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Approximate technique for solving fractional variational problems

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Abstract. The purpose of this paper is to suggest a numerical technique to solve fractional variational problems (FVPs). These problems are based on Caputo fractional derivatives. Rayleigh–Ritz method is used in this technique. First we approximate the objective function by the trapezoidal rule. Then, the unknown function is expanded in terms of the Bernstein polynomials. By this method, a system of algebraic equations is driven. We provide examples to show the effectiveness of this technique, which is considered in the current study.

Keywords. Fractional variational problems; Bernstein polynomials; Rayleigh–Ritz method.

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1. Introduction

Fractional calculus is a generalisation of classical calculus, and it has a long mathematical history. Recently, there has been a huge interest in the field of fractional calculus. There are a wide range of applications in sciences like biology, chemistry, fluid mechanics, signal processing and control theory [1–4]. Various types of approximate methods have been developed for solving fractional differential equations over the years [5–12]. It is worth pointing out that the fractional calculus of variations and the fractional optimal control problems have essential roles in science and engineering [13–16]. Hence, this subject attracts the attention of a number of scientists and engineers. Riewe [17,18] introduced fractional calculus of variations. The problem of variational calculus involving the objective function and the constraint depends on the derivatives and integrals of fractional order. These fractional operators can be defined with Caputo, Riemann–Liouville and so on. Recently, many scientists have applied different techniques to find numerical solutions of different types of fractional variational problems (FVPs). For example, Dehghan and Tatari [19] used the ADM for solving FVPs. A new approach based on operational matrix for this type of problems was carried out by Eldien [20]. In 2017, Eldein *et al* [21] published a paper in which they derived numerical solutions of FVPs using shifted Legendre polynomials. A numerical scheme is utilised to achieve the solutions of FVPs by Lotfi and Yousefi [22]. Another numerical method was introduced by Khader and Hendy [23] to solve these problems. The modified Jacobi polynomials for the solution of a class of fractional variational and optimal control problems was applied by Dehghan *et al* [24].

There are many different methods for obtaining the solution of FVPs. The purpose of this study is to use a numerical technique of the following fractional variational problems:

$$\text{Min } J[\vartheta] = \int_0^1 \zeta(t, \vartheta(t), \mathfrak{D}^{\gamma}\vartheta(t))\mathrm{d}t, \quad m-1 < \gamma < m,\ m \in \mathbb{N}, \tag{1}$$

with boundary conditions

$$\vartheta(0) = c_1, \quad \vartheta(1) = c_1, \tag{2}$$

where $\vartheta(t)$ is an unknown function and $\mathfrak{D}^{\gamma}\vartheta(t)$ indicates the Caputo fractional derivative. In this method,

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we expand the unknown function with Bernstein polynomials in [0, 1]. By using Rayleigh–Ritz method, FVPs may be converted to a system of algebraic equations.

This paper is divided into five sections. In §2 we bring a summary of fractional calculus and basic formulation of Bernstein polynomials. The numerical method for solving the FVPs is given in §3. We bring some examples in §4. In §5 conclusions are given.

Preliminaries

In this section, we indicate some main definitions of the fractional calculus and Rayleight–Ritz method that will be utilised in the current study.

1.1 *Fractional calculus*

DEFINITION 1

Let us denote $\mathfrak{I}_t^\gamma$ as the Riemann–Liouville fractional integral of order $\gamma > 0$ that is defined as [3]

$$\mathfrak{I}_t^\gamma \vartheta(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{\vartheta(\tau)}{(t-\tau)^{1-\gamma}} \mathrm{d}\tau, \quad \gamma > 0. \tag{3}$$

For the aforesaid integral, we have the following relations:

$$\mathfrak{I}_t^\gamma t^\sigma = \frac{\Gamma(\sigma+1)}{\Gamma(\gamma+\sigma+1)} t^{\gamma+\sigma}, \quad \gamma > 0,$$

$$\mathfrak{I}_t^\gamma (\lambda\vartheta(t) + \mu\varrho(t)) = \lambda \mathfrak{I}_t^\gamma \vartheta(t) + \mu \mathfrak{I}_t^\gamma \varrho(t), \quad \gamma > 0.$$

DEFINITION 2

The Caputo fractional derivative of order γ can be defined as follows [3]:

$$\mathfrak{D}_t^\gamma \vartheta(t) = \begin{cases} \dfrac{1}{\Gamma(m-\gamma)} \displaystyle\int_0^t \frac{\vartheta^{(m)}(\tau)}{(t-\tau)^{\gamma-m+1}} \mathrm{d}\tau, & m-1 < \gamma < m, \\ \vartheta^{(m)}(t), \ \gamma = m, \ m \in \mathbb{N}. \end{cases} \tag{4}$$

