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Non-linear Rayleigh–Bénard magnetoconvection in temperature-sensitive Newtonian liquids with heat source

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Abstract. The present paper aims at weak non-linear stability analysis followed by linear analysis of finite-amplitude Rayleigh–Benard magnetoconvection problem in an electrically conducting Newtonian liquid with heat source/sink. It is showed that the internal Rayleigh number, thermorheological parameter and Chandrasekhar number influence the onset of stationary convection and Nusselt number. The generalised Lorenz model derived for the problem is essentially the classical Lorenz model with its coefficients depending on the variable heat source (sink), viscosity and the applied magnetic field. The result of the parameters' influence on the critical Rayleigh number explains their effect on the Nusselt number as well. The effect of increasing strength of the magnetic field in stabilising the system and diminishing heat transport is demonstrated. But the heat source and variable viscosity work together to make the system unstable, as their effect enhances heat transfer.

Keywords. Generalised Lorenz model; Rayleigh–Bénard convection; magnetoconvection; heat source.

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1. Introduction

The Rayleigh–Bénard instability problem with internal heat generation, thermorheological effect and magnetic field have received great attention due to their implications in heat and mass transfer. The magnetic field is one of the important external mechanisms in controlling the onset of convection. The effect of magnetic field on stability analysis of the buoyancy-driven convection is discussed in several books and monographs such as Chandrasekhar [1], Nakagawa [2], Platten and Legros [3] and Drazin and Reid [4].

Magnetoconvection arises when there is an interaction of electrically conducting liquid in the presence of an external magnetic field, leading to an enhanced Lorenz force. Such type of motion can occur in natural convection. The early motivation for studying magnetoconvection is the astrophysical and geophysical problems such as the study of the sunspot (Chandrasekhar [1], Nakagawa [2], Proctor and Weiss [5]) and many other geophysical problems. Ozoe and Maruo [6] investigated the magnetoconvection of metal silicon by analysing two-dimensional numerical calculations to estimate the amount of heat transfer. The steady-state solutions were graphically shown and it was found that the flow is suppressed and elongated in a regime with high wave numbers. Siddheshwar and Pranesh [7] examined the impact of gravity and temperature modulation on thermal convection in an electrically, finitely conducting liquid with angular momentum. This examination revealed that the impact of temperature or gravity modulation is to stabilise the system by a reasonable tuning of the modulation frequency. Recently, many researchers worked on the magnetic field. Bhadauria [8] studied the effect of time-periodic heating of Rayleigh–Bénard convection in the presence of vertical magnetic field and found that the magnetic field suppresses the onset of convection. Aniss *et al* [9] and Suprio *et al* [10] stated that in a Rayleigh–Bénard convection problem, the convective rolls influence the stability of the system. Later on, Yanagisawa *et al* [11], Bhadauria and Kiran [12], Yusry *et al* [13] and Rajgopal *et al* [14] demonstrated this problem with different constraints. Recently, Ramachandramurthy and Aruna [15] studied the impact of magnetic field and variable viscosity on the stability of convection. They found that increasing the magnetic field stabilises the system, but variable viscosity has the opposite effect.

The aforementioned studies on thermal instability under the influence of magnetic field are without

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an internal heat source. However, in many practical problems, the material offers its heat source, and this leads to an enhanced heat transfer within the fluid layer. Such a situation can occur in radioactive disintegration, chemical reactions, nuclear reaction etc. It may occur even in some celestial bodies due to nuclear reaction and radioactivity, which keeps the celestial objects warm and active. Due to internal heat generation, there should be a temperature gradient between the surface and the centre of the Earth. This helps in maintaining the convective flow of heat, thereby transforming the thermal energy towards the surface of the Earth. Hence, the role of internal heat source/sink becomes very important in radioactive materials, in the fields of geophysics and astrophysics, in nuclear reactors, etc. However, there are very few studies in which the effect of internal heating on convective flow in a fluid layer has been investigated. McKenzie *et al* [16] demonstrated the thermal convection in the mantle of the Earth considering a large Prandtl number. This paper explains the geophysical information as well as convection in Boussinesq fluid of a very large Prandtl number. The effect of internal heat generation on Nusselt number at the onset of Rayleigh–Bénard convection was demonstrated by Tveitereid and Palm [17] and Clever [18]. These studies demonstrated two-dimensional finite-amplitude cellular convection and determined heat transfer coefficient as a function of two Rayleigh numbers. It is shown that the system can be stabilised with proper tuning of these parameters. Riahi *et al* [19,20] have done non-linear stability analysis of a fluid heated below and cooled above with internal heat generation and absorption. It is proved that increasing the internal heat generation enhances the onset of convection. Recently, the use of truncated Fourier series representation for the basic non-uniform temperature gradient gained attention in Rayleigh–Bénard problems as discussed in Siddheshwar and Titus [21] and Srivastava *et al* [22]. They derived an expression for non-uniform temperature gradient which was found to be sinusoidally varying with the vertical axis. This study demonstrated that in the presence of heat source, Nusselt number increases, but in the case of heat sink, it diminishes.

