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Fast transverse instability due to RF cavity impedance in the ILSF storage ring

S MASSOUDI[1], S AGHAEI[2],*, M AMNIAT-TALAB[1] and J RAHIGHI[3]

[1]Physics Department, Faculty of Sciences, Urmia University, Urmia, Iran
[2]Physics Department, Faculty of Sciences, Farhangian University, Tehran, Iran
[3]Iranian Light Source Facility (ILSF), Institute for Research in Fundamental Science (IPM), Tehran, Iran
*Corresponding author. E-mail: so_aghaei@sut.ac.ir

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Abstract. In this paper, the single bunch fast transverse instabilities of the electron beam in the ILSF storage ring due to the RF cavity impedance are investigated. We consider Satoh's formalism (that studied mode coupling instabilities based on the Vlasov equation) and Lindberg's formalisms (that studied mode coupling instabilities based on the Fokker–Planck equation). The Lindberg formalisms are rewritten to broad-band RF cavity impedance. It is shown that the zero chromaticity limits of both approaches completely coincide. In addition, we discuss the current threshold at high chromaticity that includes higher-order azimuthal and radial modes.

Keywords. Fast transverse instabilities; beam dynamics; generalised Laguerre polynomials.

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1. Introduction

Investigating the dynamics and stability of the circulating bunched beam near a long, closed curve is important for designing accelerators [1–5]. The fast transverse instability is of particular importance in large high-energy storage rings because it decreases the stored current threshold substantially. The fast transverse instability, or the transverse mode coupling instability, is caused by the mode coupling between the azimuthal modes [6–12]. To analyse fast transverse instability, the Sacherer formalism [13,14], which is based on the Vlasov equation, has been generalised in ref. [9] that includes radial and azimuthal modes. However, because of the synchrotron radiation of the electrons in the light source storage rings, the use of the Fokker–Planck equation is more suitable than the Vlasov equation [15]. The Fokker–Planck equation has been solved by expanding the radial part of the distribution function in terms of the generalised Laguerre polynomials by Suzuki to study the fast transverse instability [16].

Recently, Lindberg expanded Suzuki's approach and studied mode coupling instability based on the linearised Fokker–Planck equation [17]. He derived a theoretical model including chromaticity, both dipolar and quadrupolar transverse wakefields, and the effects of damping and diffusion due to the synchrotron radiation. Then, he applied the equations to the resistive wall impedance of the advanced photon source (APS) storage ring and compared the results with simulation using the elegant particle tracking code, which were in good agreement [18,19].

We studied longitudinal instability due to pillbox RF cavity impedance in the storage ring of the Iranian Light Source Facility (ILSF) in ref. [20]. The beam instability was discussed as a function of current and storage ring circumference. In this paper, we shall use the Satoh's and Lindberg's formalisms to study fast transverse instability in the presence of the RF cavity wakefield for ILSF.

This work is organised as follows: In §2, we present a brief explanation of Satoh's and Lindberg's approaches for deriving Vlasov and Fokker–Planck equations as a matrix eigenvalue problem, respectively. Then, we rewrite the formalisms of Lindberg approach for the RF cavity impedance. We consider ILSF storage ring parameters and apply the equations to study transverse instability of the electron beam and find the current thresholds in §3. Finally, in §4, the results of this paper are discussed.

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2. Theoretical study and formulation of transverse instability

In this section, we summarise Satoh's and Lindberg's methods to investigate transverse instability using the Vlasov and the Fokker–Planck equations respectively.

2.1 *Vlasov equation*

Vlasov equation for a particle distribution function ψ in the phase space is given by

$$\left\{\frac{\partial}{\partial t}+\omega_\beta\left[1+\left(\frac{1}{\omega_0}-\frac{\xi}{\alpha\omega_\beta}\right)\dot{\theta}\right]\frac{\partial}{\partial\phi_x} +\omega_s\left(1+\frac{1}{\omega_0}\dot{\theta}\right)\frac{\partial}{\partial\phi}\right\}\psi -\left(1+\frac{1}{\omega_0}\dot{\theta}\right)\frac{c}{E}F\beta^{1/2}\sin\phi_x\frac{\partial}{\partial r_x}\psi=0, \quad (1)$$

where ω_0 and ω_β are the revolution angular frequency and the betatron-oscillation frequency. ξ, F, α, E and c are the chromaticity, transverse force, momentum compaction factor, energy and speed of light respectively. Also, the coordinates (ϕ_x, r_x) and (ϕ, r) are related to the normalised displacement z and the azimuth θ which is measured relative to the centre of an unperturbed bunch as follows:

$$z=r_x\cos\phi_x,\quad \theta=r\cos\phi. \quad (2)$$

Equation (1) can be solved using perturbation theory. According to this theory, the distribution function ψ can be decomposed into a stationary part and a perturbed part as

$$\psi=f_0(r)g_0(r_x)+f(r_x,\phi_x)g(r,\phi)\mathrm{e}^{i(\xi/\alpha-\nu)\theta}\mathrm{e}^{-i\Omega t}. \quad (3)$$

