




The Fredholm determinant for Hulthén-distorted non-local separable potential: Application to α – α elastic scattering

U LAHA¹ ^{*}, A K BEHERA¹, M MAJUMDER¹ and J BHOI²

¹Department of Physics, National Institute of Technology, Jamshedpur 831 014, India

²Department of Physics, Veer Surendra Sai University of Technology, Burla 768 018, India

^{*}Corresponding author. E-mail: ujjwal.laha@gmail.com

MS received 23 January 2019; revised 24 April 2020; accepted 17 July 2020

Abstract. Exact analytical expression of the Fredholm determinant with outgoing wave boundary condition for motion in Hulthén-distorted non-local separable potential is constructed and expressed in the maximum reduced form. Using boundary conditions (regular and irregular), two approximate energy-dependent interactions corresponding to the parent non-local potential are also constructed. The phase shifts for the α – α elastic scattering are computed by using (i) exact expression for the Fredholm determinant and (ii) energy-dependent local interactions by exploiting the phase function method. The merits of our constructed equivalent energy-dependent potentials are judged by comparing the α – α elastic scattering phases with our exact calculation and standard data.

Keywords. Hulthén-distorted nuclear potential; Fredholm determinant; energy-dependent local potentials; phase function methods; alpha–alpha scattering phases.

PACS Nos 03.65.Nk; 21.30.Fe; 13.75.Cs; 24.10.-i

1. Introduction

The alpha–alpha (α – α) system has been treated extensively by many researchers [1–9] and many acceptable experimental data exist in the literature. Besides experiments, several theoretical models have also been proposed [10–27] which produce almost the same results though they have used different approaches. The α – α interactions, developed from the existing experimental phase shift data, show similar features. These interactions depend on ℓ but does not depend on the incident energy. The range of interaction is of the order of 5 fm in which the repulsive part is about 2 fm. For increasing values of ℓ , repulsive part gradually becomes weaker while the attractive part becomes stronger. Therefore, the theoretical analyses of α – α scattering prove that the α – α potential for all angular momenta does not exist. The ℓ dependence originates from the repulsive part of the interaction and may be considered as a standard form of velocity dependence. The studies of the α – α potential not only provide a better understanding of the α -clustered nuclei, but also give information about the basic forces between the constituents of such nuclei. Thus, one must understand the importance of quasi-hard

core in the potential through the replacement of non-local potential by its equivalent local one. Naturally, the question may arise as to what extent the energy-dependent equivalent local potential be equivalent to the non-local one. The separable non-local interactions have been used for various angular momentum states to fit α – α scattering phase shifts [28,29]. In the recent past, we proposed nuclear Hulthén potential model to study α – α system [27,30] and achieved reasonable agreement in s-wave phase shift values, particularly the phase shift changes sign at proper energy. The energy-dependent local potentials have been constructed by various groups [31,32]. Such type of analysis is generally used in optical model calculations. In these methods, one compares the characteristics of the non-local and local potentials. The phase shifts and the scattering amplitudes of both the interactions are compared in such calculations. Over the whole energy range under consideration, the energy–momentum-dependent local potentials have slight variation with the non-local potentials. The interest in the experimental situation with two potentials, sum of separable and electromagnetic one, increases. In such a situation, the short-range interaction denotes the nuclear part while the electromagnetic one originates

from the charges. The α -particles have very strong binding energy and hence used as a single particle. Thus, one can easily solve Schrödinger equation with a combination of two such potentials to have scattering phase shifts. Screened Coulomb potentials are mostly used in various branches of physics. The particles interacting through the Coulomb potential do not behave like free particles as the potential is of infinitely long range. In practice the Coulomb potential is no longer an infinitely long-range one but screened at finite distance. Thus, the standard results of scattering theory have to be modified to visualise the effect of screening. Therefore, to circumvent the difficulties which may arise in the traditional phase function method (PFM) [33] for Coulomb or Coulomb-like interaction, in reality the electromagnetic potential is replaced by a short-range screened Coulomb one.

The present study is based on the study of physical systems involving scattering by additive interactions like electromagnetic plus nuclear potentials. Initially, we treat electromagnetic plus non-local separable potential for constructing an exact analytical expression for the Fredholm determinant with outgoing wave boundary condition. In this approach, the electromagnetic effect is included rigorously. Secondly, we present two different methods for the localisation of non-local interactions: one based on simple rearrangement of the Schrödinger equation and the other by Taylor series expansion of the radial wave function to identify an equivalent local interaction. The associated irregular and regular boundary conditions are exploited in constructing energy-dependent equivalent local interactions respectively. In these cases, the electromagnetic effect is incorporated externally. The merits of our approaches to the equivalent potentials are examined by computing and comparing α - α elastic scattering phase shifts with our exact calculation and experimental data [8]. For the exact method, the expression for Fredholm determinant and for the equivalent local potentials the PFM [33,34] are utilised to compute elastic scattering phases. In §2 we construct exact analytical expression for the Fredholm determinant related to the outgoing wave boundary condition for the Hulthén plus rank N separable potential. We adapt two different approaches for the localisation of a non-local separable interaction in §3. In §4 we briefly outline the PFM and discuss our results. Finally, we conclude in §5.

