eta^\mu = \frac{\partial x^\mu}{\partial s} \quad \text{and} \quad u^\mu = \frac{\partial x^\mu}{\partial \tau}$$

and independence of parameters s and τ, we have

$$\frac{\partial u^\mu}{\partial s} = \frac{\partial}{\partial s}\frac{\partial x^\mu}{\partial \tau} = \frac{\partial}{\partial \tau}\frac{\partial x^\mu}{\partial s} = \frac{\partial \eta^\mu}{\partial \tau}. \tag{32}$$

But this is the same result that can be reached through the definition of the Lie derivative. In other words,

$$\frac{\partial u^\mu}{\partial s} = \frac{\partial \eta^\mu}{\partial \tau} \Rightarrow \mathcal{L}_{\mathbf{u}}\eta^\mu = \mathcal{L}_\eta u^\mu = 0 \\ \Rightarrow u^\nu \nabla_\nu \eta^\mu = \eta^\nu \nabla_\nu u^\mu. \tag{33}$$

Now, we substitute this result into relation (31) to get

$$\frac{d^2\eta^\mu}{d\tau^2} = u^\lambda \nabla_\lambda (\eta^\nu \nabla_\nu u^\mu). \tag{34}$$

In fact, substituting relation (33) into relation (31), gives the acceleration of the deviation vector in terms of velocity field derivative $\nabla_\nu u^\mu$. Now, we expand relation (34).

$$\frac{d^2\eta^\mu}{d\tau^2} = (u^\lambda \nabla_\lambda \eta^\nu)\nabla_\nu u^\mu + u^\lambda \eta^\nu (\nabla_\lambda \nabla_\nu u^\mu). \tag{35}$$

For the first term of the above equation, we use the result (33). This gives

$$\frac{d^2\eta^\mu}{d\tau^2} = (\eta^\lambda \nabla_\lambda u^\nu)\nabla_\nu u^\mu + u^\lambda \eta^\nu (\nabla_\lambda \nabla_\nu u^\mu). \tag{36}$$

For the second term, we use the curvature formula

$$[\nabla_\lambda \nabla_\nu - \nabla_\nu \nabla_\lambda]\, A^\mu = R^\mu_{\rho\lambda\nu} A^\rho.$$

Here, $R^\mu_{\rho\lambda\nu}$ denotes the curvature due to the mass–energy of the particle. For the field, u^μ, we have

$$[\nabla_\lambda \nabla_\nu - \nabla_\nu \nabla_\lambda]\, u^\mu = R^\mu_{\rho\lambda\nu} u^\rho \tag{37}$$

or

$$\nabla_\lambda \nabla_\nu u^\mu = \nabla_\nu \nabla_\lambda u^\mu + R^\mu_{\rho\lambda\nu} u^\rho. \tag{38}$$

By substituting this result into the second term of (36) we get

$$\begin{aligned}\frac{d^2\eta^\mu}{d\tau^2} &= (\eta^\lambda \nabla_\lambda u^\nu)\nabla_\nu u^\mu + u^\lambda \eta^\nu (\nabla_\nu \nabla_\lambda u^\mu + R^\mu_{\rho\lambda\nu} u^\rho) \\ &= (\eta^\lambda \nabla_\lambda u^\nu)\nabla_\nu u^\mu + u^\lambda \eta^\nu (\nabla_\nu \nabla_\lambda u^\mu) \\ &\quad + R^\mu_{\rho\lambda\nu} u^\rho u^\lambda \eta^\nu.\end{aligned} \tag{39}$$

The second term of the above equation is equal to

$$\begin{aligned}u^\lambda \eta^\nu (\nabla_\nu \nabla_\lambda u^\mu) &= \eta^\nu \nabla_\nu (u^\lambda \nabla_\lambda u^\mu) \\ &\quad - (\eta^\nu \nabla_\nu u^\lambda)(\nabla_\lambda u^\mu).\end{aligned} \tag{40}$$

The second term of this result cancels out the first term of (39), after substituting (40) into (39). Note that ν and λ are dummy indices and the replacement $\nu \leftrightarrow \lambda$ is allowed. Thus, relation (39) reduces to the relation

$$\frac{d^2\eta^\mu}{d\tau^2} = \eta^\lambda \nabla_\lambda (u^\nu \nabla_\nu u^\mu) + R^\mu_{\rho\lambda\nu} u^\rho u^\lambda \eta^\nu. \tag{41}$$

The expression in parentheses is equal to eq. (15). Then, we have

$$\frac{d^2\eta^\mu}{d\tau^2} = \eta^\lambda \nabla_\lambda a^\mu_{(B)} + R^\mu_{\rho\lambda\nu} u^\rho \eta^\lambda u^\nu. \tag{42}$$