Viscosity is a physical property of the fluid. It is the ratio between the shear stress and the shear strain. For most of the fluid, viscosity is assumed to be temperature-dependent and there are only a few fluids possessing constant viscosity. Viscosity is one physical property that changes with temperature variation. Hence, we consider that viscosity is temperature-dependent. The effect of variable viscosity in exponentially decreasing or in polynomial form through truncated Taylor series has been investigated by Torrance and Turcotte [23]. This study demonstrated that an exponential decrease in viscosity is good enough to represent viscosity–temperature correlation for most of the fluids. Straughan [24] studied the effect of variable viscosity on convection and he found that thermorheological parameters influence the onset of convection. Also, its effect is to destabilise as the kinematic viscosity varies even at room temperature. Siddheshwar *et al* [25] investigated Rayleigh–Bénard and Marangoni–Bénard instability problems under the influence of thermorheological effect by considering different boundary conditions and they showed that increasing thermorheological parameter destabilises the system. Besides, Wu and Libchaber [26], and Shateyi and Motsa [27] showed that when the liquid layer is heated below, the electrically conducting fluid in the presence of suspended particles is more stable than the traditional electrically conducting liquid free of suspended particles. The basic wave number is insensitive to the suspension parameters but more sensitive to the Chandrasekhar number. The viscosity of Earth's mantle is known to be very strongly temperature-dependent, which is certain to affect the pattern of convection. Thus, it is essential to include such rheology in laboratory or numerical studies. The main results established by laboratory experiments (with rigid boundaries) are as follows:

1. Experiment carried out by Giannandrea and Christensen [28] demonstrates that the dependence of Nusselt number on Rayleigh number is not greatly affected by viscosity contrast.
2. The experimental observation of Booker [29] (i.e., that rolls are stable at low viscosity contrasts, with squares at high viscosity contrasts) was verified numerically by Busse and Frick [30] in a square box with linear dependence of viscosity on temperature.

Recently, Ramachandramurthy and Aruna [31] have done the linear stability analysis of Rayleigh–Bénard–Taylor convection in a temperature-sensitive liquid with a heat source and found that increasing the thermorheological parameter and internal Rayleigh number destabilise the system (see also Aruna and Ramachandramurthy [32]). These studies have not done the non-linear stability analysis and heat transfer. Therefore, the present study deals with heat transport by Rayleigh–Bénard magnetoconvection in an electrically conducting temperature-sensitive Newtonian liquid with a heat source. We used a half-range Fourier cosine series to represent the steady temperature and viscosity. The Galerkin procedure is used to find analytical expression for the thermal Rayleigh number as a function of internal Rayleigh number R_I, thermorheological parameter V and Chandrasekhar number Q. The generalised Lorenz model derived in this study is non-linear and solved

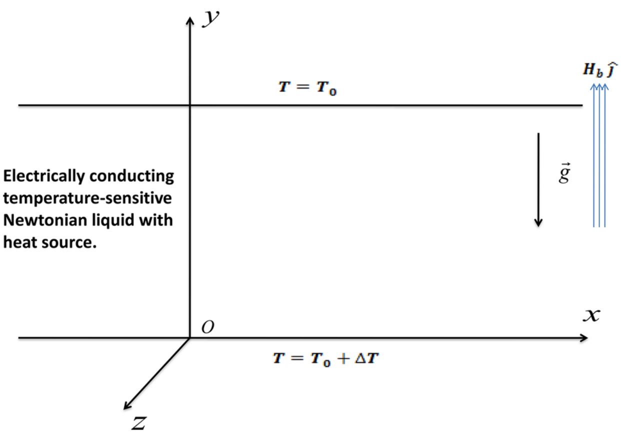

Figure 1. Schematic representation of the flow configuration.

using a numerical technique to determine the amplitudes of the system. This facilitates the quantification of heat transfer in terms of the parameters involved.

2. Mathematical formulation

Consider an electrically conducting Newtonian fluid located between two parallel infinite horizontal planes $y = 0$ and $y = d$ of depth d. The lower and upper boundaries are maintained at different temperatures that admit the temperature gradient $\Delta T > 0$ in the presence of vertical magnetic field H_0 as shown in figure 1. Cartesian coordinates have been taken with the origin at the bottom of the liquid layer, and the y-axis vertically upwards. We assume that Oberbeck–Boussinesq approximations are valid and considered only small-scale convective movements and the boundaries are considered to be stress-free and isothermal. We assume that density, dynamic viscosity and the heat source are temperature dependent. The governing system of equations to examine this situation of Rayleigh–Bénard convection in an electrically conducting Newtonian liquid in the presence of a magnetic field are given as follows:

$$\nabla \cdot \vec{q} = 0, \tag{1}$$

$$\nabla \cdot \vec{H} = 0, \tag{2}$$

$$\rho_0 \left[\frac{\partial q}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} \right] = -\nabla p + \rho(T)\vec{g} + \mu_m (\vec{H} \cdot \nabla)\vec{H} + \nabla[\mu_f(T)(\nabla \cdot \vec{q} + \nabla \cdot \vec{q}^{Tr})], \tag{3}$$

$$\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T = \chi \nabla^2 T + Q_1(T - T_0), \tag{4}$$

$$\frac{\partial \vec{H}}{\partial t} + (\vec{q} \cdot \nabla)\vec{H} = (\vec{H} \cdot \nabla)\vec{q} + \gamma_m \nabla^2 \vec{H}, \tag{5}$$

$$\rho(T) = \rho_0[1 - \alpha(T - T_0)], \tag{6}$$

$$\mu_f(T) = \mu_0 \mathrm{e}^{-\delta(T - T_0)}, \tag{7}$$

where $\vec{q} = u\hat{i} + v\hat{j}$ is the velocity vector, $\vec{H} = H_x\hat{i} + H_y\hat{j}$ is the magnetic intensity vector, ρ_0 is the reference density, $\rho(T)$ is the temperature-dependent density, $\mu_f(T)$ is the variable viscosity, μ_m is the magnetic permeability, T is the temperature distribution, t represents time, χ is the thermal diffusivity, γ_m is the magnetic viscosity, α is the coefficient of thermal expansion of the liquid and Q_1 is the strength of the heat source.