Here, f_0g_0 is the unperturbed solution ($F=0$) of eq. (1), Ω is the coherent angular frequency and ν is the time-of-flight factor [21]. By substituting eq. (3) into eq. (1) and considering the transverse force F received by a particle at location θ, for a Gaussian bunch, the following solution can be obtained:

$$\tilde{\rho}(q)=-i\frac{Ie\beta\omega_0}{4\pi E\omega_s}\sum_p Z_T(p)\tilde{\rho}(p) \times\sum_{h=0}^{\infty}\beta_h(\lambda)F_h(p\sigma_z)F_h(q\sigma_z), \quad (4)$$

where β, ω_s, Z_T are the beta function, the synchrotron oscillation frequency, the transverse impedance respectively and $\lambda=(\Omega-\omega_\beta)/\omega_s$. Here $h=|m|+2k$ is the hybrid mode number and it is obtained by the radial mode number ($k=0,1,2,\dots$) and the azimuthal mode number ($m=\dots-2,-1,0,1,2,\dots$). The function $\tilde{\rho}(p)$ is the Fourier transformation of $\rho(\theta)$ which is given by

$$\tilde{\rho}(p)=\frac{1}{2\pi}\int \mathrm{d}\theta\mathrm{e}^{-ip\theta}\rho(\theta) =\frac{\omega_s}{2\pi}\int\int r\,\mathrm{d}r\,\mathrm{d}\phi\mathrm{e}^{-ipr\cos\phi}g(r,\phi). \quad (5)$$

Also, functions $F(p\sigma_z)$ and $\beta(\lambda)$ are given by

$$F_h(p\sigma_z)=\frac{1}{\sqrt{h!}}\mathrm{e}^{-p^2\sigma_z^2/2}\left(\frac{p\sigma_z}{\sqrt{2}}\right)^h$$

$$\beta_0=\frac{1}{\lambda},\quad \beta_1=\frac{2\lambda}{\lambda^2-1},\quad \beta_2=\frac{2\lambda}{\lambda^2-4}+\frac{2}{\lambda},\quad \beta_3=\frac{2\lambda}{\lambda^2-9}+\frac{6\lambda}{\lambda^2-1},\dots. \quad (6)$$

Finally, to find the eigenmode frequency spectrum

$$\tilde{\rho}(q)=\sum_{h=0}^{\infty}\alpha_hF_h(q\sigma_z), \quad (7)$$

we have to solve the following equation:

$$\det|\delta_{hl}+\kappa\beta_h(\lambda)M_{hl}(\lambda)|=0, \quad (8)$$

where

$$\kappa=i\frac{Ie\beta\omega_0}{4\pi E\omega_s}$$

and

$$M_{hl}(\lambda)=\sum_{p=-\infty}^{\infty}Z_T(p)F_h(p\sigma_z)F_l(p\sigma_z). \quad (9)$$

It should be noted that because of the slow variation of $F(p\sigma_z)$, the approximation

$$F_k\left[\left(p+\nu-\frac{\xi}{\alpha}\right)\sigma_z\right]\sim F_k(p\sigma_z) \quad (10)$$

was used. Now, we assume a low Q transverse resonator impedance as the source of impedance, which is represented by

$$Z_T(\omega)=\frac{R_t\frac{\omega_r}{\omega}}{1-iQ\left(\frac{\omega}{\omega_r}-\frac{\omega_r}{\omega}\right)}, \quad (11)$$

where R_t is the transverse shunt impedance, Q is the quality factor and ω_r is the resonant angular frequency. Equation (11) can be written as

$$Z_T(p)=\frac{iR_tp_r}{\sqrt{4Q^2-1}}\left(\frac{1}{p-p_1}-\frac{1}{p-p_2}\right), \quad (12)$$

where

$$p_r=\frac{\omega_r}{\omega}$$

and

$$p_{1,2} = \frac{p_r}{2Q}(-i \pm \sqrt{4Q^2 - 1}). \quad (13)$$

If we substitute the impedance and the function F_h into eq. (9), the matrix elements are expressed as

$$M_{hl}(\lambda) = \frac{1}{2^{(h+l)/2}\sqrt{l!h!}} \frac{iR_t p_r}{\sqrt{4Q^2 - 1}} (S_{h+l}(p_1) - S_{h+l}(p_2)). \quad (14)$$

Here $S_{h+l}(p_{1,2})$ is an infinite sum which is given by

$$S_{h+l}(p_{1,2}) = \sum_{p=-\infty}^{p=\infty} \frac{(p\sigma_z)^{h+l} \mathrm{e}^{-p^2\sigma_z^2}}{p - p_{1,2}}. \quad (15)$$

According to Zotter's approximation [22], an analytic summation formula for $S_{h+l}(p_{1,2})$ can be evaluated as follows:

$$S_{h+l}(p_{1,2}) = z_{1,2}^{h+l} S_0(p_{1,2}) + \sum_{n=0}^{(h+l)/2-1} \Gamma\left(n + \frac{1}{2}\right) z_{1,2}^{h+l-2n-1}, \quad (16)$$

where $z_{1,2} = p_{1,2}\sigma_z$ and

$$S_0(p_{1,2}) = -\pi \exp(-z_{1,2}^2) \cot(\pi p_{1,2}) - i\pi[w(z_{1,2}) - \exp(-z_{1,2}^2)]. \quad (17)$$

w_z is the complex error function and Γ is the gamma function.