2. The Fredholm determinant $D^{(+)}(k)$

The inhomogeneous Schrödinger-type equation for the Hulthén-modified rank N separable potential is written as

$$\left[\frac{d^2}{dr^2} + k^2 - V_H(r) \right] \psi^{(+)}(k, r) = \sum_{i=1}^N \lambda_i g_i(\beta_i, r) \int_0^\infty ds g_i(\beta_i, s) \psi^{(+)}(k, s) \quad (1)$$

with

$$V_H(r) = V_0 \exp(-r/a)/(1 - \exp(-r/a)), \quad a > 0. \quad (1a)$$

The quantities $g_i(\beta_i, r)$ s, λ_i and β_i stand for the form factors, the state-dependent strength and inverse range parameters of the separable interaction. The real coupling constants λ_i s of the separable interaction may be either positive or negative. The strength parameter V_0 for the Hulthén potential is real and positive. Here $\psi^{(+)}(k, r)$ represents the physical wave function (outgoing wave boundary condition) for scattering on Hulthén plus separable potential. Henceforth, we shall use the subscript H for quantities associated with pure Hulthén interaction. For the outgoing wave boundary condition, eq. (1) may be converted into the integral equation [35]

$$\psi^{(+)}(k, r) = \psi_H^{(+)}(k, r) + \sum_{i=1}^N \lambda_i d_i(k) \times \int_0^\infty ds g_i(\beta_i, s) G_H^{(+)}(r, s) \quad (2)$$

with

$$d_i(k) = \int_0^\infty ds g_i(\beta_i, s) \psi^{(+)}(k, s). \quad (3)$$

In eq. (2) the quantity $G_H^{(+)}(r, s)$ denotes the outgoing wave or the physical Green's function related to Hulthén potential and is expressed as [33]

$$G_H^{(+)}(r, s) = -k^{-1} \Psi_H^{(+)}(k, s_<) f_H(k, s_>), \quad (4)$$

where $s_>$ and $s_<$ denote the larger and smaller values of r and s . Here $f_H(k, r)$ and $\Psi_H^{(+)}(k, r)$ represent the irregular and physical solutions of the Hulthén potential [36,37] and are written as

$$\Psi_H^{(+)}(k, r) = \frac{\Gamma(A_1 + 1) \Gamma(B_1 + 1)}{\Gamma(C_1)} ak e^{ikr} \times (1 - e^{-r/a}) F(A_1 + 1, B_1 + 1; 2; 1 - e^{-r/a}) \quad (5)$$

and

$$f_H(k, r) = e^{ikr} F(A_1, B_1; C_1; e^{-r/a}) \quad (6)$$

with

$$\begin{aligned}
 A_1 &= -iak + ia(k^2 + V_0)^{1/2}, \\
 B_1 &= -iak - ia(k^2 + V_0)^{1/2} \\
 \text{and} \\
 C_1 &= 1 - 2iak.
 \end{aligned}
 \tag{7}$$

Equation (2) can easily be solved to get

$$d_i(k) = \frac{1}{\det_N A(k)} \sum_{j=1}^N a_{ij}(k) Y_j(k),
 \tag{8}$$

where

$$\begin{aligned}
 A_{ij}(k) &= \delta_{ij} \\
 -\lambda_j \int_0^\infty \int_0^\infty dr ds g_i(\beta_i, r) G_H^{(+)}(r, s) g_j(\beta_j, s)
 \end{aligned}
 \tag{9}$$

and

$$Y_j(k) = \int_0^\infty dr \psi_H^{(+)}(k, r) g_j(\beta_j, r).
 \tag{10}$$

Here, the matrix elements $a_{ij}(k)$ s are the cofactors of $A_{ij}(k)$ s. Therefore, eq. (2) in association with eqs (8)–(10) give the expected expression for the physical solution for Hulthén-distorted rank N separable potential as

$$\begin{aligned}
 \psi^{(+)}(k, r) &= \psi_H^{(+)}(k, r) \\
 &+ \frac{1}{\det_N A(k)} \sum_{i,j=1}^N \lambda_i a_{ij}(k) Y_j(k) G_H^{(+)}(r, \beta_i)
 \end{aligned}
 \tag{11}$$

with

$$G_H^{(+)}(r, \beta_i) = \int_0^\infty ds g_i(\beta_i, s) G_H^{(+)}(r, s).
 \tag{12}$$

The quantity $\det_N A(k) = D^{(+)}(k)$ stands for the Fredholm determinant for Hulthén-modified separable potential with the physical boundary condition. For a local potential, the Fredholm determinant $D^{(+)}(k)$ is equal to the Jost function $f(k)$ (the behaviour of the irregular solution $f(k, r)$ near the origin) while for a non-local or a combination of local and non-local potentials $D^{(+)}(k)$ and $f(k)$ are not the same. They are related [38–40] by $f(k) = D^{(+)}(k)/D(k)$. Here $D(k)$, the Fredholm determinant related to the regular or irregular solution, is a real quantity. As the Jost function $f(k)$ is a complex quantity, $D^{(+)}(k)$ is also complex. Thus, the phases of the Jost function and $D^{(+)}(k)$ are equal. It is well known that the phase of the Jost function is equal to the negative of the scattering phase shift $\delta(k)$. Therefore, the quantity $\det_N A(k)$ in conjunction with eq. (9) represents a convenient starting point for calculating the elastic scattering phase shifts.