The basic state of the liquid is quiescent and the solutions are displayed below:

$$\begin{aligned} T_b &= T_0 + \Delta T f\left(\frac{y}{d}\right), \\ \rho_b\left(\frac{y}{d}\right) &= \rho_0\left(1 - \alpha \Delta T f\left(\frac{y}{d}\right)\right), \\ H_b &= H_b\left(\frac{y}{d}\right), \\ p_b\left(\frac{y}{d}\right) &= -\int \rho_b\left(\frac{y}{d}\right) g d\left(\frac{y}{d}\right) + C_0, \\ \mu_{f_b} &= \mu_0 \mathrm{e}^{-Vf(y/d)}, \end{aligned} \tag{8}$$

where

$$f\left(\frac{y}{d}\right) = \frac{1}{\sin\sqrt{R_I}} \sin\left(\sqrt{R_I}\left(1 - \frac{y}{d}\right)\right)$$

and C_0 is the constant of integration, $R_I = Q_1/d^2\chi$ is the internal Rayleigh number and $V = \delta\Delta T$ is the thermorheological parameter.

To perform finite-amplitude analysis, we impose perturbations on the basic state. Hence, we consider the following definitions:

$$\begin{aligned} &\vec{q} = q_b + \vec{q}', \quad T = T_b + T', \\ &\vec{H} = \vec{H}_b + \vec{H}', \rho = \rho_b + \rho', \\ &p = p_b + p', \quad \mu_f = \mu_{f_b} + \mu_f', \end{aligned} \tag{9}$$

where the primes indicate a perturbed quantity. We now consider the x, y components of the momentum equations and we restrict our study for the two-dimensional flow. Therefore, we introduce the stream function ψ' and the magnetic potential function ϕ' as follows:

$$u' = \frac{\partial \psi'}{\partial y}, \quad v' = -\frac{\partial \psi'}{\partial x}, \tag{10}$$

$$H_x' = \frac{\partial \phi'}{\partial y}, \quad H_y' = -\frac{\partial \phi'}{\partial x}. \tag{11}$$

Eliminating the pressure in eq. (3) and using eqs (10) and (11), we get the following component equations:

$$\rho_0 \frac{\partial}{\partial t}(\nabla^2 \psi') = \frac{\partial \mu_{f_b}}{\partial y}\frac{\partial}{\partial y}(\nabla^2 \psi') + \mu_{f_b}\nabla^4\psi' + \mu_m H_b \frac{\partial}{\partial y}(\nabla^2 \phi') + \mu_m \frac{\partial H_b}{\partial y}\frac{\partial^2 \phi'}{\partial y^2} - \rho_0 \alpha g \frac{\partial T'}{\partial x} - \frac{\partial(\psi', \nabla^2\psi')}{\partial (x,y)} - \frac{\partial(\phi', \nabla^2\phi')}{\partial (x,y)}, \quad (12)$$

$$\frac{\partial T'}{\partial t} = -\frac{\partial \psi'}{\partial x}\frac{\mathrm{d}T_b}{\mathrm{d}y} + \chi \nabla^2 T' + Q_1 T' + \frac{\partial\left(\psi', T'\right)}{\partial (x,y)}, \quad (13)$$

$$\frac{\partial \phi'}{\partial t} = \gamma_m \nabla^2 \phi' + H_b \frac{\partial \psi'}{\partial y} - \frac{\partial\left(\phi', \psi'\right)}{\partial (x,y)}. \quad (14)$$

We non-dimensionalise eqs (12)–(14) using the following definitions:

$$(X, Y) = \left(\frac{x}{d}, \frac{y}{d}\right), \quad \tau = \frac{\chi}{d^2}t, \quad \Psi = \frac{\psi'}{\chi}, \quad \Theta = \frac{T'}{\Delta T}, \quad \Phi = \frac{\phi'}{d H_b} \quad (15)$$

and obtain the dimensionless equations in the form

$$\frac{1}{Pr}\frac{\partial}{\partial \tau}(\nabla^2\Psi) = \frac{\partial \mu_{f_b}}{\partial Y}\frac{\partial}{\partial Y}(\nabla^2\Psi) + \mu_{f_b}\nabla^4\Psi - R_E\frac{\partial \Theta}{\partial X} + \frac{1}{Pr}\frac{\partial(\Psi, \nabla^2\Psi)}{\partial (X,Y)} + QPm\frac{\partial}{\partial Y}(\nabla^2\Phi) - QPm\frac{\partial(\Phi, \nabla^2\Phi)}{\partial (X,Y)}, \quad (16)$$

$$\frac{\partial \Theta}{\partial \tau} = -\frac{\partial \Psi}{\partial X}f'(Y) + \nabla^2\Theta + R_I\Theta + \frac{\partial\,(\Psi, \Theta)}{\partial\,(X,Y)}, \quad (17)$$

$$\frac{\partial \Phi}{\partial \tau} = Pm\nabla^2\Phi + \frac{\partial \Psi}{\partial Y} - \frac{\partial\,(\Phi, \Psi)}{\partial\,(X,Y)}, \quad (18)$$