Relation (42) can be expressed in terms of quantum potential:

$$\begin{aligned}\frac{d^2\eta^\mu}{d\tau^2} &= \eta^\lambda \nabla_\lambda \left(-\frac{1}{2} u^\mu u^\nu \nabla_\nu \mathcal{Q} + \frac{1}{2}\nabla^\mu \mathcal{Q}\right) \\ &\quad + R^\mu_{\rho\lambda\nu} u^\rho \eta^\lambda u^\nu,\end{aligned} \tag{43}$$

where we have used relation (29). Now, we need its non-relativistic limit to demonstrate the correctness of our previous arguments about the role of quantum and gravitational forces in the objective gravitational collapse of the wave function.

At the non-relativistic limit, we can take $u^i \simeq \delta^\mu_0$, $\tau \to t$, $\nabla_\mu \to \partial_\mu$ and $R^\mu_{0\rho 0} = \partial^\mu \partial_\rho \varphi(x)$, where $\varphi(x)$ is the Newtonian gravitational potential. Also the non-relativistic Bohmian quantum potential has no explicit dependence on time. In other words, $\partial_0 Q = 0$ (see relation (4)). Then, eq. (43) takes the form

$$\frac{\partial^2 \eta^i}{\partial t^2} = \eta^j \partial_j \left(\frac{\partial^i Q}{m} - \partial^i \varphi(\mathbf{x})\right). \tag{44}$$

One of the possibilities for having constant deviation or parallel trajectories in the ensemble in the non-relativistic domain, is that in the above equation we take

$$\partial^i Q = m \partial^i \varphi(\mathbf{x}), \quad i = 1, 2, 3 \quad \text{or} \quad \nabla Q = m \nabla \varphi. \tag{45}$$

This relation represents the equivalence of Bohmian quantum force and self-gravitational force of the particle for constant deviation between trajectories. Thus, the validity of our physical argument, i.e. the equality of self-gravitational force with the quantum force in the transition regime is confirmed here. Originally, the evolution of the wave packet is not stationary but if the mass of the particle is large enough to produce enough gravity, then the equality of the two forces makes the wave packet stationary. For obtaining an objective criterion for the transition from the quantum world to the classical world, we act as follows. For simplicity, we do calculations for a one-dimensional stationary wave packet, with the width σ_0. If we calculate the average quantum potential for a stationary one-dimensional wave packet

$$\psi(x,t) = (2\pi\sigma_0^2)^{-1/4} e^{-x^2/4\sigma_0^2} e^{iEt/\hbar}$$

with

$$R_s(x) = (2\pi\sigma_0^2)^{-1/4} e^{-x^2/4\sigma_0^2},$$

we get

$$\begin{aligned}\langle Q\rangle_s &= \int_{-\infty}^{+\infty} R_s^2 Q_s \, dx \\ &= \int_{-\infty}^{+\infty} R_s^2 \left(-\frac{\hbar^2}{2m}\frac{\nabla^2 R_s}{R_s}\right) dx \sim \frac{\hbar^2}{2m\sigma_0^2}\end{aligned} \tag{46}$$

which is the average quantum potential of the particle, when it is described by a stationary wave packet with the width σ_0. In Bohmian quantum mechanics, for a stationary wave packet we have $\mathbf{p} = \nabla S(t) = 0$, because for the stationary wave functions, the phase of the wave is a function of time only. Therefore, the kinetic energy of the particle vanishes, and the energy of the particle is due to the quantum potential completely, while in standard quantum mechanics it is due to the kinetic term $\mathbf{p}^2/2m$. The result of (46), i.e. $\hbar^2/2m\sigma_0^2$, is exactly the same as kinetic term in standard quantum mechanics. There, it is obtained through the relation

$$\langle K\rangle_s = \int_{-\infty}^{+\infty} \psi_s^* \left(\frac{\hat{\mathbf{p}}^2}{2m}\right) \psi_s \, \mathrm{d}x \sim \frac{\hbar^2}{2m\sigma_0^2}. \tag{47}$$

This result has been obtained in ref. [1].

