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Quasi-exact and asymptotic iterative solutions of Dirac equation in the presence of some scalar potentials

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Abstract. In this paper, the Dirac equation in the presence of some scalar potentials based on $sl(2)$ Lie algebra is solved by quasi-exact solvability theory. The configuration of the classes III and VI potentials in the Turbiner's classification is constructed. Then, the Bethe ansatz equations are calculated so that the energy eigenvalues and eigenfunctions are obtained. Also, we study the problem by using asymptotic iteration method. Finally, we compare the results obtained by these two methods.

Keywords. Dirac equation; quasi-exact solvability; supersymmetric quantum mechanics; asymptotic iteration method.

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1. Introduction

In the context of relativistic quantum mechanics and quantum field theory, Lorentz scalar potential has had considerable interest, and has important use in these areas [1–6]. But, some forms of Lorentz scalar potential result in exactly solvable problems.

Recently, much efforts were put in to solve the Dirac equation using various methods [7–23]. However, a few potentials can be solved exactly. On the other hand, quasi-exact solvability approach prepares a procedure for deriving a part of the energy spectrum but not the whole spectrum [23]. Also, supersymmetric quantum mechanics has attracted considerable attention in the past years [24–30].

A general method for constructing quasi-exactly solvable problems based on $sl(2)$ Lie algebra has been presented in refs [31–33]. Moreover, construction of all the one-dimensional quasi-exactly solvable models has been done. In fact, factorisability or equivalently supersymmetric structure of the Hamiltonian of the problem plays an important role in this method. At the same time, this method proposes some configurations for the potential that lead to the derivation of a part of the spectrum but not the whole spectrum. Also, possible forms of the Lorentz scalar potential which lead to the quasi-exactly solvable forms of the corresponding Dirac equation are determined. On the other hand, the asymptotic iteration method was proposed [34,35] for solving eigenvalue problems. So, this method can be used to investigate Dirac equation with various potentials.

Turbiner classified the quasi-exactly solvable models based on $sl(2)$ Lie algebra in ten classes [36]. When Dirac equation with the Lorentz scalar potential is factorised, seven classes of quasi-exactly solvable potential can be identified. These classes are classes I to VI and class X in the framework of the Turbiner's classification. The construction of class I potential has been performed in ref. [37]. Meanwhile, the constructions of classes II, IV, V and X have been studied in ref. [38].

In what follows, we investigate the construction of classes III and VI by means of the factorisabillity of the Dirac equation with the Lorentz scalar potential. Meanwhile, the Dirac equation with these potentials is studied by asymptotic iteration method. In fact, we solve the Dirac equation by two methods, that is, quasi-exactly solvable (QES) theory and asymptotic iteration method (AIM).

2. $2+1$-Dimensional Dirac equation

Quasi-exactly solvable theory is studied in this section on the basis of ref. [37].

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The Hamiltonian of (2 + 1)-dimensional Dirac equation can be written as

$$H = \boldsymbol{\alpha} \cdot \mathbf{p} + \beta(m + V_s) \tag{1}$$

in which $\boldsymbol{\alpha}$ and β are Dirac matrices and $V_s(x)$ is the scalar potential. Meanwhile, the following representation of the Dirac matrices is used:

$$\beta = \sigma_z, \quad \alpha_y = \sigma_y \quad \text{and} \quad \alpha_x = \sigma_x, \tag{2}$$

where σ_x, σ_y and σ_z are the Pauli matrices.

As the potential is only a function of x, the wave function is given by

$$\psi = \mathrm{e}^{ik_y y} \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix}, \tag{3}$$

where k_y is a real constant and $f_\pm$ are some real functions of x. By substituting relations (2) and (3) in the Dirac equation, one may have the following matrix equation:

$$\begin{pmatrix} U(x) & p_x - ik_y \\ p_x + ik_y & -U(x) \end{pmatrix} \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix} = E \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix}, \tag{4}$$

where $U = m + V_s(x)$. In order to construct the supersymmetric form of eq. (4), the wave function and the Hamiltonian should be transformed by a unitary matrix T as

$$\begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix} \longrightarrow \begin{pmatrix} i\psi_-(x) \\ i\psi_+(x) \end{pmatrix} \equiv T^+ \begin{pmatrix} f_-(x) \\ f_+(x) \end{pmatrix},$$