where $Pm = \gamma_m/\chi$ represents the magnetic Prandtl number, $Pr = \gamma/\chi$ is the Prandtl number, $R_E = \alpha g \Delta T d^3/\gamma\chi$ is the thermal Rayleigh number, $Q = \mu_m {H_b}^2 d^2/\rho_0\gamma\gamma_m$ is the Chandrasekhar number, $f'(Y) = -(\sqrt{R_I}/\sin(\sqrt{R_I}))\cos[\sqrt{R_I}\,(1-Y)]$ is the non-uniform temperature gradient and its Fourier cosine series in (0, 1) is given by

$$f'(Y) = 1 + 2\sum_{n=1}^{\infty}\frac{R_I}{R_I - n^2\pi^2}\cos(n\pi Y).$$

The stress-free, isothermal boundary conditions for solving eqs (16)–(18) are

$$\Psi = \nabla^2\Psi = \Theta = D\Phi = 0 \quad \text{at} \quad Y = 0, 1. \quad (19)$$

In the case of a free surface, the boundary conditions on velocity will depend on whether we consider surface tension or not. If the free surface does not deform in the direction normal to itself, we must require that that vertical component of velocity vanishes at the boundaries ($v = 0$). In the absence of surface tension, the conditions on velocity at the free surface is $v = (\mathrm{d}^2 v/\mathrm{d}y^2) = 0$ and this condition is called the stress-free condition. If the bounding wall of the fluid layer has high heat conductivity and large heat capacity, the temperature, in this case, would be uniform and unchanging in time, i.e. the boundary temperature would be unperturbed by any flow or temperature perturbations in the fluid. Thus $T = 0$ at the boundaries. This boundary condition is known as isothermal or boundary condition of the first kind. This condition is also known as the Dirichlet condition. Also at the boundaries $D\Phi = 0$.

In §2.1, we discuss linear stability analysis, which is very useful in the study of local nonlinear stability analysis. Also, we analyse the heat transfer using the Nusselt number by varying different parameters involved in the problem.

2.1 *Linear stability analysis*

In order to obtain an analytical expression for the thermal Rayleigh number R_E, the linearised version of eqs (16)–(18) is considered along with the boundary conditions mentioned in eq. (19). All non-linear terms and the Jacobians $\partial\,(\Psi, \Theta)/\partial\,(X, Y)$ and $\partial\,(\Phi, \Psi)/\partial\,(X, Y)$ from eqs (16)–(18) are removed. The neglect of Jacobian is essential to remove the product of amplitudes in the non-linear terms. As discussed in Chandrasekhar's linear theory [1], the solutions of these equations are assumed to be periodic waves of the form

$$\Psi(X, Y, \tau) = \Psi_0 e^{i\omega\tau}\sin(\pi\alpha X)\sin(\pi Y)$$
$$\Theta(X, Y, \tau) = \Theta_0 e^{i\omega\tau}\cos(\pi\alpha X)\sin(\pi Y)$$
$$\Phi(X, Y, \tau) = \Phi_0 e^{i\omega\tau}\cos(\pi\alpha X)\sin(\pi Y) \quad (20)$$

and we obtain the expression for thermal Rayleigh number in the form

$$R_E = \frac{\eta_1^2(\eta_1^2 - R_I)(4\pi^2 - R_I)}{4\pi^2\alpha^2} \times \left(\frac{1+\alpha^2}{2}a_0 + \frac{1-\alpha^2}{2}a_2 + \frac{Q}{\eta_1^2}\right), \tag{21}$$

where

$$\eta_1^2 = \pi^2(1+\alpha^2),$$
$$a_0 = 2\mu_0 \int_0^1 e^{\frac{V}{\sin\sqrt{R_I}}\sin[\sqrt{R_I}(Y-1)]}\, \mathrm{d}Y,$$
$$a_2 = 2\mu_0 \int_0^1 e^{\frac{V}{\sin\sqrt{R_I}}\sin[\sqrt{R_I}(Y-1)]} \cos 2\pi Y\, \mathrm{d}Y.$$

In eq. (21), α is the scaled horizontal wave number. The quantities Ψ_0, Θ_0 and Φ_0 are respectively amplitudes of the stream function, temperature distribution and magnetic potential and ω is the frequency. In order to derive an expression for R_E, we substitute eq. (20) into the linearised version of eqs (16)–(18) (after dropping asterisks) and integrate with respect to X between the limits $[0, \frac{2\pi}{\pi\alpha}]$ and with respect to Y in [0, 1], resulting in a set of homogeneous linear system of equations in Ψ_0, Θ_0 and Φ_0. For the systems of non-trivial solution, the determinant of the coefficient matrix vanishes. In the event of obtaining the non-trivial solution, this expression R_E is derived.

It is now clear that R_E defined by eq. (21) is the thermal Rayleigh number of the marginal stationary state. The scaled horizontal wave number α_c for the preferred mode satisfies the equation

$$(a_0 - a_2)\pi^4\alpha^6 + \frac{1}{2}\begin{pmatrix} 3a_0\pi^4 - a_2\pi^4 \\ -R_I a_0\pi^2 + R_I a_2\pi^2 \end{pmatrix}\alpha^4 - \frac{1}{2}\begin{pmatrix} (a_0+a_2)\pi^4 + 2Q_1\pi^2 \\ -R_I a_0\pi^2 - R_I a_2\pi^2 + 2QR_I \end{pmatrix} = 0. \tag{22}$$

In the next section, we perform non-linear stability analysis, using truncated, yet representative Fourier modes for the basic temperature gradient and viscosity. We obtain an analytically intractable Lorenz model which is thus solved numerically by classical Runge–Kutta method of the fourth-order. This helps to quantify heat transfer coefficient Nusselt number Nu in terms of Chandrasekhar number Q, thermorheological parameter V and internal Rayleigh number R_I.