2.2 *Fokker–Planck equation*

Fokker–Planck equation in terms of the transverse action-angle variables, (J, Ψ) for the distribution function ψ is given by [17]

$$\frac{\partial\psi}{\partial s} + \{\psi, H\} = \frac{2}{c\tau_z}\left[\sigma_\delta^2 \frac{\partial^2\psi}{\partial p_z^2} + p_z \frac{\partial\psi}{\partial p_z} + \psi\right] + \frac{2}{c\tau_x}\left[\varepsilon_0 J \frac{\partial^2\psi}{\partial J^2} + \frac{\varepsilon_0}{4J}\frac{\partial^2\psi}{\partial\Psi^2} + (\varepsilon_0 + J)\frac{\partial\psi}{\partial J} + \psi\right], \quad (18)$$

where τ_z and τ_x are the longitudinal and transvers damping time, respectively, σ_δ is the equilibrium energy spread, ε_0 is the equilibrium emittance, p_z is the negative energy deviation, $z = s - ct$ is the longitudinal coordinates and $\{,\}$ denotes the Poisson bracket. The Hamiltonian presented in eq. (18) is the ring-averaged Hamiltonian and contains the synchrotron, chromatic and transverse wake potentials as

$$H = \frac{\omega_\beta}{c} J + H_z(z, p_z) + V_{\mathrm{wake}}(z, \bar{\Psi}, J). \quad (19)$$

Here $H_z = \frac{\alpha}{2}p_z^2 + V_z(z)$ is the longitudinal Hamiltonian. In the usual RF system, we have

$$V_z = \frac{\omega_s^2}{2\alpha c^2} z^2$$

and H_z describes simple harmonic motion. Also, it should be noted that the chromatic part of the Hamiltonian was eliminated by replacing the betatron phase via $\Psi \to \bar{\Psi} - k_\xi z$. $k_\xi z = (2\pi\xi/\alpha C)z$ is called the head-tail phase [23,24], which arises because of the linear dependence of the betatron frequency on energy. If we neglect the wake potential, the solution of eq. (18) becomes a Gaussian in energy p_z and a decaying exponential in action J as follows:

$$\psi_0 = f_0(J)g_0(H_z) = \frac{\exp(-H_z/\alpha\sigma_\delta^2)}{2\pi\sigma_\delta\sigma_z} \frac{\exp(-J/\varepsilon_0)}{2\pi\varepsilon_0}. \quad (20)$$

By considering the wakefield potential as a perturbative potential, the first-order perturbed part of the distribution function can be written as follows:

$$\psi_1 = -\sqrt{\frac{1}{2}J} f_0'(J)\mathrm{e}^{i\bar{\Psi}} \tilde{g}_1(z, p_z)\mathrm{e}^{-i\Omega s/c}\mathrm{e}^{-i\omega_\beta s/c}, \quad (21)$$

where Ω represents the frequency difference in the oscillation from the betatron frequency. Also, the wakefield potential in terms of the distribution function $\psi = \psi_0 + \psi_1$ is given by

$$V_{\mathrm{wake}} = \chi J \int \mathrm{d}z'\, \mathrm{d}p_z' g_0(z', p_z') W_Q^\beta(z - z') + \chi\sqrt{2J} \cos(\bar{\Psi} - k_\xi z)\mathrm{e}^{-i(\Omega+\omega_\beta)s/c} \times \int \mathrm{d}z'\, p_z' \tilde{g}_1(z', p_z')\mathrm{e}^{ik_x i z'} W_D^\beta(z - z'). \quad (22)$$

Here W_D and W_Q are the total beta function-weighted dipolar and quadropolar wake fields and χ is the coupling constant which is defined as follows:

$$\chi = \frac{2\pi I}{\gamma Z_0 c I_{\mathrm{A}}}, \quad (23)$$

where $I = ecN/C$ is the average bunch current, $I_{\mathrm{A}} = 4\pi\varepsilon_0 mc^3/e$ is the Alfven current and $Z_0 = 1/(\epsilon_0 c)$ is the impedance of free space.

Finally, using the perturbed distribution function and expanding $\tilde{g}_1$ in terms of Gauss–Laguerre functions, eq.

(18) is reduced to the following linear mode equation:

$$\left[\Omega - m\omega_s + \frac{i}{\tau_x} + \frac{i}{\tau_z}(2p+m)\right] a_p^m + \sum_{n,q}(D+Q)_{p,q}^{m,n} a_q^n = i\left(R_p^m a_{p+1}^{m-2} + T_p^m a_{p-1}^{m+2}\right), \quad (24)$$

where R and T are the diffusive coupling matrices and D and Q are the coupling matrices associated with the dipole and quadrupole wakefields respectively, which are defined as follows:

$$R_p^m = \frac{1}{2\tau_z}\sqrt{(p+1)(p+m)} \quad (25)$$

$$T_p^m = \frac{1}{2\tau_z}\sqrt{p(p+m+1)} \quad (26)$$

$$D_{p,q}^{m,n} = \frac{i^{m-n+1} cI}{I_A \gamma Z_0} \int \mathrm{d}k Z_D^\beta (k+k_\xi) \mathrm{e}^{-k^2\sigma_z^2} \times \frac{(k\sigma_z/\sqrt{2})^{2p+m}}{\sqrt{p!(p+m)!}} \frac{(k\sigma_z/\sqrt{2})^{2q+n}}{\sqrt{q!(q+n)!}}, \quad (27)$$

$$Q_{p,q}^{m,n} = \frac{i^{n-m+1} cI}{I_A \gamma Z_0} \sqrt{\frac{p!}{(p+m)!}\frac{q!}{(q+n)!}} \times \int \mathrm{d}k Z_Q^\beta(k) \mathrm{e}^{-k^2\sigma_z^2/2} \int_0^\infty \mathrm{d}y \mathrm{e}^{-y} y^{(m+n)/2} \times J_{n-m}(k\sigma_z\sqrt{2y}) L_p^m(y) L_q^n(y). \quad (28)$$