In the following we shall construct exact analytical expressions for the single and double transforms of the

physical Hulthén Green’s function $G_H^{(+)}(r, s)$ by the form factors of the separable potential under consideration. Here we shall consider a two-term separable interaction for the nuclear part of the α – α interaction and study the effect of electromagnetic distortion on the nuclear scattering. The rank-two potential possesses an attractive and a repulsive component. The strong repulsion between the nucleons which is a consequence of the Pauli exclusion principle is thus reproduced in a phenomenological way. The two-term separable potential is written as

$$V(r, s) = \lambda_1 g_1(\alpha, r) g_1(\alpha, s) + \lambda_2 g_2(\beta, r) g_2(\beta, s)
 \tag{13}$$

with

$$g_1(\alpha, r) = e^{-\alpha r}; \quad g_2(\beta, r) = e^{-\beta r}.
 \tag{14}$$

From eqs (9), (12) and (14) it is clear that one needs to calculate the single and double Laplace transforms of the physical Hulthén Green’s function $G_H^{(+)}(r, s)$. The Fredholm determinant for the Hulthén-modified rank-two separable potential is given by

$$\begin{aligned}
 D^{(+)}(k) &= \left[1 - \lambda_1 G_H^{(+)}(\alpha, \alpha) \right] \left[1 - \lambda_2 G_H^{(+)}(\beta, \beta) \right] \\
 &- \lambda_1 \lambda_2 G_H^{(+)}(\alpha, \beta) G_H^{(+)}(\beta, \alpha),
 \end{aligned}
 \tag{15}$$

where

$$\begin{aligned}
 G_H^{(+)}(\alpha, \beta) &= G_H^{(+)}(\beta, \alpha) \\
 &= \int_0^\infty \int_0^\infty dr dr' g_1(\alpha, r) G_H^{(+)}(r, r') g_2(\beta, r').
 \end{aligned}
 \tag{16}$$

Combining eqs (4)–(12) the single Laplace transform of the physical Hulthén Green’s function is expressed as

$$\begin{aligned}
 G_H^{(+)}(r, \beta) &= -k^{-1} \left[f_H(k, r) \int_0^r ds e^{-\beta s} \Psi_H^{(+)}(k, s) \right. \\
 &\quad \left. + \Psi_H^{(+)}(k, r) \int_r^\infty ds e^{-\beta s} f_H(k, s) \right] \\
 &= T_1(r, \beta) + T_2(r, \beta)
 \end{aligned}
 \tag{17}$$

with

$$\begin{aligned}
 T_1(r, \beta) &= -k^{-1} \left[f_H(k, r) \int_0^r ds e^{-\beta s} \Psi_H^{(+)}(k, s) \right. \\
 &\quad \left. - \Psi_H^{(+)}(k, r) \int_0^r ds e^{-\beta s} f_H(k, s) \right]
 \end{aligned}
 \tag{18}$$

and

$$T_2(r, \beta) = -k^{-1} \Psi_H^{(+)}(k, r) \int_0^\infty ds e^{-\beta s} f_H(k, s).
 \tag{19}$$

Substitution of the solutions $\Psi_H^{(+)}(k, r)$, $f_H(k, r)$ and transformation of the associated Gaussian hypergeometric functions in eq. (18) by the following relation [41–43]

$$\begin{aligned}
 & {}_2F_1(a, b; c; z) \\
 &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} {}_2F_1(a, b; a \\
 &+ b - c + 1; 1 - z) + (1 - z)^{c-a-b} \\
 &\times \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \\
 &\times {}_2F_1(c-a, c-b; c-a-b+1; 1-z) \quad (20)
 \end{aligned}$$

leads to

$$\begin{aligned}
 T_1(r, \beta) &= \lim_{\varepsilon \rightarrow 0} a e^{ikr} [(1 - e^{-r/a}) {}_2F_1(A_1 + 1, B_1 \\
 &+ 1; 2; 1 - e^{-r/a}) \int_0^r ds e^{-(\beta-ik)s} \\
 &\times {}_2F_1(A_1, B_1; \varepsilon; 1 - e^{-s/a}) \\
 &- {}_2F_1(A_1, B_1; \varepsilon; 1 - e^{-s/a}) \\
 &\times \int_0^r ds e^{-(\beta-ik)s} (1 - e^{-s/a}) \\
 &\times {}_2F_1(A_1 + 1, B_1 + 1; 2; 1 - e^{-s/a})] \quad (21)
 \end{aligned}$$

and

$$\begin{aligned}
 T_2(r, \beta) &= -a e^{ikr} (1 - e^{-r/a}) \frac{\Gamma(A_1 + 1)\Gamma(B_1 + 1)}{\Gamma(C_1)} \\
 &\times {}_2F_1(A_1 + 1, B_1 + 1; 2; 1 - e^{-r/a}) \\
 &\times \int_0^\infty ds e^{-(\beta-ik)s} {}_2F_1(A_1, B_1; C_1; e^{-s/a}) \quad (22)
 \end{aligned}$$