2.2 *Local non-linear stability analysis using minimal representation of Fourier series*

Minimal Fourier modes which describe the unsteady, two-dimensional non-linear Rayleigh–Bénard convective flow are given by

$$\left.\begin{aligned} \Psi(X,Y,\tau) &= A(\tau)\sin(\pi\alpha X)\sin(\pi Y), \\ \Theta(X,Y,\tau) &= B(\tau)\cos(\pi\alpha X)\sin(\pi Y) \\ &\quad -C(\tau)\sin(2\pi Y), \\ \Phi(X,Y,\tau) &= D(\tau)\sin(\pi\alpha X)\cos(\pi Y) \\ &\quad -E(\tau)\sin(2\pi\alpha x), \end{aligned}\right\} \tag{23}$$

where the time-dependent amplitudes $A(\tau), B(\tau), C(\tau), D(\tau)$ and $E(\tau)$ are to be determined from the dynamics of the system. Substituting eq. (23) into the eqs (16)–(18) following standard orthonormal procedure yields the generalised Lorenz model of fifth order as follows:

$$\frac{\mathrm{d}A}{\mathrm{d}\tau_1} = -Pr\left[\frac{a_0}{2} + \frac{2\pi^2 - \eta_1^{-2}}{2\eta_1^{-2}}a_2\right]A + \frac{\pi Pr\alpha R_E}{\eta_1^4}B - \frac{\pi Pr Pm Q}{\eta_1^2}D, \tag{24}$$

$$\frac{\mathrm{d}B}{\mathrm{d}\tau_1} = \frac{-\pi\alpha}{\eta_1^2}A - B - \frac{\pi^2\alpha}{\eta_1^2}AC, \tag{25}$$

$$\frac{\mathrm{d}C}{\mathrm{d}\tau_1} = -\frac{4\pi^2}{\eta_1^2}C + \frac{\pi^2\alpha}{2\eta_1^2}AB, \tag{26}$$

$$\frac{\mathrm{d}D}{\mathrm{d}\tau_1} = \frac{\pi}{\eta_1^2}A - PmD + \frac{\pi^2\alpha}{\eta_1^2}AE, \tag{27}$$

$$\frac{\mathrm{d}E}{\mathrm{d}\tau_1} = -\frac{4\pi^2\alpha^2}{\eta_1^2}PmE - \frac{\pi^2\alpha}{\eta_1^2}AD, \tag{28}$$

where

$$\tau_1 = \eta_1^2\tau.$$

Quite evidently, the generalised Lorenz model (GELM) of eqs (24)–(28) has the form of the classical Lorenz model but its coefficients depend on the parameters characterising variable heat source, viscosity and applied magnetic field. Due to the complexity of the autonomous system which is non-linear, it is not possible to obtain analytical solution for the general time-dependent variables. It is thus solved numerically using the Runge–Kutta method for the quantification of heat transfer.

2.3 *Analytical study of heat transport*

In any non-linear Rayleigh–Bénard convection problems, the quantification of heat transport by means of parameters involved is much important. Thus, the Nusselt number, Nu, calculated at the lower boundary characterising the heat transport is given by

$$Nu(\tau) = \frac{\text{Heat transport by conduction} + \text{Heat transport by convection}}{\text{Heat transport by conduction}},$$

$$Nu(\tau) = 1 + \frac{\left[\int_{X=0}^{2/\alpha_c} \frac{d\Theta}{dY} dX\right]_{Y=0}}{\left[\int_{X=0}^{2/\alpha_c} \frac{d}{dY}\left(\frac{\sin[\sqrt{R_I}(1-Y)]}{\sin\sqrt{R_I}}\right) dX\right]_{Y=0}}. \quad (29)$$

Substituting eq. (23) in eq. (29) and completing the integration, the expression of $Nu(\tau)$ is obtained in the following form:

$$Nu(\tau) = 1 + 2\pi\frac{\tan\sqrt{R_I}}{\sqrt{R_I}}C(\tau). \quad (30)$$

The second term on the right side of eq. (30) is the convective contribution to the heat transport.

3. Results and discussion

The effect of temperature-sensitive viscosity, heat source or sink on heat transport by Rayleigh–Bénard magnetoconvection in a finitely electrically conducting Newtonian liquid is studied by performing linear and weekly non-linear stability analysis using generalised Lorenz models. The effect of internal heat source/sink, temperature-dependent viscosity and the applied magnetic field appears in the form of internal Rayleigh number R_I, thermorheological parameter V and Chandrasekhar number Q respectively. The non-dimensional number R_E is the thermal Rayleigh number which is the eigenvalue of the problem.

3.1 *Linear stability analysis*

Some of the important highlights of linear stability analysis are

1. finding an analytical expression for Rayleigh number for the stationary state using truncated Fourier cosine series representation for the basic non-uniform temperature gradient and viscosity,
2. plotting neutral stability curves for the stationary state of convection. This helps to analyse the stability of the system,
3. effect of the oscillatory motions is neglected in linear theory.

The focus in this paper is the impact of variable heat source, viscosity and the presence of magnetic field on the onset of Rayleigh–Bénard convection. Therefore, we considered only those values of R_I, V and Q that do not dominate the buoyancy in effecting the convection. If these parameters dominate the buoyancy convection, then one can deal with this differently.