In the above equations, $Z_D^\beta(k)$ and $Z_Q^\beta(k)$ are the transverse dipolar and quadropolar components of impedance, respectively. The indices m, n are angular mode numbers, p, q are radial mode numbers and take the values $p = 0, 1, 2, \ldots, \infty$ and $-p < m < \infty$. We assume a broadband impedance as eq. (11). This impedance can be expressed as

$$Z(k+k_\xi) = \frac{iR_t k_r}{\sqrt{4Q^2-1}}\left(\frac{1}{k-k_1} - \frac{1}{k-k_2}\right), \quad (29)$$

where $k_r = \omega_r/c$ and

$$k_{1,2} = \frac{k_r}{2Q}(-i \pm \sqrt{4Q^2-1}) - k_\xi. \quad (30)$$

Therefore, the matrix D elements, eq. (27), are given by

$$D_{p,q}^{m,n} = \frac{i^{m-n+2} cI}{I_A\gamma Z_0} \frac{\beta R_t k_r}{\sqrt{4Q^2-1}} \frac{1}{2^{p+q+\frac{m+n}{2}}} \times \frac{1}{\sqrt{p!(p+m)!}} \frac{1}{\sqrt{q!(q+n)!}} \times (S_{2p+2q+m+n}(k_1) - S_{2p+2q+m+n}(k_2)). \quad (31)$$

It should be noted that the quadrupolar matrix Q is a real symmetric matrix with purely real eigenvalues [17]. Here we only study the transverse collective effects due to a purely dipolar impedance Z_D. According to eqs (3) and (21), it is clear that in both approaches, the amplitude of the distribution function increases with time if the imaginary part of Ω is positive. Therefore, one can solve eqs (8) or (24) and evaluate the transverse beam instability conditions. In the next section, we want to consider RF cavity impedance and study the transverse instability of the ILSF storage ring using two approaches based on Vlasov and Fokker–Planck equations.

3. Numerical results and discussion

Here we apply the formalism developed in §2 to study transverse instability in the ILSF storage ring and discuss the thresholds. The main parameters of the ILSF storage ring are given in table 1. For instance, we assume that the number of RF cavity installed in the storage ring is 20. So, we estimate that the total transverse shunt impedance is equal to $R_t = 980$ kΩ/m which corresponds to 49 kΩ/m per cell [25].

First, we study instability using Vlasov equation. We take the first three terms in eq. (8) which contain six modes, $h = 0$ $(m = 0, k = 0)$, $h = 1$ $(m = -1, k = 0)$ $(m = 0, k = 1)$, $h = 2$ $(m = -2, k = 0)$ $(m =$

Table 1. Main parameters of the ILSF storage ring.

Parameter	Unit	Value
Energy (E)	GeV	3
Circumference (C)	m	528
Momentum compaction factor (α)	–	1.53×10^{-4}
Damping time (τ_x, τ_y, τ_z)	ms	19.709, 19.716, 9.859
Resonant frequency ($f_r = \omega_r/2\pi$)	MHz	27000
Quality factor (Q)	–	1
Synchrotron tune ($\nu_s = \omega_s/\omega_0$)	–	0.01
Betatron tune ($\nu_\beta = \omega_\beta/\omega_0$)	–	44.16
Beta function (β)	m	17.8