$$H \to T^+ H T = \begin{pmatrix} k_y & i\left(-\frac{\mathrm{d}}{\mathrm{d}x} + U(x)\right) \\ -i\left(-\frac{\mathrm{d}}{\mathrm{d}x} + U(x)\right) & -k_y \end{pmatrix}, \tag{5}$$

where

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}. \tag{6}$$

So, the Dirac equation is factorised as follows;

$$\left(-\frac{\mathrm{d}}{\mathrm{d}x} + U\right)\psi_+ = (E - k_y)\psi_-,$$
$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + U\right)\psi_- = (E + k_y)\psi_+ \tag{7}$$

or

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U^2 - U'\right)\psi_- = \epsilon\psi_-,$$
$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + U^2 + U'\right)\psi_+ = \epsilon\psi_+, \tag{8}$$

where $\epsilon = E^2 - k_y^2$. In fact, $U(x)$ is the superpotential in the above equation. We can solve factorised equations (8) by obtaining just one of the component of the wave function, for example, the upper component. Then, the lower component may be derived by using suitable supercharge operator on the upper component.

Quasi-exact solvability theory suggests an imaginary gauge transformation on the function $\psi_-(x)$, i.e.

$$\psi_-(x) = \varphi \mathrm{e}^{-g(x)}, \tag{9}$$

where $g(x)$ is the gauge function. On the other hand, the function $\varphi(x)$ satisfies the following equation:

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + 2g'\frac{\mathrm{d}\varphi}{\mathrm{d}x} + (V(x) + g'' - g'^2)\varphi = \epsilon\varphi, \tag{10}$$

in which $V(x) = U^2 - U'$ [39]. In fact, the phase factor e^{-g} plays an important role in the asymptotic behaviours of the wave function of the physical systems which leads to the normalisation condition. Indeed, the functional shape of the potential $V(x)$ ensures the quasi-exact solvability of the systems which makes the gauge transformed Hamiltonian to be written down as a quadratic combination of the generators of the $sl(2)$ Lie algebra. Moreover, one can construct seven classes of quasi-exactly solvable potentials. Actually, in Turbiner's classification, the mentioned classes are classes I to VI and class X. In ref. [37], the construction of class I has been studied. The constructions of classes II, IV, V and X have been analysed in ref. [38]. In this paper, classes III and VI are examined in detail.

3. Class III

According to the Turbiner's classification, the quasi-exactly solvable potential in class III takes the following form:

$$V = d^2\mathrm{e}^{4\alpha x} + 2ad\mathrm{e}^{3\alpha x} + [a^2 + 2d(b + \alpha)]\mathrm{e}^{2\alpha x}$$
$$+ (2ab + \alpha a + \lambda)\mathrm{e}^{\alpha x} + b^2. \tag{11}$$

where a, b, d, λ and α are some parameters which guarantee the normalisation condition of the wave function. The corresponding gauge function of the potential is given by

$$g(x) = -\frac{d}{2\alpha}\mathrm{e}^{2\alpha x} - \frac{a}{\alpha}\mathrm{e}^{\alpha x} - bx. \tag{12}$$

The quasi-exact potential that leads to ground state, has the following form:

$$V(\lambda = 0) = U_0^2 - U_0', \tag{13}$$

where

$$U_0 = g'(x) = -d\mathrm{e}^{2\alpha x} - a\mathrm{e}^{\alpha x} - b. \tag{14}$$

Note that the supersymmetric partner potential of class III is given by

$$\tilde{V}_0 = d^2 e^{4\alpha x} + 2ade^{3\alpha x} + (a^2 + 2db - 2d\alpha)e^{2\alpha x} + (2ab - a\alpha)e^{\alpha x} + b^2. \quad (15)$$

It can be shown that the energy eigenvalues, the wave functions and the S-matrices of the partner potentials are related [41]. Here, we only study one of these potentials. Considering the above potential, eq. (10) will be quasi-exactly solvable when its corresponding operator can be written as a linear combination of the generators of $sl(2)$ Lie algebra. So for this purpose, first of all, we use the change of variable $z = e^{-\alpha x}$ in eq. (10). Then we have

$$-\alpha z^3 \frac{d^2\varphi}{dz^2} + [(2b-\alpha)z^2 + 2az + 2d]\frac{d\varphi}{dz} + \left(\frac{\lambda}{\alpha} - \frac{\epsilon z}{\alpha}\right)\varphi = 0. \quad (16)$$

Now, we solve the above differential equation using two methods, i.e. QES and AIM.