Transformations of the independent variables $y = (1 - e^{-s/a})$ and $z = e^{-s/a}$ in eqs (21) and (22) respectively along with the series expansion of z' in eq. (21) as

$$(1 - z)^{(\beta+ik)a-1} = \sum_{n=0}^\infty \frac{\Gamma(n+1 - (\beta+ik)a)}{\Gamma(1 - (\beta+ik)a)} \frac{z^n}{n!} \quad (23)$$

we obtain

$$\begin{aligned}
 T_1(r, \beta) &= a^2 e^{ikr} (1 - e^{-r/a}) \\
 &\times \sum_{n=0}^\infty \frac{\Gamma(n+1 - (\beta+ik)a)}{\Gamma(1 - (\beta+ik)a)} \frac{z^n}{n!} \\
 &\times f_{n+1}(A_1 + 1, B_1 + 1; 2; 1 - e^{r/a}) \quad (24)
 \end{aligned}$$

and

$$\begin{aligned}
 T_2(r, \beta) &= -a e^{ikr} (1 - e^{-r/a}) \frac{\Gamma(A_1 + 1)\Gamma(B_1 + 1)}{(\beta - ik)\Gamma(C_1)} \\
 &\times {}_2F_1(A_1 + 1, B_1 + 1; 2; 1 - e^{-r/a}) \\
 &\times {}_3F_2(A_1, B_1, (\beta - ik)a; \\
 &C_1; (\beta - ik)a + 1; 1). \quad (25)
 \end{aligned}$$

Here the function $f_n(\mu, \chi; \gamma; z)$, the solution of the inhomogeneous Gaussian hypergeometric equation [44] reads as

$$\begin{aligned}
 f_n(\mu, \chi; \gamma; z) &= \frac{z^n}{n(n+c-1)} \\
 &\times {}_3F_2(1, n + \mu, n + \chi; n + 1, n + \gamma; z). \quad (26)
 \end{aligned}$$

The series is fully convergent when $|z| < 1$; it also converges when $|z| = 1$ provided that $\text{Re}(c - a - b) > 0$. In deriving expressions (24) and (25) we applied the following standard integrals [44]:

$$\begin{aligned}
 f_\sigma(\mu, \chi; \gamma; z) &= \frac{1}{(\gamma - 1)} \left[{}_2F_1(\mu, \chi; \gamma; z) \right. \\
 &\times \int_0^z dz' z'^{\sigma-1} (1 - z')^{\mu+\chi-\gamma} \\
 &\times {}_2F_1(\mu - \gamma + 1, \chi - \gamma + 1; 2 - \gamma; z') \\
 &- z^{1-\gamma} {}_2F_1(\mu - \gamma + 1, \chi - \gamma + 1; 2 - \gamma; z) \\
 &\left. \times \int_0^z dz' z'^{\sigma+\gamma-2} (1 - z')^{\mu+\chi-\gamma} {}_2F_1(\mu, \chi; \gamma; z') \right] \quad (27)
 \end{aligned}$$

and

$$\begin{aligned}
 & \int_0^1 dx x^{\rho-1} (1 - x)^{\sigma-1} {}_2F_1(a_1, b_1; c_1; x) \\
 &= \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho + \sigma)} {}_3F_2(a_1, b_1, \rho; c_1; \rho + \sigma; 1); \\
 & [\text{Re } \rho > 0, \text{ Re } \sigma > 0, \text{ Re}(c_1 + \sigma - a_1 - b_1) > 0]. \quad (28)
 \end{aligned}$$

Equation (17), in conjunction with eqs (24) and (25), produces the desired expression for the single Laplace transform of the physical Hulthén Green’s function. The double Laplace transform can be calculated by integrating the expression for $G_H^{(+)}(r, \beta)$. Thus, the double Laplace transform is given by

$$G_H^{(+)}(\alpha, \beta) = \int_0^\infty dr e^{-\alpha r} G_H^{(+)}(r, \beta). \quad (29)$$

Substituting the values of $G_H^{(+)}(r, \beta)$ in terms of eqs (24) and (25) and changing the independent variable $y = (1 - e^{-s/a})$ one gets

$$\begin{aligned}
 G_H^{(+)}(\alpha, \beta) &= a^3 \\
 &\times \sum_{n=0}^{\infty} \frac{\Gamma(n+1-(\beta+ik)a)\Gamma(n+2)\Gamma((\alpha-ik)a)}{\Gamma(1-(\beta+ik)a)\Gamma(n+1)\Gamma(n+2-(\alpha-ik)a)} \\
 &\times f_{n+1}(A_1+1, B_1+1; 2+(\alpha-ik)a; 1) \\
 &- a^2 \frac{\Gamma(A_1+1)\Gamma(B_1+1)\Gamma((\alpha-ik)a)}{(\beta-ik)\Gamma(C_1)\Gamma(2+(\alpha-ik)a)} \\
 &\times {}_3F_2(A_1, B_1, (\beta-ik)a; 1+(\beta-ik)a, C_1; 1).
 \end{aligned} \tag{30}$$

The above equation is derived by using eq. (28) along with the following integral [44]:

$$\begin{aligned}
 &\int_0^1 dz z^{c_1-1} (1-z)^{v-1} f_{\sigma}(a_1, b_1; c_1; pz) \\
 &= \frac{\Gamma(\sigma+c_1-1)\Gamma(v)}{\Gamma(\sigma+c_1+v-1)} f_{\sigma}(a, b_1; c_1+v; p) \\
 &\times [\operatorname{Re} v > 0, \operatorname{Re} \sigma > 0, \operatorname{Re}(\sigma+c_1) > 1, |p| < 1].
 \end{aligned} \tag{31}$$