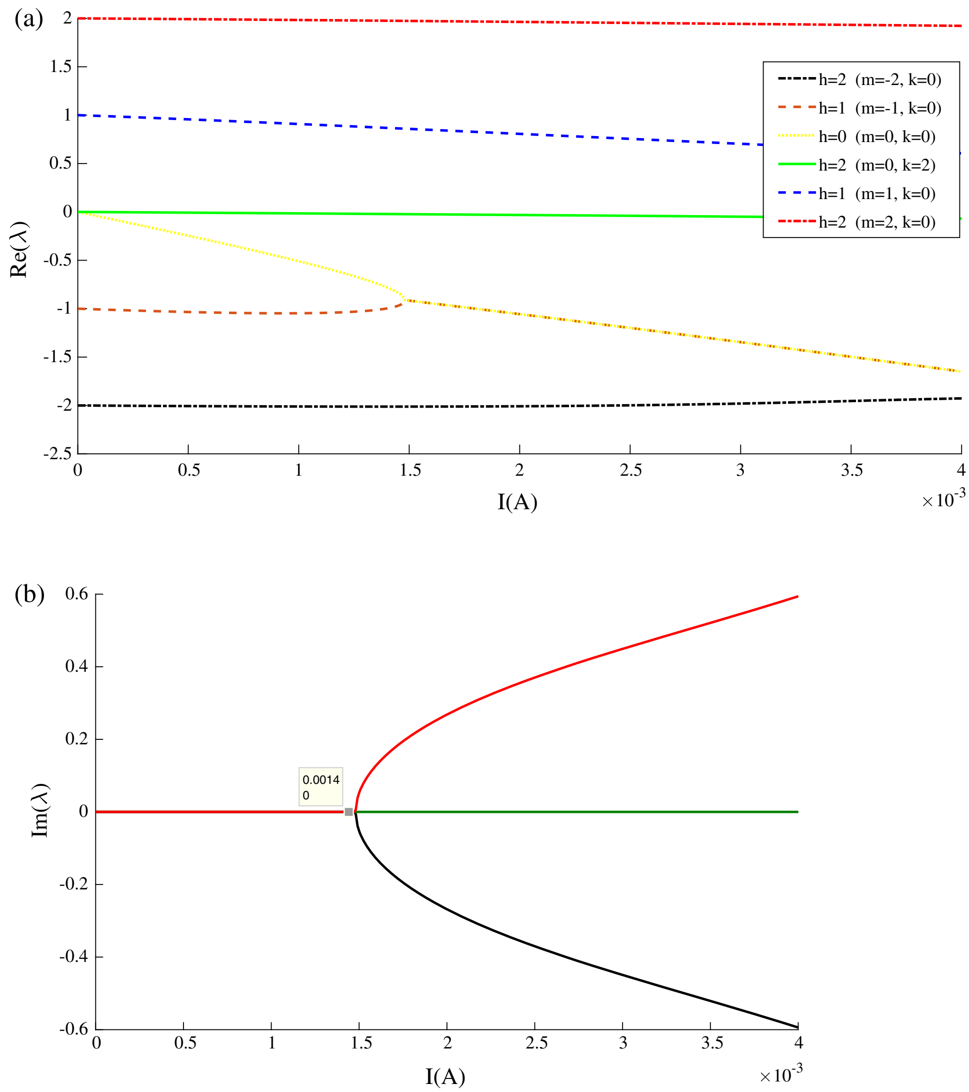

Figure 1. (**a**) Real and (**b**) imaginary parts of $\lambda = (\Omega - \omega_\beta)/\omega_s$ vs. the bunch current at $\sigma_z = 3$ mm and $\xi_x = 0$ based on the Vlasov equation for six modes.

$0, k = 2)$ $(m = 2, k = 0)$, and solve the following equation:

$$\begin{vmatrix} \kappa M_{11} + 1/\beta_1 & \kappa M_{12} & \kappa M_{13} \\ \kappa M_{21} & \kappa M_{22} + 1/\beta_2 & \kappa M_{23} \\ \kappa M_{31} & \kappa M_{32} & \kappa M_{33} + 1/\beta_3 \end{vmatrix} = 0. \quad (32)$$

The results are given in figures 1 and 3. Figure 1a shows the real part of $\lambda = (\Omega - \omega_\beta)/\omega_s$ as a function of bunch current at the bunch length $\sigma_z = 3$ mm and is a diagram of mode coupling. A mode coupling is observed between $h = 0$ $(m = 0, k = 0)$ and $h = 1$ $(m = -1, k = 0)$ modes. On the other hand, figure 1b shows the imaginary part of λ vs. bunch current. Since instability occurs if the imaginary part of λ is positive, it is clear that fast instability starts to occur just at the point that the mode coupling has been caused. Therefore, the threshold bunch current for 3 mm bunch length is obtained at 1.4 mA. Also, we take the first four terms in eq. (8) which contain ten modes and the results are shown in figure 2. It is clear that the threshold current is independent of the number of modes at zero chromaticity. Figure 3 shows threshold current vs. bunch length. It can be seen that the threshold current is changing slowly if the bunch length is within the range of several millimetres.

Neglecting chromatic and tune dependence in eq. (10) significantly limits the range of application of the study. It is appropriate to use available numerical codes to study transverse mode coupling instability based on Vlasov equation at non-zero chromaticity. For example, the MOSES code can be used to generate the bunch mode spectrum as a function of bunch current,