3.1 *QES*

One may rewrite eq. (16) as

$$T_{\rm III}\varphi = 0, \quad (17)$$

where

$$T_{\rm III} = -\alpha z^3 \frac{d^2}{dz^2} + [(2b-\alpha)z^2 + 2az + 2d]\frac{d}{dz} + \frac{\lambda}{\alpha} - \frac{\epsilon z}{\alpha}. \quad (18)$$

Now, we write down the differential operator $T_{\rm III}$ in terms of the generators of the $sl(2)$ Lie algebra, that is, J^+, J^0 and J^-. In fact, the most general symmetry algebra for the one-dimensional quasi-exactly solvable systems is $sl(2)$ Lie algebra [40] is

$$T_{\rm III} = -\alpha J^+ J^0 + (2b - 3N\alpha/2)J^+ + 2aJ^0 + 2dJ^- + \left(-\alpha N^2 + 2bN - \frac{\epsilon}{\alpha}\right)z + \frac{\lambda}{\alpha} + aN, \quad (19)$$

where

$$J^+ = z^2\frac{d}{dz} - Nz,$$
$$J^0 = z\frac{d}{dz} - \frac{N}{2}, \quad N = 0, 1, 2, \ldots$$
$$J^- = \frac{d}{dz}. \quad (20)$$

It is seen that eq. (10), with the class III potential, will be quasi-exactly solvable when

$$-\alpha N^2 + 2bN - \frac{\epsilon}{\alpha} = 0. \quad (21)$$

In other words, the energy should be equal to

$$\epsilon = \alpha N(2b - \alpha N). \quad (22)$$

It must be mentioned that the quasi-exactly solvable Hamiltonian has an algebraic sector with $N+1$ eigenvalues and eigenfunctions. The eigenfunctions can be given as

$$\psi_- = \varphi e^{-g(x)} = \prod_{i=1}^{N}(z - z_i)e^{-\int U_0(x)dx}, \quad (23)$$

in which z_i $(i = 1, 2, \ldots, N)$ are some parameters which are derived by the Bethe ansatz equations. Furthermore, eq. (23) may be written as follows:

$$\psi_- = e^{-\int U_N(x,\{z_i\})dx}, \quad (24)$$

$$U_N(x, \{z_i\}) = U_0(x) - \sum_{i=1}^{N}\frac{z'(x)}{z(x) - z_i}. \quad (25)$$

It is evident that there are $N+1$ possible functions U_N for the $N+1$ sets of eigenfunctions ψ. Considering $\varphi(z) = \prod_{i=1}^{N}(z - z_i)$, the Bethe ansatz equations are deduced for relation (16). The operator $T_{\rm III}$ in eq. (18) is factorisable as products of a and a^+ and the ground state is given by: $a\phi = 0$. Accordingly, the operator a is defined as

$$a = \frac{d}{dz} - \sum_{i=1}^{N}\frac{1}{z - z_i} \quad (26)$$

which satisfies the condition $a\varphi = 0$. Also, the operator a^+ is defined as

$$a^+ = -\alpha z^3\frac{d}{dz} + [(2b-\alpha)z^2 + 2az + 2d] - \alpha z^3\sum_{j=1}^{N}\frac{1}{z - z_j}. \quad (27)$$

It is shown that

$$a^+ a = T_{\rm III} - \frac{\lambda}{\alpha} + \frac{\epsilon z}{\alpha} + \alpha z^3\sum_{i\neq j}^{N}\frac{1}{(z - z_i)(z - z_j)} - [(2b-\alpha)z^2 + 2az + 2d] \times \sum_{i=1}^{N}\frac{1}{(z - z_i)}. \quad (28)$$

On the other hand

$$T_{\rm III}\varphi = 0. \quad (29)$$

So, we have

$$\left\{-\frac{\lambda}{\alpha}+\frac{\epsilon z}{\alpha}+\alpha z^3\sum_{i\neq j}^{N}\frac{1}{(z-z_i)(z-z_j)}\right.$$
$$\left.-[(2b-\alpha)z^2+2az+2d]\sum_{i=1}^{N}\frac{1}{(z-z_i)}\right\}\varphi=0. \tag{30}$$