We consider the non-linear basic state dimensionless temperature distribution in the following form:

$$\theta_b(Y) = \frac{T_b(Y) - T_0}{\Delta T} = \frac{\sin(\sqrt{R_I}(1-Y))}{\sin(\sqrt{R_I})}, \quad (31)$$

which helps us to analyse the effect of heat source/sink in the problem (see figure 2).

The basic temperature distribution in the presence of heat source/sink is given by the above equation and when $R_I = 0$, the basic temperature distribution is linear and it is given by $\theta_b = 1 - Y$. It is obvious that the curves with heat source and sink are not symmetric about the line $\theta_b = 1 - Y$. Therefore, one can come to a conclusion that predictions and analysis on the problem with heat sink cannot be obtained from the heat source through any transformation unlike in the case of a constant heat source. Figure 3 shows variation of thermal Rayleigh number against wave number. It is clear from these plots that the internal Rayleigh number R_I, thermorheological parameter V and Chandrasekhar number Q influence the onset of convection.

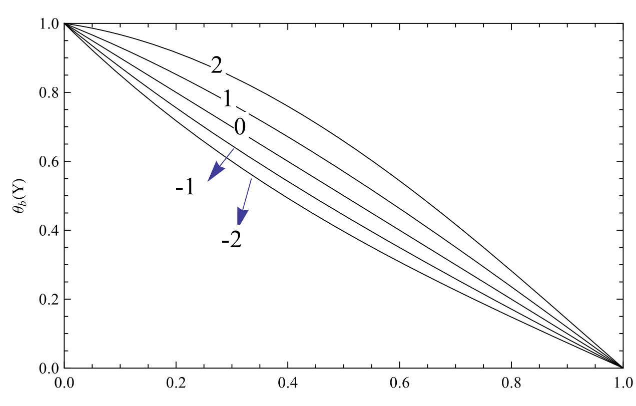

Figure 2. Temperature profile of the quiescent basic state for different values of R_I.

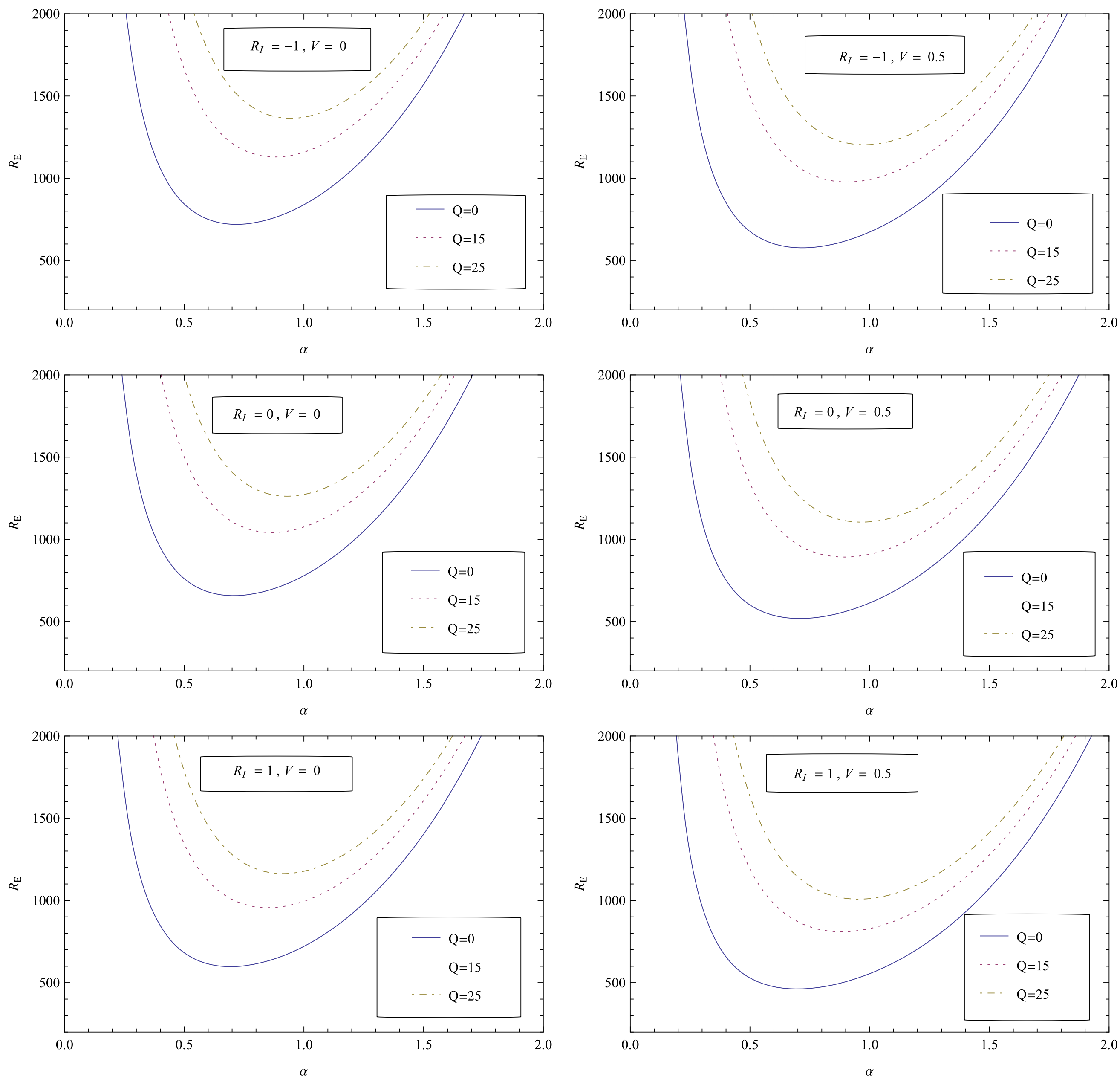

Figure 3. Plots of thermal Rayleigh number R_E vs. wave number α (stability curves).