Using relation (28) and [44]

$$\begin{aligned}
 &{}_3F_2(\alpha, \beta, \gamma; \delta, \chi; 1) \\
 &= \frac{\Gamma(\delta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)} \\
 &\times {}_3F_2(\alpha, \beta, \chi-\gamma; \alpha+\beta-\gamma+1, \chi; 1) \\
 &+ \frac{\Gamma(\delta)\Gamma(\chi)\Gamma(\alpha+\beta-\delta)\Gamma(\delta+\chi-\alpha-\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\chi-\gamma)\Gamma(\gamma+\chi-\alpha-\beta)} \\
 &\times {}_3F_2(\delta-\alpha, \delta-\beta, \delta+\chi-\alpha-\beta-\gamma; \\
 &\delta-\alpha-\beta+1, \delta+\chi-\alpha-\beta; 1).
 \end{aligned} \tag{32}$$

In eq. (30) one has

$$\begin{aligned}
 G_H^{(+)}(\alpha, \beta) &= -a^3 \frac{\Gamma((\alpha-ik)a)}{(1+A_1)\Gamma(1-(\beta+ik)a)} \sum_{n=0}^{\infty} \frac{(n+1)\Gamma(n+1-(\beta+ik)a)}{\Gamma(n+3+(\alpha-ik)a)} \\
 &\times {}_3F_2(1, n+A_1+2, 1-B_1+(\alpha-ik)a; A_1+2, n+3+(\alpha-ik)a; 1) \\
 &+ a^3 \frac{\Gamma((\alpha+ik)a)\Gamma((\alpha-ik)a)}{(A_1+1)(B_1+1)\Gamma(1+(\alpha-ik)a-B_1)\Gamma(1+(\alpha-ik)a-A_1)} \\
 &\times {}_3F_2(1, 2, 1-(\beta+ik)a; A_1+2, B_1+2; 1) \\
 &- a^2 \frac{\Gamma(A_1+1)\Gamma(B_1+1)\Gamma((\alpha+ik)a)\Gamma((\alpha-ik)a)}{(\beta-ik)\Gamma(C_1)\Gamma(1+(\alpha-ik)a-B_1)\Gamma(1+(\alpha-ik)a-A_1)} \\
 &\times {}_3F_2(A_1, B_1, (\beta-ik)a; 1+(\beta-ik)a, C_1; 1)
 \end{aligned} \tag{33}$$

Further judicious application of the following transformations [45] of the ${}_3F_2(*)$ functions

$$\begin{aligned}
 &{}_3F_2(\alpha, \beta, \gamma; \delta, \chi; 1) \\
 &= \frac{\Gamma(s)\Gamma(\delta)\Gamma(\chi)}{\Gamma(\alpha)\Gamma(s+\gamma)\Gamma(s+\beta)} \\
 &\times {}_3F_2(s, \delta-\alpha, \chi-\alpha; s+\beta, s+\gamma; 1);
 \end{aligned}$$

$$s = \chi + \delta - \alpha - \beta - \gamma \tag{34}$$

and

$$\begin{aligned}
 &{}_3F_2(\alpha, \beta, \gamma; \delta, \chi; 1) \\
 &= \frac{\Gamma(\chi)\Gamma(\delta+\chi-\alpha-\beta-\gamma)}{\Gamma(\chi-\gamma)\Gamma(\delta+\chi-\alpha-\beta)} \\
 &\times {}_3F_2(\delta-\alpha, \delta-\beta, \gamma; \delta, \delta+\chi-\alpha-\beta; 1)
 \end{aligned} \tag{35}$$

in eq. (33) leads to

$$\begin{aligned}
 G_H^{(+)}(\alpha, \beta) &= -a^3 \frac{\Gamma((\alpha-ik)a)}{(A_1+1)(B_1+1)\Gamma(1-(\beta+ik)a)} \\
 &\times \sum_{n=0}^{\infty} \frac{(n+1)\Gamma(n+1-(\beta+ik)a)}{\Gamma(n+2+(\alpha-ik)a)} \\
 &\times {}_3F_2(-n, 1, 1-(\alpha+ik)a; A_1+2, B_1+2; 1).
 \end{aligned} \tag{36}$$

The ${}_3F_2(*)$ function involved in the above expression is a polynomial. Thus, to remove the aesthetically displeasing sum in eq. (36), some algebraic manipulations and rearrangements of the terms are made to have

$$\begin{aligned}
 G_H^{(+)}(\alpha, \beta) &= - \frac{a^2}{(A_1+1)(B_1+1)(\beta+\alpha)((\beta+\alpha)a-1)} \\
 &\times {}_4F_3(1, 2, 1-(\alpha+ik)a, 1 \\
 &- (\beta+ik)a; A_1+2, B_1+2, 2-(\beta+\alpha)a; 1).
 \end{aligned} \tag{37}$$

Relatively recently, we have derived another closed form expression for the double Laplace transform [46] of the Hulthén Green’s function in terms of a different ${}_4F_3(*)$ function but are unable to relate these two expressions. However, one can easily compute scattering phase shifts for charged hadron systems by exploiting eq. (15) together with eq. (37).