Now, we obtain the Bethe ansatz equations corresponding to the class III potential which, in turn, leads to the roots z_i's of the function φ and the energy eigenvalues

$$(2b-\alpha)z_i^2+2az_i+2d-2\sum_{j\neq i}^{N}\frac{\alpha z_i^3}{z_i-z_j}=0, \tag{31}$$

$$\lambda=2d\alpha\sum_{i=1}^{N}\frac{1}{z_i}. \tag{32}$$

For example, let us consider the case $N=1$. In this case, the energy eigenvalue is given by

$$\epsilon=\alpha(2b-\alpha) \tag{33}$$

and the wave function is

$$\psi_-(x)=(z-z_1)\mathrm{e}^{-g(x)}. \tag{34}$$

Also, by using eq. (7), the lower component of the wave function, ψ_+, can be obtained as

$$\psi_+(x)=0. \tag{35}$$

The root z_1 is obtained by the following relation:

$$(2b-\alpha)z_1^2+2az_1+2d=0. \tag{36}$$

Thus, we have two roots

$$z_1^{\mp}=\frac{-2a\mp\sqrt{4a^2-8d(2b-\alpha)}}{2(2b-\alpha)}. \tag{37}$$

Also, two values are derived for the parameter λ:

$$\lambda^{\mp}=\frac{2d\alpha}{z^{\mp}}. \tag{38}$$

In fact, z_1^- and z_1^+ correspond to the ground state and the first excited state, respectively. In other words, the potential $V_1^{(0)}(x,\lambda^-)$ yields the ground state while $V_1^{(1)}(x,\lambda^+)$ yields the first excited state. Also, the numerical results of the energy eigenvalues for $N=1$ and 2 are given in table 1. Considering relation (25), the mentioned potentials are given by the following superpotentials:

$$U_1^{(0,1)}(x)=U_0(x)-\frac{z'(x)}{z(x)-z_1^{\mp}}=-d\mathrm{e}^{2\alpha x}-a\mathrm{e}^{\alpha x}-b$$
$$+\frac{\alpha\mathrm{e}^{-\alpha x}}{\mathrm{e}^{-\alpha x}-z_1^{\mp}}. \tag{39}$$

Therefore, the scalar potentials that result in the solvability of the ground state as well as the first excited state are given by

$$V_{s1}^{(0,1)}(x,\{z_1\})=U_1^{(0,1)^2}-U_1^{(0,1)'}. \tag{40}$$

Note that the same procedure is used for higher N. However, for a given N, $N+1$ different configurations of potentials are produced, so that every potential calculates one of the states.

3.2 *AIM*

Eqution (16) can be rewritten as follows:

$$\frac{\mathrm{d}^2\varphi(z)}{\mathrm{d}z^2}=\lambda_0(z)\frac{\mathrm{d}\varphi(z)}{\mathrm{d}z}+s_0(z)\varphi(z), \tag{41}$$

where

$$\lambda_0(z)=\frac{(2b-\alpha)z^2+2az+2d}{\alpha z^3},$$
$$s_0(z)=\frac{\lambda-\epsilon z}{\alpha^2z^3}. \tag{42}$$

It should be mentioned that $\lambda_0(z)$ and $s_0(z)$ have sufficiently many continuous derivatives. According to AIM [34], eq. (41) has a general solution as

$$\varphi(z)=\exp\left(-\int^{z}\beta\,\mathrm{d}t\right)\left[C_2+C_1\int^{z}\exp\left(\int^{t}(\lambda_0(\tau)\right.\right.$$
$$\left.\left.+2\beta(\tau))\,\mathrm{d}t\right)\right] \tag{43}$$

if for some $n>0$, we have

$$\frac{s_n}{\lambda_n}=\frac{s_{n-1}}{\lambda_{n-1}}\equiv\beta, \tag{44}$$

where

$$\lambda_k=\lambda'_{k-1}+s_{k-1}+\lambda_0\lambda_{k-1},$$
$$s_k=s'_{k-1}+s_0\lambda_{k-1}, \tag{45}$$

for $k=1,2,3,\ldots,n$. Also, for the eigenvalue problems, the energy eigenvalues can be determined from the quantisation condition given by eq. (44). The energy eigenvalues and parameter λ are computed for $N=1$ and 2 by using the AIM and compared by the QES as shown in table 1. To obtain the numerical results, the constant parameters are taken as: $a=5$, $b=3$ and $\alpha=d=1$. The values obtained by QES are equal to the AIM values.