The gravitational self-energy for a particle with mass m is defined as

$$U_g(\mathbf{x}) = -Gm^2 \int \frac{|\psi(\mathbf{x}', t)|^2}{|\mathbf{x}' - \mathbf{x}|} \mathrm{d}^3 x', \tag{48}$$

where $|\psi(\mathbf{x}', t)|^2 = R_s^2(\mathbf{x}')$. The average gravitational self-energy of the particle in one dimension, with width σ_0, is

$$\begin{aligned}\langle U_g\rangle &= \int_{-\infty}^{+\infty} R_s^2 U_g \, \mathrm{d}x \\ &= -Gm^2 \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{R_s^2(x') R_s^2(x)}{|x' - x|} \mathrm{d}x \, \mathrm{d}x' \\ &\sim -\frac{Gm^2}{\sigma_0}. \end{aligned} \tag{49}$$

It is obvious from relations (48) and (49) that the average quantum potential and average self-gravitational energy are functions of σ_0. Because we are actually dealing with average values, we obtain the critical width for which the average gravitational force and quantum force become equal:

$$\frac{\mathrm{d}\langle Q\rangle_s}{\mathrm{d}\sigma_0} = \frac{\mathrm{d}\langle U_g\rangle}{\mathrm{d}\sigma_0} \Rightarrow (\sigma_0)_{\text{critical}} \sim \frac{\hbar^2}{Gm^3}, \tag{50}$$

where the derivative has been taken with respect to the probabilistic width σ_0. The above relation is the famous result of Diosi that has been obtained here from a geometrical approach related to Bohmian trajectories (see ref. [1]). This study reveals the role of gravity in the wave function reduction in the context of Bohmian quantum mechanics [2,5].

It is necessary to mention that our approach is in the field of actual collapse similar to those of the Penrose, Karolyhazi and Diosi approaches, not in the context of decoherence approach. Let us make the point more clear. As in those approaches in which gravity is responsible for reducing wave function, so is in our approach. But, there are technical differences between our approach and their's. They consider uncertainty in the structure of space–time (an ensemble of metrics). This has been considered specially in the works of Karolyhazy and Penrose [1–3,5–8]. But, we work in a single space–time. Furthermore, we use the definition and ability of the concept of Bohmian trajectories which is not possible in the standard quantum mechanics. Instead of an ensemble of metrics (dispersion in metric) we are faced with an ensemble of trajectories which leads to a mass distribution in the configuration space of the particle and consequently provides a self-gravitational force. When the self-gravitational force is equal to the quantum force, then we have parallel trajectories. At this moment, the particle is at the threshold of classical world. This equilibrium leads to a criterion (characteristic width (50)) for transition from the quantum domain to the classical domain. This criterion also gives a critical mass $m_c = (\hbar^2/G\sigma_0)^{1/3}$ for which the transition occurs. Naturally, for masses grater than the critical mass, the gravitational force overcomes the quantum force and classical behaviour appears. Specially when $m \to \infty$, $\sigma_0 \to 0$ we have

$$\psi = \lim_{\sigma_0 \to 0} \frac{1}{\sigma_0^{1/4}} \mathrm{e}^{-x^2/\sigma_0^2} \to \delta(x). \tag{51}$$

Then, the wave packet tends to a Dirac delta function and we have more certainty in the position of the particle (classical domain-wave function reduction). In contrast, for masses less than the critical mass, the quantum force overcomes the gravitational force and the deviation between trajectories increases (wave function is dispersive in the usual language) and quantum features become significant. In fact, we have shown the existence of a dynamical procedure for the wave function collapse which is real and is not due to the environmental effects like in decoherence interpretation. In all these approaches, the superposition breaks down through the gravitational effects. Thus, the system does not have a unitary (but norm-preserving) evolution [5].

By imposing operator $(\nabla\cdot)$ on both sides of condition (45), we get the Poisson-like equation for the quantum potential as

$$\nabla^2 Q = 4\pi G m \rho$$

or

$$-\frac{\hbar^2}{2m^2} \nabla^2 \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}}\right) = 4\pi G \rho. \tag{52}$$

This is a non-linear differential equation which expresses that, in the transition from quantum world to the classical world, quantum information (Q) is replaced by gravitational information (φ). Equation (52) can be solved for different distributions ρ to get further results. We have derived this relation in ref. [35] from another point of view. Then we investigated the behaviour of ρ in the wave function reduction with respect to the mass variations. Relation (52) is the consequence of gravitational wave function reduction in Bohm's causal quantum theory. This may be a starting point for further studies in this context.

4. Conclusion

In this study, we argued how it is possible to obtain a criterion for the gravitational objective wave function reduction in the Bohmian context. It was done based on Bohmian trajectories and geometrical concepts. Finally, in addition to the famous result of ref. [1], an interesting nonlinear equation, i.e. eq. (52), was obtained. It represents a Poisson-like equation for the Bohmian quantum potential in the reduction process. The solutions of eq. (52) can be investigated numerically or analytically. In fact, we have stated that quantum information reduces to the gravitational information in the reduction process. It is displayed as the balance between average self-gravitational force of the particle and its average quantum force. Now, we have a geometrical and a dynamical notion about the objective gravitational wave function reduction. This geometrical study has never been done before.

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