For this derivative when $m - 1 < \gamma < m$, we have

$$\mathfrak{D}^\gamma \lambda = 0, \quad \lambda \text{ is a constant},$$

$$\mathfrak{D}_t^\gamma t^\sigma = \begin{cases} 0, & \sigma \in \mathbb{N}_0, \ \sigma < \lceil \gamma \rceil \\ \dfrac{\Gamma(\sigma+1)}{\Gamma(1+\sigma-\gamma)} t^{\sigma-\gamma}, & \sigma \in \mathbb{N}_0, \ \sigma \geq \lceil \gamma \rceil, \\ & \text{or } \sigma \notin \mathbb{N}_0, \ \sigma > \lfloor \gamma \rfloor, \end{cases}$$

$$\mathfrak{D}^\gamma \mathfrak{I}^\gamma \vartheta(t) = \vartheta(t),$$

$$\mathfrak{I}^\gamma \mathfrak{D}^\gamma \vartheta(t) = \vartheta(t) - \sum_{i=0}^{m-1} \vartheta^{(i)}(0) \frac{t^i}{i!},$$

$$\mathfrak{D}_t^\gamma (\lambda\vartheta(t) + \mu\varrho(t)) = \lambda \mathfrak{D}_t^\gamma \vartheta(t) + \mu \mathfrak{D}_t^\gamma \rho(t).$$

$\lceil \gamma \rceil$ indicates the smallest integer greater than or equal to γ and $\lfloor \gamma \rfloor$ denotes the largest integer less than or equal to γ and $\mathbb{N}_0 = 0, 1, 2, \ldots$.

1.2 *Rayleigh–Ritz method*

The Rayleigh–Ritz method [25] is based on a part of mathematics called calculus of variations. In this method, we try to minimise a special class of functions called a functional $J[\vartheta]$. To this end,

$$\vartheta(t) = \vartheta_n(t) = \sum_{l=0}^{n} \vartheta_l \mathfrak{b}_l^n(t), \tag{5}$$

where $\{\mathfrak{b}_l^n(t)\}_{l=0}^n$ is a denoted family of functions that vanish where fundamental boundary conditions are enforced. Further, function $\mathfrak{b}_0^n(t)$ is defined so that $\vartheta_n(t)$ satisfies identically the essential boundary conditions. Therefore, we have an n-dimensional subspace Y_n that is defined by (5). Extermination of $J[\vartheta]$ over Y_n gives

$$\delta J[\vartheta_n] = \delta J \left[\sum_{l=0}^{n} \vartheta_l \mathfrak{b}_l^n \right] = 0. \tag{6}$$

We can rewrite the extermination statement as a function of parameters ϑ_l, $l = 0, 1, \ldots, n$, by

$$\delta J(\vartheta_0, \vartheta_1, \ldots, \vartheta_n) = 0. \tag{7}$$

In the light of the integral scalar function $J(\vartheta_0, \vartheta_1, \ldots, \vartheta_n)$, we have an extremum over Y_n if and only if

$$\frac{\partial J}{\partial \vartheta_0} \delta\vartheta_0 + \frac{\partial J}{\partial \vartheta_1} \delta\vartheta_1 + \cdots + \frac{\partial J}{\partial \vartheta_n} \delta\vartheta_n = 0. \tag{8}$$

According to variations $\delta\vartheta_l$, $l = 0, 1, \ldots, n$ are arbitrary, and as a consequence

$$\frac{\partial J}{\partial \vartheta_l} = 0, \quad l = 0, 1, \ldots, n. \tag{9}$$

By solving the above equation, ϑ_l can easily be obtained. Also an approximate solution of the variational problem is derived by (5).

2. Bernstein polynomials

In this section, we discuss Bernstein polynomials and their properties.

DEFINITION 3

The Bernstein polynomial $\mathfrak{b}_l^n(t)$ is given by [6]

$$\mathfrak{b}_l^n(t) = \binom{n}{l} t^l (1-t)^{n-l}, \quad 0 \leq l \leq n, \tag{10}$$

where $\mathfrak{b}_l^n(t)$ of degree n is defined on [0, 1]. By using the binomial expansion of $(1-t)^{n-l}$, we get following relation:

$$\mathfrak{b}_l^n(t) = \sum_{\eta=l}^{n}(-1)^{\eta-l}\binom{n}{l}\binom{n-l}{\eta-l}t^{\eta},\quad l = 0, 1, \ldots, n. \tag{11}$$

Now, we define

$$\Psi(t) = \left[\mathfrak{b}_1^n(t), \mathfrak{b}_2^n(t), \ldots, \mathfrak{b}_n^n(t)\right]^T, \tag{12}$$

$$\Pi_n = [1, t, \ldots, t^n]^T, \tag{13}$$

and we can demonstrate $\Psi(t)$ in matrix form below:

$$\Psi(t) = \Omega\Pi_n(t), \tag{14}$$

where $\Omega = (\varpi_{l,j})_{l,j=1}^{n+1}$ is a matrix of order $(n+1)$ given by

$$\varpi_{l+1,j+1} = \begin{cases} (-1)^{j-l}\binom{n}{l}\binom{n-l}{j-l}, & l \leq j, \\ & l, j = 0, 1, \ldots, n, \\ 0, & l > j. \end{cases} \tag{15}$$