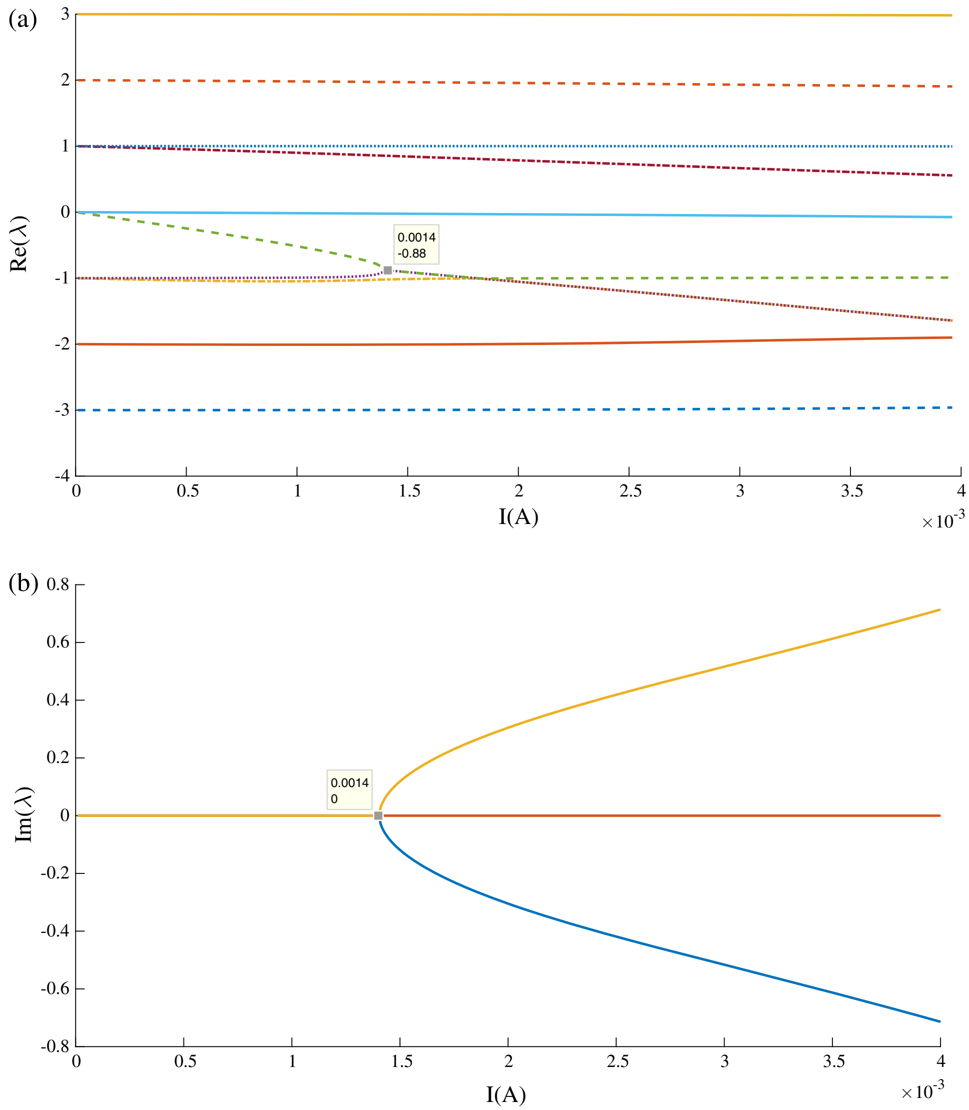

Figure 2. (**a**) Real and (**b**) imaginary parts of $\lambda = (\Omega - \omega_\beta)/\omega_s$ vs. the bunch current at $\sigma_z = 3$ mm and $\xi_x = 0$ based on the Vlasov equation for ten modes.

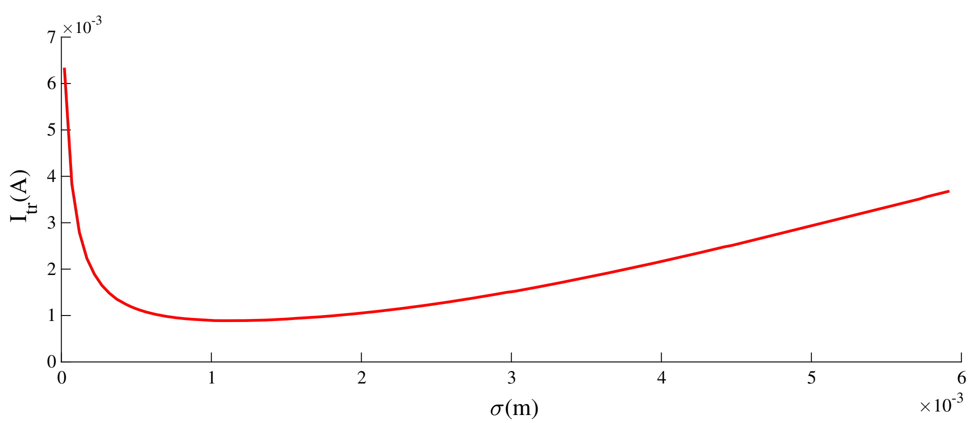

Figure 3. The threshold current vs. the bunch length based on the Vlasov equation at zero chromaticity ($\xi_x = 0$).

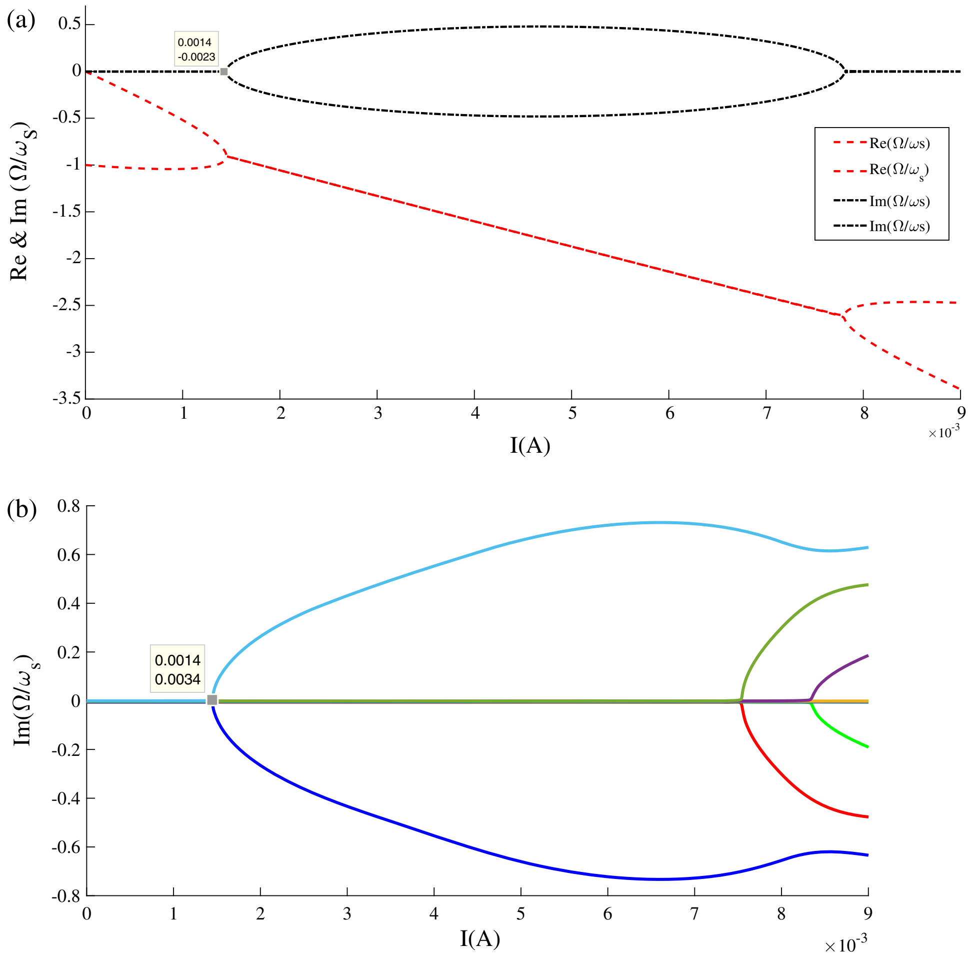

Figure 4. (**a**) Imaginary and real parts of Ω/ω_s vs. bunch current for 4 mode and (**b**) imaginary part of Ω/ω_s vs. bunch current for 16 mode according to the Fokker–Planck equation at zero chromaticity ($\xi_x = 0$) and $\sigma_z = 3$ mm.