3. Energy-dependent equivalent local potentials

Equation (1) without the electromagnetic interaction i.e. $V_H(r) = 0$ may be rewritten as

$$\left[\frac{d^2}{dr^2} + k^2 \right] \psi(k, r) = \left[\frac{1}{\psi(k, r)} \sum_{i=1}^N \lambda_i g_i(\beta_i, r) d_i(\beta_i, k) \right] \psi(k, r) \tag{38}$$

with

$$d_i(\beta_i, k) = \int_0^\infty ds g_i(\beta_i, s) \psi(k, s). \tag{39}$$

Equation (38) resembles Schrödinger wave equation with a local potential and the term on the right-hand side is regarded as an approximate velocity-dependent equivalent local interaction for the rank- N separable potential identified as

$$V_{EQ}(k, r) = \frac{1}{\psi(k, r)} \sum_{i=1}^N \lambda_i g_i(\beta_i, r) d_i(\beta_i, k). \tag{40}$$

In the following, the irregular boundary condition will be applied to construct the energy-dependent equivalent local potential and the quantity $\psi(k, r)$ be replaced by its standard notation $f(k, r)$. The irregular solution for the potential defined in eq. (13) is written as

$$f(k, r) = e^{ikr} + \lambda_1 d_1(k) I_1(r, \alpha) + \lambda_2 d_2(k) I_2(r, \beta), \tag{41}$$

where

$$I_1(r, \alpha) = \int_r^\infty ds e^{-\alpha s} G_0^{(1)}(r, s) \tag{42}$$

and

$$I_2(r, \beta) = \int_r^\infty ds e^{-\beta s} G_0^{(1)}(r, s) \tag{43}$$

with the free-particle irregular Green's function [33]

$$G_0^{(1)}(r, s) = \begin{cases} -k^{-1} \sin k(r - s); & s < r, \\ 0; & s > r. \end{cases} \tag{44}$$

Substitution of eq. (44) in eqs (42) and (43), and evaluation of simple integrals yield

$$I_1(r, \alpha) = -\frac{1}{k(\alpha^2 + k^2)} [\alpha \sin kr - k \cos kr] \tag{45}$$

and

$$I_2(r, \beta) = -\frac{1}{k(\beta^2 + k^2)} [\beta \sin kr - k \cos kr]. \tag{46}$$

In analogy with eq. (8) with irregular boundary condition we have

$$d_1(k) = \frac{1}{(\alpha - ik)D(k)} \tag{47}$$

and

$$d_2(k) = \frac{1}{(\beta - ik)D(k)} \tag{48}$$

with the Fredholm determinant associated with the irregular solution

$$D(k) = \left[1 - \lambda_1 G_0^{(1)}(\alpha, \alpha) \right] \left[1 - \lambda_2 G_0^{(1)}(\beta, \beta) \right] - \lambda_1 \lambda_2 G_0^{(1)}(\alpha, \beta) G_0^{(1)}(\beta, \alpha). \tag{49}$$

The quantity $G_0^{(1)}(\alpha, \beta)$ stands for the double Laplace transform of the free-particle irregular Green's function given by

$$G_0^{(1)}(\alpha, \beta) = [(\alpha + \beta)(\beta^2 + k^2)]^{-1}. \tag{50}$$

Thus, eq. (40) in conjunction with eqs (41)–(50) reproduces the energy-dependent equivalent local potential, hereby denoted as $V_{EQ}^{(1)}(k, r)$, for the two-term non-local separable interaction under consideration. In general, the irregular solution is a complex quantity and obviously, the potential generated from it is also complex in nature. However, one can also construct an equivalent potential with regular boundary condition by following the same method and the potential will be real. Thus, one may conclude that the result of inversion from non-local to local potentials is independent of the boundary conditions.

In the following we shall use Taylor series expansion of the regular solution $u(k, r)$ for the non-local separable potential and write the radial equation as

$$\left[\frac{d^2}{dr^2} + k^2 \right] u(k, r) = \lambda_1 e^{-\alpha r} \sum_{n=0}^\infty \int_0^\infty ds \frac{(r-s)^n}{\Gamma(n+1)} e^{-\alpha s} \frac{d^n}{dr^n} u(k, s) + \lambda_2 e^{-\beta r} \sum_{n=0}^\infty \int_0^\infty ds \frac{(r-s)^n}{\Gamma(n+1)} e^{-\beta s} \frac{d^n}{dr^n} u(k, s). \tag{51}$$