The set of Bernstein polynomials $\{\mathfrak{b}_0^n(t), \mathfrak{b}_1^n(t), \ldots, \mathfrak{b}_n^n(t)\}$ in Hilbert space $L^2[0, 1]$ is a complete basis [26]. Consequently, any function can be demonstrated in terms of Bernstein basis as follows:

$$\vartheta(t) = \sum_{l=0}^{n}\vartheta_l\mathfrak{b}_l^n(t) = \boldsymbol{\vartheta}^T\Psi(t), \tag{16}$$

where $\boldsymbol{\vartheta}^T = [\vartheta_0, \vartheta_1, \ldots, \vartheta_n]$. Then $\boldsymbol{\vartheta}^T$ can be derived as follows:

$$\boldsymbol{\vartheta}^T = \left(\int_0^1 \vartheta(t)\Psi(t)^T\,\mathrm{d}t\right)\Theta^{-1}, \tag{17}$$

where Θ is indicated as the dual matrix of $\Psi(t)$ which is specified as

$$\Theta = \int_0^1 \Psi(t)\Psi(t)^T\,\mathrm{d}x. \tag{18}$$

Matrix Θ is invertible and symmetric [27]. Using

$$\int_0^1 (1-t)^r t^i\,\mathrm{d}t = \frac{1}{(r+i+1)\binom{r+i}{i}};\quad i, r \in \mathbb{N}\cup\{0\}, \tag{19}$$

element of Θ can be obtained as follows:

$$\begin{aligned}\Theta_{(i+1),(j+1)} &= \int_0^1 \mathfrak{b}_i^n(t)\mathfrak{b}_j^n(t)\mathrm{d}t \\ &= \binom{n}{i}\binom{n}{j}\int_0^1 (1-t)^{2n-(i+j)}t^{i+j}\,\mathrm{d}t \\ &= \frac{\binom{n}{i}\binom{n}{j}}{(2n+1)\binom{2n}{i+j}},\quad i, j = 1, 2, ..., n.\end{aligned} \tag{20}$$

Now, we discuss the approximate formula of the fractional Caputo derivative.

Theorem 1. *Let $\vartheta(t)$ be approximated Bernstein polynomials as (16), then*

$$\mathfrak{D}^{\gamma}\vartheta(t) = \sum_{l=0}^{n}\sum_{\eta=\lceil\gamma\rceil}^{n}\vartheta_l\,\omega_{l,\eta}t^{\eta-\gamma},\quad m-1 < \gamma < m, \tag{21}$$

where

$$\omega_{l,\eta} = (-1)^{\eta-l}\binom{n}{l}\binom{n-l}{\eta-l}\frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\gamma)}. \tag{22}$$

Proof. Since the Caputo's fractional differentiation is a linear operation, we have

$$\mathfrak{D}^{\gamma}\vartheta(t) = \sum_{l=0}^{n}\vartheta_l\mathfrak{D}^{\gamma}\mathfrak{b}_l^n(t), \tag{23}$$

using (5) and (11)

$$\begin{aligned}\mathfrak{D}^{\gamma}\mathfrak{b}_l^n(t) &= \sum_{\eta=l}^{n}(-1)^{\eta-l}\binom{n}{l}\binom{n-l}{\eta-l}\mathfrak{D}^{\gamma}(t^{\eta}) \\ &= \sum_{\eta=\lceil\gamma\rceil}^{n}(-1)^{\eta-l}\binom{n}{l}\binom{n-l}{\eta-l} \\ &\quad\times\frac{\Gamma(\eta+1)}{\Gamma(\eta+1-\gamma)}t^{\eta-\gamma}.\end{aligned} \tag{24}$$

Lemma 1. *We assume $\vartheta(t) : [0, 1] \to \mathbb{R}$ is $n+1$ times continuously differentiable, and*

$$S_n = \mathrm{span}\{\mathfrak{b}_0^n(t), \mathfrak{b}_1^n(t), \ldots, \mathfrak{b}_n^n(t)\}.$$

If $\vartheta_n(t) = \boldsymbol{\vartheta}^T\Psi(t)$ is the best approximation $\vartheta(t)$ out of S_n, then

$$\|\vartheta(t) - \boldsymbol{\vartheta}^T\Psi(t)\|_{L^2[0,1]} \leq \frac{\hat{\varepsilon}}{(n+1)!\sqrt{2n+3}}, \tag{25}$$

where
$\hat{\varepsilon} = \max_{t\in[0,1]}|\vartheta^{(n+1)}(t)|$.