Table 1. Comparison between λ computed by AIM and QES.

	QES	AIM
$N = 1$	−8.8730	−8.8730
$\epsilon = 5$	−1.1270	−1.1270
$N = 2$	−18.7505	−18.7505
$\epsilon = 8$	−9.4420	−9.4420
	−1.8075	−1.8075

4. Class VI

Considering the Turbiner's classification, the general form of the quasi-exactly solvable potential in class VI is

$$V(x) = a^2x^6 + 2abx^4 + [b^2 - (2k+3)a]x^2 - b, \tag{46}$$

where the parameters a, b and k are responsible for the normalisation condition of the wave function. The gauge function $g(x)$ associated with the potential is given by

$$g(x) = \frac{ax^4}{4} + \frac{bx^2}{2}. \tag{47}$$

The quasi-exactly solvable potential which yields the ground state with energy parameters $\epsilon = E^2 - k_y^2$, has the following form:

$$\begin{aligned} V_0 &= g'^2 - g'' \\ &= U_0^2 - U_0', \end{aligned} \tag{48}$$

where

$$U_0(x) = g'(x) = ax^3 + bx \tag{49}$$

and the supersymmetric partner potential of class VI is given by

$$\begin{aligned} \tilde{V}_0 &= g'^2 + g'' \\ &= a^2x^6 + abx^4 + (b^2 + 3a)x^2 + b. \end{aligned} \tag{50}$$

Again, similar to the previous section, we study only one of the partner potentials. In order to see that eq. (10) with the above potential is quasi-exactly solvable, we should derive the Bethe ansatz equations. First of all, we do the change of variable $z = x^2$ in eq. (10). Then, we have

$$\begin{aligned} &-4z\frac{\mathrm{d}^2\varphi}{\mathrm{d}z^2} \\ &\quad +2(2az^2 + 2bz - 1)\frac{\mathrm{d}\varphi}{\mathrm{d}z} + (-2kaz - \epsilon + b)\varphi = 0. \end{aligned} \tag{51}$$

The energy eigenvalues of the above differential equation are obtained by using the QES and AIM.

4.1 *QES*

Equation (51) may be written as

$$T_{\mathrm{VI}}\varphi = 0, \tag{52}$$

where

$$\begin{aligned} T_{\mathrm{VI}} &= -4z\frac{\mathrm{d}^2}{\mathrm{d}z^2} \\ &\quad +2(2az^2 + 2bz - 1)\frac{\mathrm{d}}{\mathrm{d}z} + (-2kaz - \epsilon + b). \end{aligned} \tag{53}$$

Indeed, the differential operator T_{VI} can be expressed in terms of the $sl(2)$ generators J^+, J^0 and J^- as

$$\begin{aligned} T_{\mathrm{VI}} &= -4J^0J^- + 4aJ^+ + 4bJ^0 - 2(N + 2 - \nu)J^- \\ &\quad +\text{const.}, \end{aligned} \tag{54}$$

where $N = [k/2]$. Also, for the odd cases $\nu = 0$ and for the even cases $\nu = 1$, the quasi-exactly solvable Hamiltonian possesses an algebraic sector with $N + 1$ eigenvalues and eigenfunctions.

Now, we can derive the Bethe ansatz equations. The operator a is defined as

$$a = \frac{\mathrm{d}}{\mathrm{d}z} - \sum_i \frac{1}{z - z_i}. \tag{55}$$

It is seen that

$$a\varphi(z) = 0. \tag{56}$$

Regarding the form of the Hamiltonian, the operator a^+ is defined as

$$\begin{aligned} a^+ &= -4z\frac{\mathrm{d}}{\mathrm{d}z} + 2(2az^2 + 2bz - a) \\ &\quad -4z\sum_j \frac{1}{z - z_j}. \end{aligned} \tag{57}$$