One can suppress or advance the onset of convection using these parameters. Increasing Chandrasekhar number Q leads to increase in the values of R_{E_c} and α_c. On the other hand, if V increases the critical Rayleigh number R_{E_c} and the basic wave number α_c diminishes. This is because the temperature-dependent viscosity destabilises the system. The negative values of R_I indicate heat sink (heat absorption) and the positive values indicate heat source (heat generation). The impact of internal Rayleigh number R_I is the same as the impact of thermorheological parameter V. Also, table 1 demonstrates the variation of R_{E_c}, α_c with respect to the parameters and there is an excellent agreement between these values with Chandrasekhar [1] in the case of $V = R_I = 0$.

3.2 *Nonlinear stability analysis*

Some of the important highlights of non-linear stability analysis are

1. derivation of useful autonomous system of differential equations by standard orthogonalisation procedure for the Galerkin expansion (generalised Lorenz model),

Table 1. Critical values of Rayleigh and wave number for different parameters.

	$V = 0$		$V = 0.5$	
	R_{E_c}	α_c	R_{E_c}	α_c
	$R_I = -1$			
$Q = 0$	719.46	0.7184	576.86	0.7185
$Q = 15$	1129.34	0.8790	977.13	0.9056
$Q = 25$	1364.30	0.9446	1203.69	0.9778
	$R_I = 0$			
$Q = 0$	657.51	0.7071	518.50	0.7085
$Q = 15$	1041.51	0.8652	892.30	0.8947
$Q = 25$	1262.49	0.9297	1104.74	0.9661
	$R_I = 1$			
$Q = 0$	597.31	0.6944	461.684	0.6975
$Q = 15$	955.85	0.8498	809.432	0.8827
$Q = 25$	1163.08	0.9131	1007.93	0.9536