Proof. According to the Taylor expansion, we have [10]

$$\begin{aligned}\varphi(t) =& \vartheta(t_0) + \vartheta'(t_0)(t-t_0) + \vartheta''(t_0)\frac{(t-t_0)^2}{2!} + \cdots \\ &+ \vartheta^{(n)}(t_0)\frac{(t-t_0)^n}{n!}.\end{aligned} \tag{26}$$

Also, we know

$$|\vartheta(t) - \varphi(t)| \leqslant |\vartheta^{(n+1)}(\zeta)|\frac{(t-t_0)^{n+1}}{(n+1)!}. \tag{27}$$

Here $\zeta \in (0,1)$. As $\boldsymbol{\vartheta}^T\Psi(t)$ is the best approximation for $\vartheta(t)$, then using (27), we have

$$\begin{aligned}\|\vartheta(t)-\boldsymbol{\vartheta}^T\Psi(t)\|_2^2 &\le \|\vartheta-\varphi(t)\|_2^2\\ &=\int_0^1 |\vartheta(t)-\varphi(t)|^2\,\mathrm{d}t\\ &\le \int_0^1\left[\vartheta^{(n+1)}(\zeta)\frac{(t-t_0)^{n+1}}{(n+1)!}\right]^2\,\mathrm{d}t\\ &\le \frac{\hat{\varepsilon}^2}{(n+1)!^2}\int_0^1 (t-t_0)^{2n+2}\,\mathrm{d}t\\ &=\frac{\hat{\varepsilon}^2}{(n+1)!^2(2n+3)}. \end{aligned} \tag{28}$$

By taking the square roots we have the upper bound and this complete proof.

This Lemma indicates that if $\vartheta \in C^{n+1}$ then

$$\lim_{n\to\infty}\vartheta_n(t)=\vartheta(t).$$

3. Solving the fractional variational problem

Now, let us investigate the fractional variational problem (1). At first, we approximate $\vartheta(t)$ and $D^\gamma\vartheta(t)$ using eqs (16) and (21) respectively. Then FVPs (1) can be converted as follows:

$$J[\vartheta]=\int_0^1\zeta\left(t,\sum_{l=0}^{n}\vartheta_l\mathfrak{b}_l^n(t),\right.\\ \left.\times\sum_{l=0}^{n}\sum_{\eta=\lceil\gamma\rceil}^{n}\vartheta_l\,\omega_{l,\eta}t^{\eta-\gamma}\right)\mathrm{d}t, \tag{29}$$

where $\omega_{l,\eta}$ is defined in (22).

Now, by using the trapezoidal integration technique, we can derive the following relation:

$$J(\vartheta_0,\vartheta_1,\ldots,\vartheta_n)\cong\frac{h}{2}\left(\Upsilon(t_0)\right.\\ \left.+2\sum_{k=1}^{N-1}\Upsilon(t_k)+\Upsilon(t_N)\right), \tag{30}$$

where

$$\Upsilon(t)=\zeta\left(t,\sum_{l=0}^{n}\vartheta_l\mathfrak{b}_l^n(t),\sum_{l=0}^{n}\sum_{\eta=\lceil\gamma\rceil}^{n}\vartheta_l\,\omega_{l,\eta}t^{\eta-\gamma}\right),$$

$$h=\frac{1}{N},$$

for an arbitrary integer N, $t_k=kh,\ k=0,1,\ldots,N$.

Consequently, by using of Rayleigh–Ritz method, we get $n-1$ algebraic equations

$$\frac{\partial J}{\partial\vartheta_1}=0,\quad \frac{\partial J}{\partial\vartheta_2}=0,\ldots,\frac{\partial J}{\partial\vartheta_{n-1}}=0. \tag{31}$$

By replacing eq. (16) in the boundary conditions (4), we achieve a system $n+1$ of algebraic equations in the unknowns $\vartheta_0,\vartheta_1,\ldots,\vartheta_n$. By solving this system, the approximate values of $\vartheta(t)$ will be obtained from

$$\vartheta(t)=\sum_{l=0}^{n}\vartheta_l\mathfrak{b}_l^n(t). \tag{32}$$

4. Illustrative examples

In this section, we give some examples of the fractional variational problems involving a Caputo fractional derivative. We obtained the error as follows:

$$\mathrm{Error}\{\vartheta,\vartheta_N\}=\int_0^1(\vartheta(t)-\vartheta_N(t))^2\,\mathrm{d}t, \tag{33}$$

where ϑ and ϑ_N are exact and approximate solutions respectively. Based on the results obtained, we can conclude that our method is very simple and accurate.

Example 1. Consider the FVP [28,29]:

$$\mathrm{Min}\ J[\vartheta]=\frac{1}{2}\int_0^1(\mathfrak{D}_t^\gamma\vartheta(t)-g(t))^2\,\mathrm{d}t,\quad 0<\gamma<1, \tag{34}$$

where

$$g(t)=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)}t^{\beta-\gamma}, \tag{35}$$

subject to

$$\vartheta(0)=0,\quad \vartheta(1)=1. \tag{36}$$

The exact solution of eq. (34) is $\vartheta(t)=t^\beta$. Similar to the method described in the previous section, we can expand $\vartheta(t)$ as follows:

$$\vartheta(t)=\sum_{l=0}^{n}\vartheta_l\mathfrak{b}_l^n(t). \tag{37}$$

By replacing eq. (37) in eq. (34) and using eq. (21), the FVP (34) is converted to the following approximated formula:

$$J[\vartheta]=\frac{1}{2}\int_0^1\left(\sum_{l=0}^{n}\sum_{\eta=\lceil\gamma\rceil}^{n}\vartheta_l\,\omega_{l,\eta}t^{\eta-\gamma}\right.\\ \left.-\frac{\Gamma(\beta+1)t^{\beta-\gamma}}{\Gamma(\beta-\gamma+1)}\right)^2\mathrm{d}t, \tag{38}$$