The right-hand side of eq. (3) involves an infinite sum but we restrict ourselves up to $n = 2$ which generates energy-momentum-dependent local potential corresponding to the non-local one [34]. For $n = 2$ we arrive at

$$\left[\frac{d^2}{dr^2} + k^2 \right] u(k, r) = V_1^{(T)}(k, r)u(k, r) + V_2^{(T)}(r) \frac{d}{dr} u(k, r), \quad (52)$$

where the quantities $V_1^{(T)}(k, r)$ and $V_2^{(T)}(r)$ are given by

$$V_1^{(T)}(k, r) = \frac{K_0(r) - k^2 K_2(r)}{1 - K_2(r)} \quad (53)$$

and

$$V_2^{(T)}(r) = \frac{K_1(r)}{1 - K_2(r)} \quad (54)$$

with

$$K_0(r) = \lambda_1 e^{-\alpha r} \int_0^\infty ds e^{-\alpha s} - \lambda_2 e^{-\beta r} \int_0^\infty ds e^{-\beta s}, \quad (55)$$

$$K_1(r) = \lambda_1 e^{-\alpha r} \int_0^\infty ds (r - s) e^{-\alpha s} + \lambda_2 e^{-\beta r} \int_0^\infty ds (r - s) e^{-\beta s} \quad (56)$$

and

$$K_2(r) = \frac{\lambda_1 e^{-\alpha r}}{2} \int_0^\infty ds (r - s)^2 e^{-\alpha s} - \frac{\lambda_2 e^{-\beta r}}{2} \int_0^\infty ds (r - s)^2 e^{-\beta s}. \quad (57)$$

Evaluation of the above integrals leads to

$$K_0(r) = \frac{\lambda_1}{\alpha} e^{-\alpha r} + \frac{\lambda_2}{\beta} e^{-\beta r}, \quad (58)$$

$$K_1(r) = \frac{\lambda_1}{\alpha^2} (\alpha r - 1) e^{-\alpha r} + \frac{\lambda_2}{\beta^2} (\beta r - 1) e^{-\beta r} \quad (59)$$

and

$$K_2(r) = \frac{\lambda_1}{2\alpha^3} (1 + (\alpha r - 1)^2) e^{-\alpha r} + \frac{\lambda_2}{2\beta^3} (1 + (\beta r - 1)^2) e^{-\beta r}. \quad (60)$$

For charged hadron scattering one has to add the electromagnetic potential to the nuclear part of the interaction. Thus, the resultant interaction for the α - α system is obtained by adding the atomic Hulthén potential $V_H(r)$ to the respective energy-dependent local potentials. The s-wave α - α elastic scattering phase shifts in such situations will be computed by the judicious applications of the PFMs [33,34].

4. Results and discussion

The phase function method, in general, represents a powerful tool to compute the scattering phase shifts for quantum mechanical problems involving local [33,34] and non-local interactions [29,47,48]. In this approach the radial wave function of the Schrödinger equation is separated into an amplitude part $\alpha(k, r)$ and an oscillating part with variable phase $\delta(k, r)$. The function $\delta(k, r)$ is called the phase function. For the conventional s-wave Schrödinger equation with a local potential $V(r)$, the phase function $\delta(k, r)$ obeys the differential equation [33]

$$\delta'(k, r) = -k^{-1} V(r) [\sin(kr + \delta(k, r))]^2. \quad (61)$$

The scattering phase shift $\delta(k)$ is obtained by solving the equation from $r = 0$ to $r = \infty$ with the initial condition $\delta(k, 0) = 0$. Finally, one gets the phase shift $\delta(k) = \lim_{r \rightarrow \infty} \delta(k, r)$. On the other hand, for the Schrödinger equation expressed in eq. (52) the phase equation is given by [34]

$$\delta'(k, r) = -k^{-1} \left[V_1^{(T)}(k, r) \sin(kr + \delta(k, r)) + k V_2^{(T)}(r) \cos(kr + \delta(k, r)) \right] \sin(kr + \delta(k, r)). \quad (62)$$

The elastic scattering phases induced by the Hulthén-distorted α - α potentials can easily be computed by exploiting eqs (15) and (37), eqs (40) and (61) and eqs (53), (54) and (62) respectively. Rahaman *et al* [28] computed the s-wave scattering phase shifts using $\lambda_1 = -2.4 \text{ fm}^{-3}$, $\lambda_2 = 20 \text{ fm}^{-3}$, $\alpha = 0.4 \text{ fm}^{-1}$ and $\beta = 1.3 \text{ fm}^{-1}$ and obtained good agreement in the low energy range. Beyond 5 MeV, their results differ slightly from the experimental data [8] and the phase shift changes sign at about 22 MeV which is 2 MeV above the experimental result. Using the parameters of Rahaman *et al* [28] we computed the α - α scattering phase shifts [29] for Coulomb plus separable potential through the modified PFM [29,48] and observed that the phase shift values change sign at about 17 MeV, 3 MeV below the experimental data [8]. Keeping this in view, we have modified the parameters $\lambda_1 = -2.72 \text{ fm}^{-3}$, $\lambda_2 = 20 \text{ fm}^{-3}$, $\alpha = 0.5 \text{ fm}^{-1}$ and $\beta = 1.3 \text{ fm}^{-1}$ such that the phase shift for the α - α system changes sign at the correct energy for Hulthén-modified separable potential. We portrayed our results along with the experimental data [8] and those of Laha *et al* [29] in figure 1. Here we have chosen to work with $V_0 a = 0.2758 \text{ fm}^{-1}$ and $a = 250 \text{ fm}$. We designate our phase shifts as $\delta_{EX}(k)$, $\delta_{EQ}^{(I)}(k)$ and $\delta_{EQ}^{(T)}(k)$ for the exact calculation of $D^{(+)}(k)$, for energy-dependent potentials with irregular solution and by Taylor expansion methods respectively.