If we write down the Hamiltonian in terms of the operators a and a^+, we shall have

$$\begin{aligned} &\left[a^+a + 2kaz - b + \epsilon + 4z\sum_{i\neq j}\frac{1}{(z - z_i)(z - z_j)}\right. \\ &\quad \left. -4(2az^2 + 2bz - 1)\sum_i \frac{1}{z - z_i}\right]\varphi = 0. \end{aligned} \tag{58}$$

Equation (56) leads to

$$\begin{aligned} &\left[2kaz - b + \epsilon + 4z\sum_{i\neq j}\frac{1}{(z - z_i)(z - z_j)}\right. \\ &\quad \left. -2(2az^2 + 2bz - 1)\sum_i \frac{1}{z - z_i}\right]\varphi = 0. \end{aligned} \tag{59}$$

The above equation gives the roots z_i as follows:

$$2az_i^2 + 2bz_i - 1 - 2\sum_{i\neq j} \frac{z_i}{z_i - z_j} = 0. \qquad (60)$$

Also, the energy parameters is given by

$$\epsilon = \sum_j \frac{2}{z_j} + b. \qquad (61)$$

Equations (60) and (61) are called Bethe ansatz equations corresponding to eq. (51). For example, we construct U_1. In this case, $N = 1$ and consequently $\varphi(z) = z - z_1$. The upper and lower components of the wave function are obtained as

$$\psi_- = (z - z_1)e^{-g(x)},$$
$$\psi_+ = 0. \qquad (62)$$

On the basis of eq. (60), the root z_1 satisfies the following relation:

$$2az_1^2 + 2bz_1 - 1 = 0. \qquad (63)$$

So, we have two roots

$$z_1^\pm = \frac{-b \pm \sqrt{b^2 + 2a}}{2a}. \qquad (64)$$

Also, the energy parameters are

$$\epsilon^\pm = \frac{4a}{-b \pm \sqrt{b^2 + 2a}} + b. \qquad (65)$$

In fact, z_1^- and z_1^+ are associated with the ground state and the first excited state, respectively. Considering eq. (24), the superpotential can be constructed as follows:

$$U_1(x) = U_0 - \frac{h'(x)}{h(x) - z_1^-} = ax^3 + bx - \frac{2x}{x^2 + |z_1^-|}, \qquad (66)$$

where $h(x) = x^2$. The energies of the ground and the first excited states are given by $E^2 - k_y^2 = 0$ and $E^2 - k_y^2 = \epsilon^+ - \epsilon^- = 4\sqrt{b^2 + 2a}$, respectively. It must be mentioned that the same procedure is followed for higher N for which $N + 1$ different configurations of potentials are generated. The numerical values of the energy eigenvalues for $N = 1$ and 2 are given in table 2.

4.2 *AIM*

In order to find the solution of the Dirac equation in the presence of the class VI QES potential by using the asymptotic iteration method, we consider the differential equation (51) as

$$\frac{d^2\varphi(z)}{dz^2} = \lambda_0(z)\frac{d\varphi}{dz} + s_0(z)\varphi, \qquad (67)$$

Table 2. Comparison between the energy eigenvalues computed by QES and AIM.

	QES	AIM
$N = 1$	1.1010	1.1010
$k = 2$	10.8990	10.8990
$N = 2$	−0.1363	−0.1363
$k = 4$	7.9308	7.9308
	22.2055	22.2055

where

$$\lambda_0(z) = \frac{2az^2 + 2bz - 1}{2z},$$
$$s_0(z) = \frac{-2kaz - \epsilon + b}{4z}. \qquad (68)$$

The enegry eigenvalue ϵ can be obtained by considering the quantisation condition (44). The numerical solutions for $N = 1$ and 2 are presented in table 2. The constant parameters are taken as $a = 1$ and $b = 2$. It is clear that the values obtained by the QES and AIM are equal.

5. Conclusions

The Dirac equation with classes III and VI potentials in the Turbiner's classification was factorised and the associated operators were expressed in terms of the generators of the $sl(2)$ Lie algebra. The corresponding Bethe ansatz equations were obtained. So, the wave functions and the energy eigenvalues of the ground and the first excited states were obtained. Meanwhile, the energy eigenvalues were computed using the asymptotic iteration method. It is noticed that the two methods lead to the same results.

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