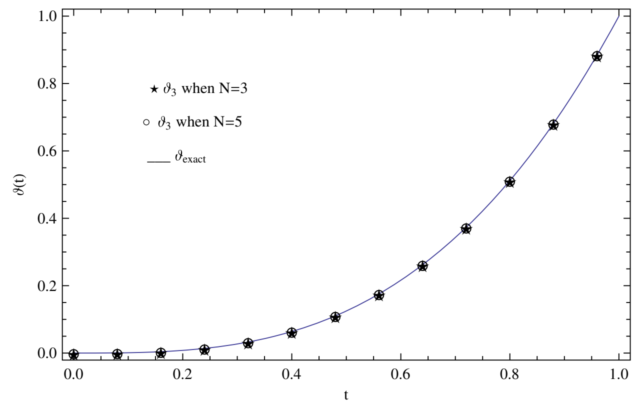

Figure 1. Exact and approximate solutions of $\vartheta(t)$ when $\gamma = 0.75$, $\beta = 3$ with $n = 3$ and $N = 3, 5$ for Example 1.

where $\omega_{l,\eta}$ is defined in eq. (22). Now, by using the trapezoidal integration technique, we get

$$J(\vartheta_0, \vartheta_1, \ldots, \vartheta_n) \cong \frac{h}{2}\left(\Upsilon(t_0) + 2\sum_{k=1}^{N-1}\Upsilon(t_k) + \Upsilon(t_N)\right), \quad (39)$$

where

$$\Upsilon(t) = \frac{1}{2}\left(\sum_{l=0}^{n}\sum_{\eta=\lceil\gamma\rceil}^{n} \vartheta_l\, \omega_{l,\eta} t^{\eta-\gamma} - \frac{\Gamma(\beta+1)}{\Gamma(\beta-\gamma+1)} t^{\beta-\gamma}\right)^2, \quad h = \frac{1}{N},$$

for an arbitrary integer N, $t_k = kh,\ k = 0, 1, \ldots, N$.

As before, by applying the Rayleigh–Ritz method, we have $n-1$ algebraic equations

$$\frac{\partial J}{\partial \vartheta_1} = 0, \quad \frac{\partial J}{\partial \vartheta_2} = 0, \ldots, \frac{\partial J}{\partial \vartheta_{n-1}} = 0. \quad (40)$$

Also, by expanding the boundary conditions (36) we have

$$\sum_{l=0}^{n} \vartheta_l \mathfrak{b}_l^n(0) = 0, \quad \sum_{l=0}^{n} \vartheta_l \mathfrak{b}_l^n(1) = 1. \quad (41)$$

From eqs (40) and (41) we have a system $n+1$ of algebraic equations. As was argued above, we can obtain approximate values of $\vartheta(t)$ from (32).

A comparison between them is given and all the results are summarised. In figure 1, the behaviour of the exact and approximate solutions for $\gamma = 0.75$, $\beta = 3$, $n = 3$ and different values of N are compared. Also, the behaviour of the exact and approximate solutions for $\gamma = 0.5$, $\beta = 0.8$, $n = 6$ and different values of N are shown in figure 2. Ordokhani and Rahimkhani [29] applied numerical solution based on the Müntz–Legendre polynomials combined with the Gauss–Legendre rule for $\beta = 1$ and some different values of γ to solve this problem. Figures 3 and 4 compare the absolutes errors $\vartheta(t)$ of the method in [29] and our method for $\gamma = 0.2, 0.99$ with $n = 3$ and $N = 7$. The absolute error for $\vartheta(t)$ for $\gamma = 0.6$ with our method is equal to zero. By comparing the obtained results, one can see that our method is more accurate. Table 1 provides the errors computed with (33) for different values of γ, N and $n = 3$. The absolute errors achieved with $N = 12$ and different values of γ and β are calculated in table 2. With these results, we conclude that the suggested method is accurate.