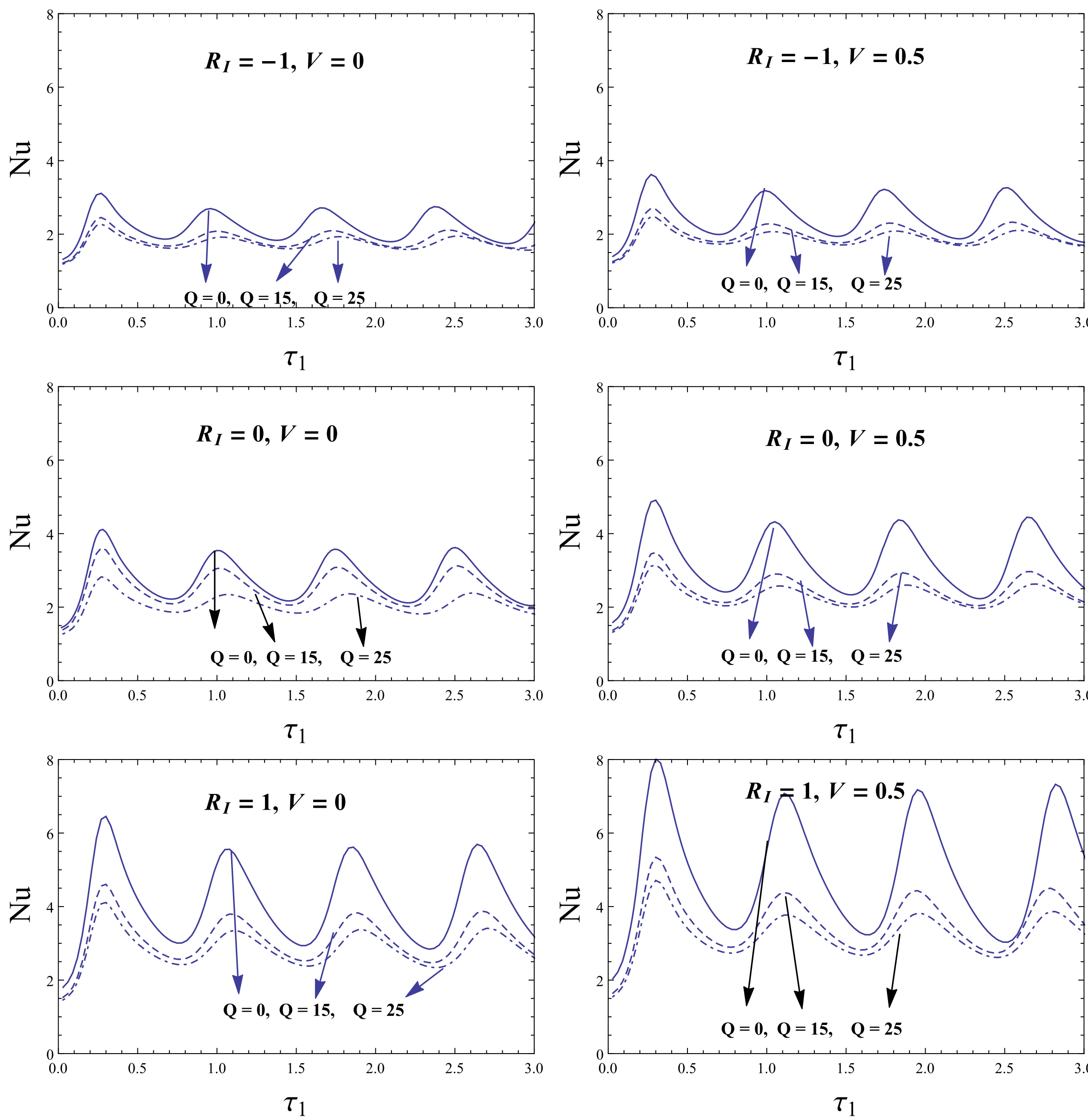

Figure 4. Variation of Nusselt number Nu with scaled time τ_1 for $Pr = Pm = 1$.

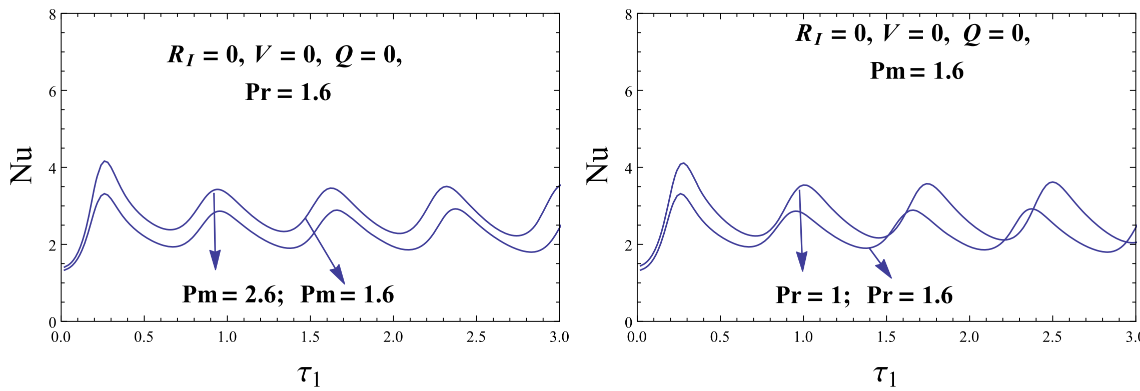

Figure 5. Variation of Nusselt number Nu with scaled time τ_1.

2. quantification of heat transfer (Nusselt number) by solving the Lorenz model using fourth-order Runge–Kutta method,
3. the effect of Prandtl and magnetic Prandtl numbers on heat transfer.

Under non-linear stability analysis, quantification of heat transfer is much important. The analytic expression of Nusselt number $Nu(\tau)$ involves convective term $C(\tau)$ that capture the heat flux. Moreover $C(\tau)$ is coupled with all other time-dependent amplitudes and parameters involved in the problem. The non-linear state with heat source/sink is likely to be stationary because the convective flow is characterised by two non-dimensional numbers: The Rayleigh number R_E and the Prandtl number Pr. The non-linear stationary rolls appear as primary instability in Boussinesq fluid confined between two thermally conducting boundaries, when Rayleigh number increases just above its critical value R_{E_c}, that is at the onset of convection (Chandrasekhar [1]). Quite evidently, the Lorenz model display is the arrangement of coupled first-order differential conditions. These are not amenable to the analytical solution having non-linear terms and variable coefficients. Therefore, these equations have been solved numerically utilising the classical Runge–Kutta fourth-order technique with an adaptive step size of $h = 0.001$. The initial amplitudes of the Lorenz system are assumed to be $A(0) = B(0) = C(0) = D(0) = E(0) = 5$. Figure 4 demonstrates the variation of Nusselt number with the evaluation of time. The plots are drawn with different values of Q, R_I, V as shown in the figure. We see that these parameters influence the heat transfer coefficient as the critical Rayleigh and wave number change with these parameters.

We see from these plots, the effect of increasing Chandrasekhar number Q on Nusselt number Nu. The Nusselt number Nu decreases with increasing Q and vice versa. This is due to the decrease in heat transport in the presence of a magnetic field. As the variable viscosity parameter V and internal heat source parameter R_I increase, their effect is to increase the Nusselt number Nu and hence it enhances the heat transfer. But for negative values of R_I, heat transport decreases because of internal heat absorption.

From figure 5 we draw the following important results:

1. $Nu(Pr = 1.0) > Nu(Pr = 1.6)$,
2. $Nu(Pm = 1.6) < Nu(Pr = 2.6)$.

4. Conclusion

The paper presents an analytical study of unsteady, non-linear Rayleigh–Bénard magnetoconvection in an electrically conducting temperature-dependent Newtonian liquid with a variable heat source. Analytical expression for the external Rayleigh number R_E is determined with respect to Q, V and R_I by applying Galerkin technique. The linear analysis concludes that increasing the internal Rayleigh number R_I and thermorheological parameter V destabilises the system, whereas increasing Chandrasekhar number Q stabilises the same. Non-linear stability analysis involves the derivation of the generalised Lorenz model and its solution using the classical Runge–Kutta method to quantify heat transfer. This new result is also demonstrated in the paper. It is shown that the effect of increase in the strength of magnetic field increases the critical Rayleigh number and hence diminishes heat transport, whereas the effect of increase in internal Rayleigh number and thermorheological parameter together decreases the critical Rayleigh number and hence it enhances the heat transfer.

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