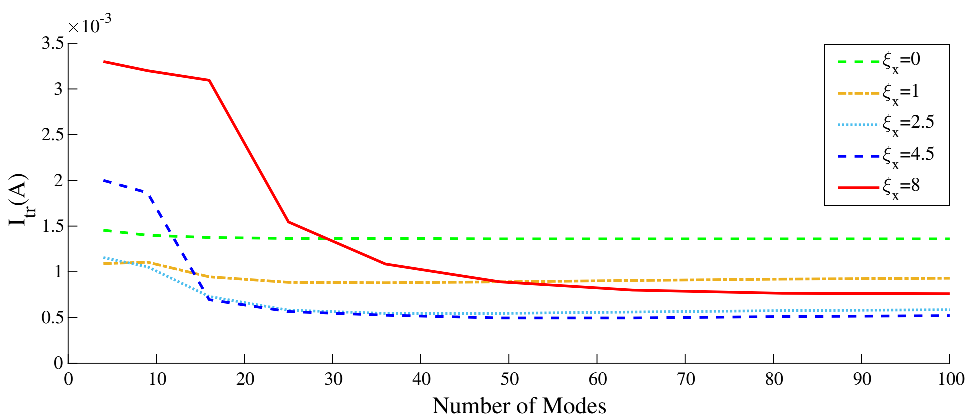

Figure 5. Threshold current vs. number of modes for five different chromaticities.

for a bunch interacting with a transverse broadband impedance at different chromaticities [26,27]. On the other hand, in Vlasov equation, the synchrotron radiation of the electron beam is not considered. As the electron beam in the light source has a synchrotron radiation, for a closer study of instability, it is appropriate to study instability using Fokker–Planck equation. As discussed in §2, the dynamic study of the electron beam in terms of Fokker–Planck equation was reduced to solve an eigenvalue problem (eq. (24)). According to the result of ref. [17], most of the instability is due to a purely dipolar impedance. Therefore, we consider only Z_D in eq.

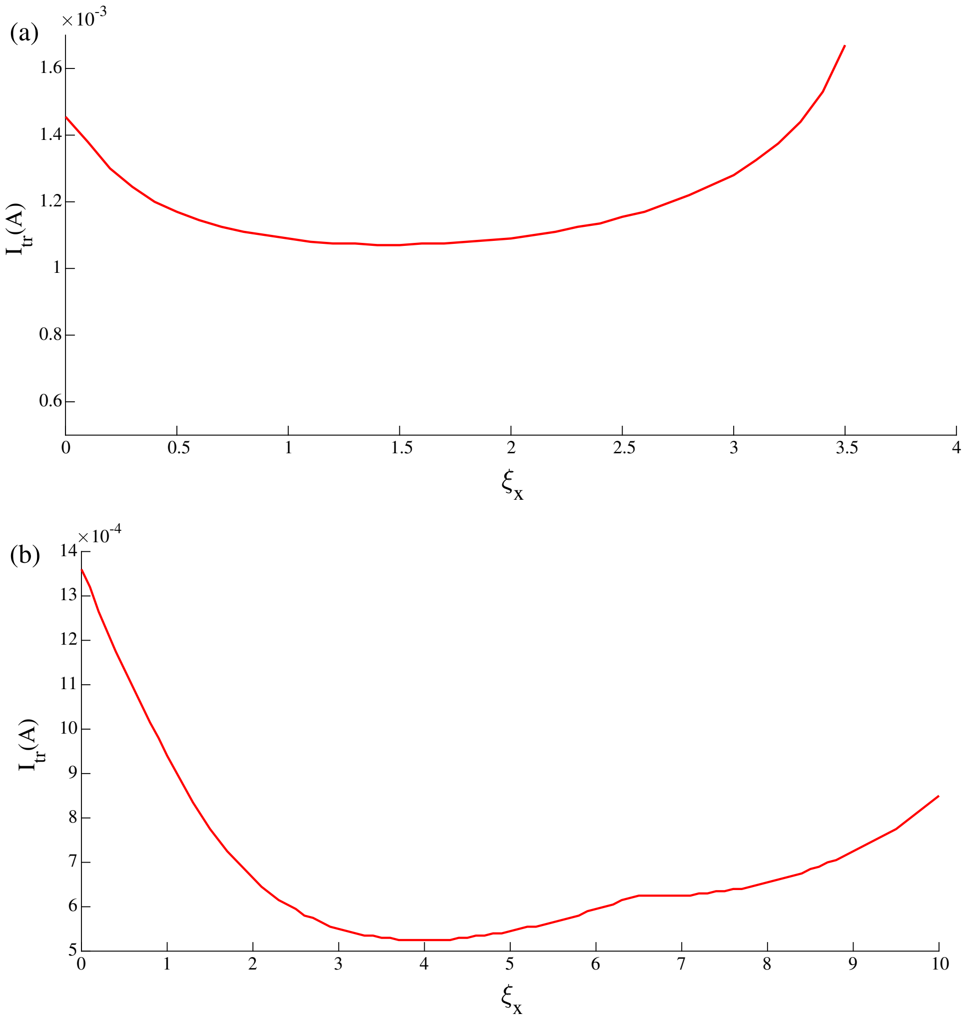

Figure 6. Threshold current vs. chromaticity (**a**) for two modes theory and (**b**) for 676 modes at $0 < k_\xi \sigma_z < 2.3$.