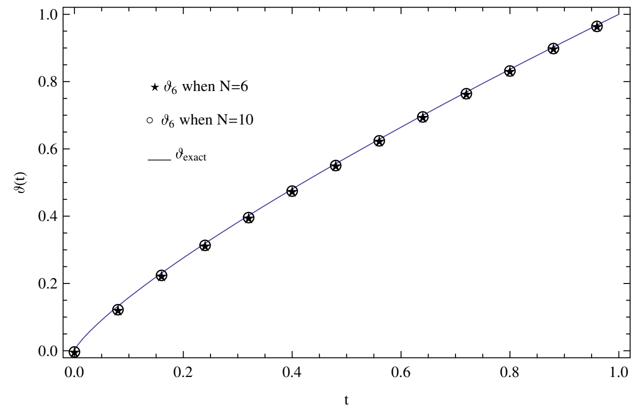

Figure 2. Exact and approximate solutions of $\vartheta(t)$ when $\gamma = 0.5$, $\beta = 0.8$ with $n = 6$ and $N = 6, 10$ for Example 1.

Example 2. Consider the FVP [30]:

$$\text{Min } J[\vartheta] = \int_0^1 (\mathfrak{D}_t^\gamma \vartheta(t) - \vartheta'^2(t))^2\, dt, \quad 0 < \gamma < 1, \quad (42)$$

with boundary conditions

$$\vartheta(0) = 0, \quad \vartheta(1) = 1. \quad (43)$$

The accurate solution of eq. (42) is

$$\vartheta(t) = -\frac{1}{2\Gamma(3-\gamma)}(1-t)^{2-\gamma} + \left(1 - \frac{1}{2\Gamma(3-\gamma)}\right)t + \frac{1}{2\Gamma(3-\gamma)}.$$

Similar to the previous example, the exact solution and results obtained from this method for $n = 3$, $\gamma = 0.75$ and different values of N are plotted in figure 5. Table 3 compares the error obtained with (33) using the

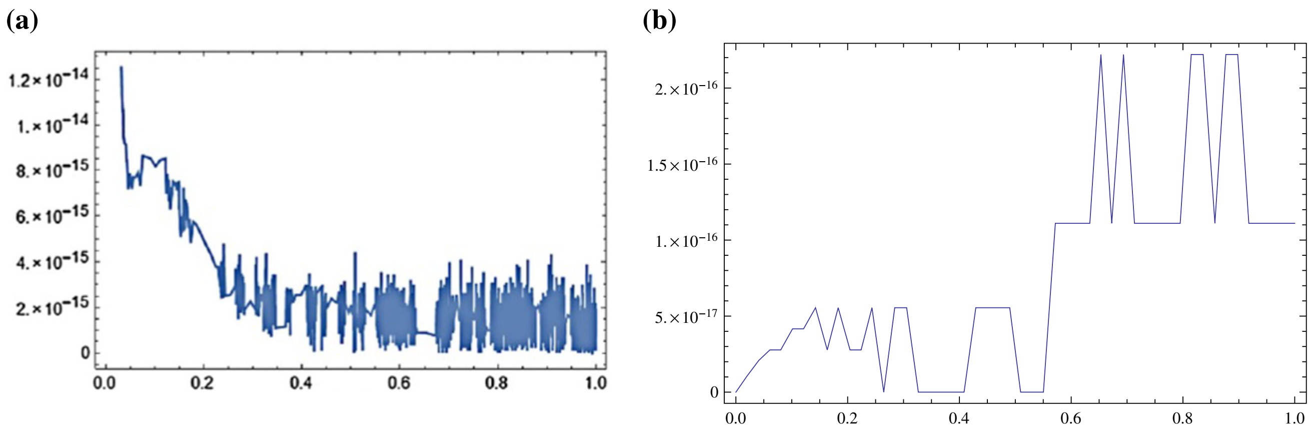

Figure 3. The absolute errors of $\vartheta(t)$ for $\beta = 1$, $\gamma = 0.2$. (**a**) Method in [29] and (**b**) our method for Example 1.

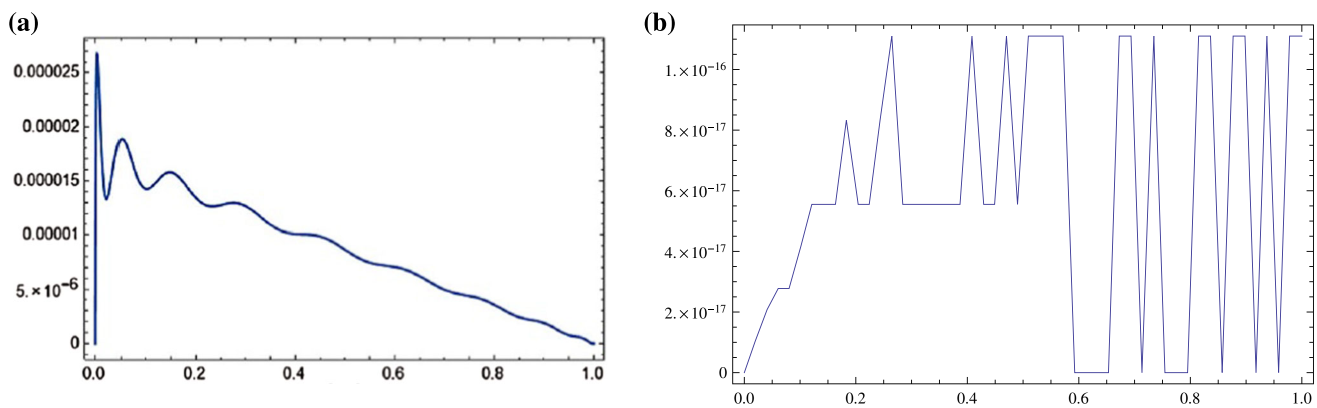

Figure 4. The absolute errors of $\vartheta(t)$ for $\beta = 1$ and $\gamma = 0.99$. (**a**) Method in [29] and (**b**) our method for Example 1.

Table 1. Errors obtained for $\vartheta(t)$ with $\beta = 1$ and different choices of N and γ for Example 1.

N	$\gamma = 0.39$	$\gamma = 0.6$	$\gamma = 0.95$
3	1.87824×10^{-33}	7.51296×10^{-33}	1.8782×10^{-33}
5	1.75648×10^{-33}	6.3912×10^{-34}	0.0000
7	9.3912×10^{-34}	0.0000	0.0000

Table 2. Absolute errors obtained for $\vartheta(t)$ with $n = 12$ and different choices of β and γ for Example 1.

	$\beta = 0.8$			$\beta = 0.9$		
t	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$	$\gamma = 0.25$	$\gamma = 0.5$	$\gamma = 0.75$
0.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.1	6.99942×10^{-4}	1.27799×10^{-3}	5.05307×10^{-4}	2.26724×10^{-4}	4.21273×10^{-4}	1.72064×10^{-4}
0.2	3.84831×10^{-4}	8.84276×10^{-4}	5.97134×10^{-4}	1.24769×10^{-4}	2.9030×10^{-4}	1.99458×10^{-4}
0.3	2.78071×10^{-4}	7.28392×10^{-4}	5.46798×10^{-4}	8.9911×10^{-5}	2.38854×10^{-4}	1.8228×10^{-4}
0.4	2.17636×10^{-4}	6.12118×10^{-4}	5.02155×10^{-4}	7.0442×10^{-5}	2.00701×10^{-4}	1.67258×10^{-4}
0.5	1.87011×10^{-4}	5.59292×10^{-4}	4.86258×10^{-4}	6.04598×10^{-5}	1.8330×10^{-4}	1.61849×10^{-4}
0.6	1.57172×10^{-4}	4.96172×10^{-4}	4.58179×10^{-4}	5.08552×10^{-5}	1.62616×10^{-4}	1.52459×10^{-4}
0.7	1.46504×10^{-4}	4.74180×10^{-4}	4.49812×10^{-4}	4.73533×10^{-5}	1.55368×10^{-4}	1.49626×10^{-4}
0.8	1.19007×10^{-4}	4.14386×10^{-4}	4.17545×10^{-4}	3.85104×10^{-5}	1.35789×10^{-4}	1.38875×10^{-4}
0.9	1.47240×10^{-4}	4.80411×10^{-4}	4.77729×10^{-4}	4.75779×10^{-5}	1.5738×10^{-4}	1.58851×10^{-4}
1	9.09495×10^{-13}	2.04636×10^{-12}	5.91172×10^{-12}	1.59162×10^{-12}	3.86535×10^{-12}	1.13687×10^{-13}

presented method and the proposed methods in [30] for different values of γ and n.

Example 3. Here, we consider another problem [23,25,28,29]:

$$\text{Min } J[\vartheta] = \frac{1}{2}\int_0^1 (\mathfrak{D}_t^{\gamma}\vartheta(t))^2\,\mathrm{d}t, \quad 0 < \gamma < 1, \tag{44}$$

with boundary conditions

$$\vartheta(0) = 0, \quad \vartheta(1) = 1. \tag{45}$$

The exact solution for this example is

$$\vartheta(t) = \frac{1}{2\gamma - 1}\int_0^t \frac{\mathrm{d}z}{[(1-z)(t-z)^{1-\gamma}}. \tag{46}$$

When $\gamma = 1$, the solution is t. The numerical results of eq. (44) when $n = 3$ and $N = 10$ for $\gamma = 0.65, 0.75, 0.85, 1$ and the exact solution are shown in figure 6. Also, figure 7 indicates the exact solution and the approximate solution for different values of $n = 3$ and $N = 5, 7$ when $\gamma = 1$. The absolute errors achieved when $n = 10$ and $\gamma = 1$ are shown in figure 8. From these figures, we can conclude that the numerical results obtained by using the proposed method are in excellent agreement with the exact solution and the numerical results obtained in [23,25]. Table 4 shows the error obtained with (33) using the presented method.