(24). For the limit of zero chromaticity, we take the first four terms in eq. (24). In this case, mode coupling would not occur and the following matrix equation is obtained:

$$\begin{bmatrix} \frac{i}{\tau_x} - D_{0,0}^{0,0} & D_{0,1}^{0,-1} & D_{0,1}^{0,0} & D_{0,1}^{0,1} \\ D_{1,0}^{-1,0} & \omega_s + \frac{i}{\tau_x} + \frac{i}{\tau_z} - D_{1,1}^{-1,-1} & D_{1,1}^{-1,0} & D_{1,1}^{-1,1} \\ D_{1,0}^{0,0} & D_{1,1}^{0,-1} & \frac{i}{\tau_x} + \frac{2i}{\tau_z} - D_{1,1}^{0,0} & D_{1,1}^{0,1} \\ D_{1,0}^{1,0} & D_{1,1}^{1,-1} & D_{1,1}^{1,0} & -\omega_s + \frac{i}{\tau_x} + \frac{3i}{\tau_z} + D_{1,1}^{1,1} \end{bmatrix}$$

$$\times \begin{bmatrix} a_0^0 \\ a_1^{-1} \\ a_1^0 \\ a_1^1 \end{bmatrix} = -\Omega \begin{bmatrix} a_0^0 \\ a_1^{-1} \\ a_1^0 \\ a_1^0 \\ a_p^m \end{bmatrix}. \quad (33)$$

The results of the solution of eq. (33) for some cases are plotted in figures 4 and 6. Figure 4a shows the imaginary and real parts of Ω/ω_s in terms of bunch current without mode coupling at zero chromaticity. This figure shows the starting instability point when current $I_{\mathrm{tr}} = 1.45$ mA which is in good agreement with the result of the previous approach (figure 1). In figure 4b, we take the first 16 terms in eq. (24) and it is clear that instability is due to the coupling of the first modes and the threshold current is not changed. However, at high chromaticity, we should take higher-order azimuthal and radial modes. We plot in figure 5 the instability threshold current as a function of mode number for five different chromaticities. It is clear that matrix size in eq. (24) required to obtain convergence increases with ξ_x. Figure 6 shows threshold current vs. chromaticity. Figure 6a is plotted by two-mode approximation. The two-mode approximation predicts that the beam is stable at any current when $\xi_x > 3.5$. Hence, we conclude that when the chromaticity becomes sufficiently large, the two-mode approximation fails. For higher values of ξ_x, it is necessary to solve the problem with large matrices. We consider 676 radial and azimuthal modes in figure 6b. Comparing figures 6a and 6b shows that the two-mode approximation is no longer valid when the head-tail phase $k_\xi x \gtrsim 0.2$. Also, figure 6b shows that the instability threshold current first decreases and then increases. The minimum threshold current occurred at $\xi \sim 4$ and the head-tail phase at the minimum threshold current is $k_\xi \sigma_z \sim 1$.

In the Lindberg's approach, the longitudinal motion is approximated as a simple harmonic oscillator and the longitudinal impedance is given entirely by

$$V_z = \frac{\omega_s^2}{2\alpha c^2} z^2.$$

Adding another longitudinal impedance distorts potential V_z that depends on the current profile of the bunch. Also, the equilibrium distribution function must be self-consistent with the total V_z, and so the situation becomes more complicated. In ref. [17], it was shown that the Fokker–Planck-based equations can be extended to approximately include a longitudinal impedance provided the resulting V_z is only a small perturbation.

For most modern storage rings, there are many other large sources of transverse impedance such as the small gap insertion devices. It is well known that transverse mode coupling instability threshold depends strongly on the tapers and gap size for these sections. By obtaining transverse impedance of these sections, the threshold current can be easily investigated.

4. Conclusions

For most modern storage rings, there are many sources of transverse impedance such as the RF cavities, small-gap insertion devices and resistive wall impedance of beam pipe. To analyse mode coupling and investigate transverse instability of the electron beam in the ILSF storage ring due to RF cavity impedance, two approaches are considered. The first approach is based on Vlasov equation, which was solved for a Gaussian bunch perturbed by transverse impedance. This formalism includes the radial and azimuthal modes and instability starts to occur when the first azimuthal modes are coupled. We apply this formalism to ILSF storage ring parameters and obtained the current threshold. Because of the synchrotron radiation of electron beam in the storage ring, to study the instability more precisely, we have used the approach based on the Fokker–Planck equation as well. In this method, the Fokker–Planck equation was solved by expansion in terms of the generalised Laguerre polynomials including chromaticity, both dipolar and quadrupolar transverse wakefields and the effects of damping and diffusion due to the synchrotron radiation. We have rewritten this formalism in the presence of broadband RF cavity impedance and used the ILSF parameters to find threshold current vs. chromaticity. It is shown that in 3 mm bunch length, at chromaticity $\xi_x \sim 4$, the instability is starting to occur at the minimum current and the head-tail phase that arises because of the dependence of the betatron frequency on the energy for non-zero chromaticity is equal to $k_\xi \sigma_z \sim 1$ at this threshold current.

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