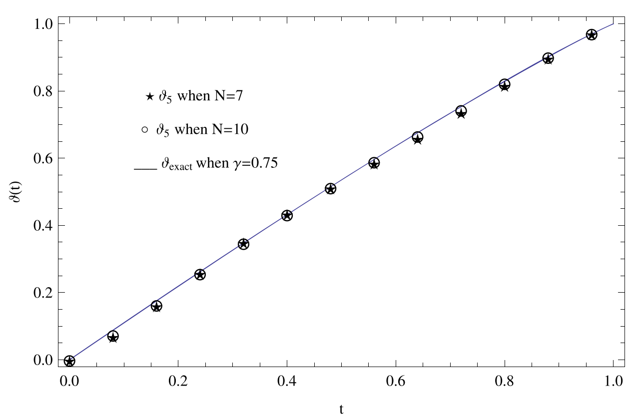

Figure 5. Exact and approximate solutions of $\vartheta(t)$ at $\gamma = 0.75$ when $n = 5$ and $N = 7$ and 10 for Example 2.

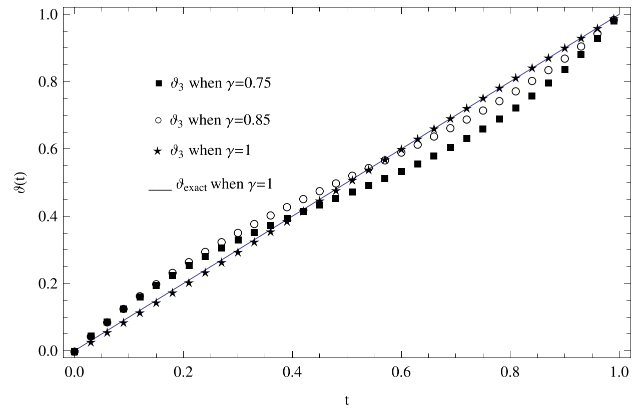

Figure 6. Exact and approximate solutions of $\vartheta(t)$ when $n = 3$, $N = 10$ and different values of γ for Example 3.

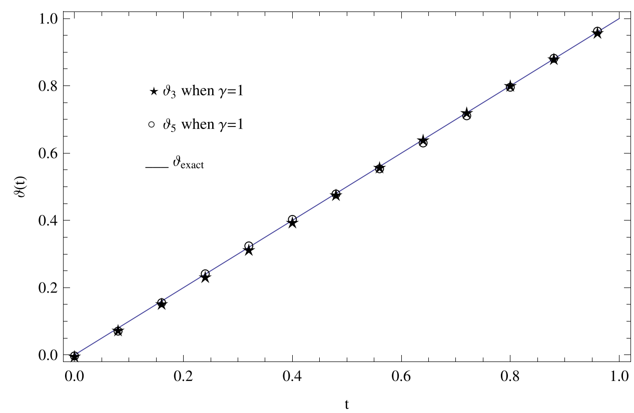

Figure 7. Exact and approximate solutions of $\vartheta(t)$ when $N = 10$, $\gamma = 1$ and $n = 3, 5$ for Example 3.

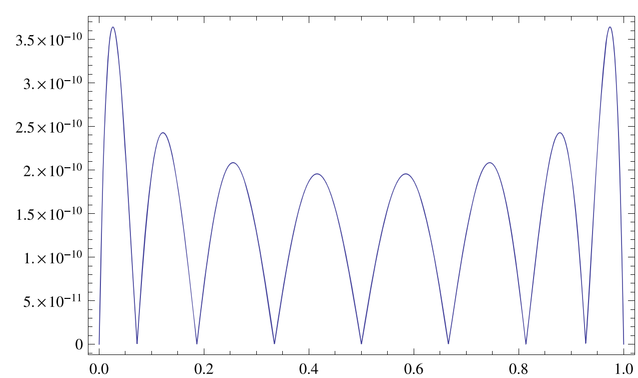

Figure 8. The absolute errors of $\vartheta(t)$ when $n = 10$ and $\gamma = 1$ for Example 3.

Table 3. The comparison of the errors obtained for $\vartheta(t)$ with different choices of n and γ for Example 2.

γ	Ref. [30]	$n = 3$	Ref. [30]	$n = 4$	Ref. [30]	$n = 6$
0.25	1.7×10^{-7}	1.77667×10^{-7}	1.9×10^{-8}	2.97279×10^{-8}	9.1×10^{-10}	5.3251×10^{-9}
0.5	9.7×10^{-6}	9.71836×10^{-7}	1.5×10^{-7}	1.93249×10^{-7}	1.1×10^{-8}	2.28348×10^{-8}
0.75	1.6×10^{-6}	1.89985×10^{-6}	3.4×10^{-7}	6.72651×10^{-7}	3.5×10^{-8}	1.36255×10^{-7}

Table 4. Errors obtained for $\vartheta(t)$ with different choices of γ for Example 3.

γ	n
0.85	3.63355×10^{-4}
0.99	8.9545×10^{-7}
1	1.44768×10^{-20}

5. Conclusion

The purpose of the current paper was to determine approximate solutions of the optimal control problem by a nonlinear Volterra integral equation. We applied Bernstein polynomials for getting approximate solutions. Using operational matrices of integration and product, the given problem was reduced to a set of algebraic equations. Illustrative examples were presented to show the applicability and validity of the approach. We used Mathematica for computations.

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