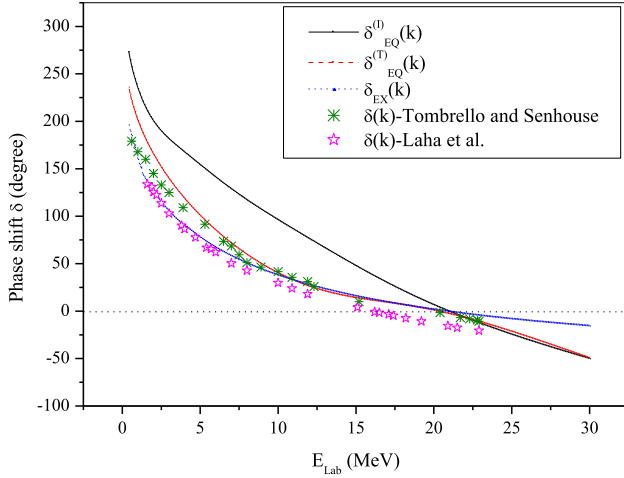


Figure 1. Alpha–alpha phase shifts. Experimental data are from refs [8,29]. The zero line is represented by the black dotted line.

As expected, the s-wave phase shifts for the α – α system are positive at low energies and become negative at high energies. The s-wave phase shifts based on the exact calculation of $D^{(+)}(k)$ for Hulthén-modified separable non-local potential and those computed with equivalent potential by Taylor series expansion change their signs at $E_{\text{lab}} = 20.2$ MeV and 20.5 MeV respectively, in agreement with experimental result [8], while that for the energy-dependent equivalent interaction in terms of irregular solution at $E_{\text{lab}} = 21.6$ MeV. It is observed that our phase shift values for $\delta_{\text{EX}}(k)$ and $\delta_{\text{EQ}}^{(T)}(k)$ compare well with that of ref. [8] over the entire range of energy under consideration. On the other hand, the phase shifts $\delta_{\text{EQ}}^{(I)}(k)$, computed via the real part of the

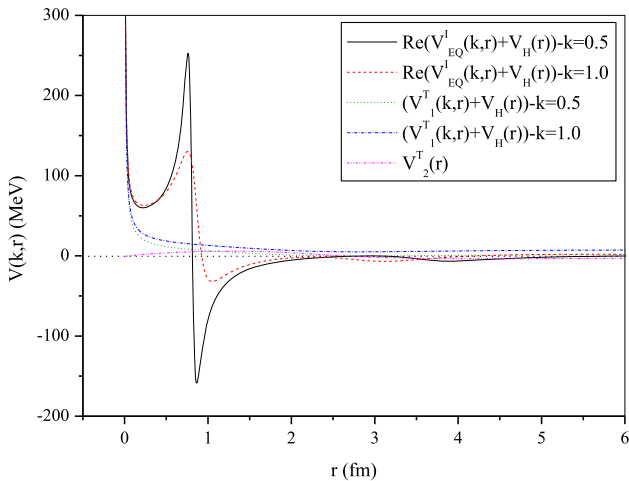


Figure 2. Energy-dependent and independent α – α potential components. The zero line is represented by the black dotted line.

interaction, although produce the correct nature of the s-wave phase but differ significantly from those of refs [8,49] up to 20 MeV. However, beyond 20 MeV they are in better agreement with refs [8,49]. The existence of the imaginary part usually means either the absorption or the production of flux. We have verified that the imaginary part of our potential gives the indication of resonance at the energy interval 9 to 10 KeV which is far below the unstable Be^8 ground-state energy. In figure 2 we have also plotted the energy-dependent interactions ($V_1^{(T)}(k, r) + V_H(r)$), and the real part of the potential ($V_{\text{EQ}}^{(I)}(k, r) + V_H(r)$) for two different momenta to show their variations with energies. Also, $V_2^{(T)}(r)$, which is independent of energy, is plotted in the same figure. It is observed that these equivalent potentials to the parent non-local one possess repulsive cores. Looking closely into figure 2 it is noticed that our equivalent potentials indicate slow convergence with distance as energy increases. On the other hand, the real attractive part of the potential ($V_{\text{EQ}}^{(I)}(k, r) + V_H(r)$) is stronger at low energies while the repulsive part dominates as energy increases, thereby produces slightly larger values of phase shifts at low energies and produces better result at high energies. The energy-independent part $V_2^{(T)}(r)$ has no repulsive core and is weaker compared to its counterpart $V_1^{(T)}(k, r)$.

5. Conclusion

It is believed that knowing the basic feature of the interactions, much phenomenological potential can be constructed with a number of free parameters which have their importance in the cluster model of light nuclei. Through the superposition of repulsive and attractive square wells, Gaussian or Woods–Saxon shapes, a number of α – α potentials have been constructed earlier. Apart from square well and Gaussian shapes, we consider here a two-term non-local potential and other two approximate energy-dependent local potentials constructed from the same non-local one. The merit of our constructed interactions is judged through the phase shift calculations by the judicious use of the PFMs [33,34]. It is noticed that the energy-dependent potential developed via the Taylor expansion is superior to that generated from irregular solution. From our observation it is clear that the quality of the energy-dependent equivalent potential, obtained by Taylor series expansion of the wave function, depends effectively on the difference $(r - s)$ of the coordinates compared with the wavelength of the radial wave function. It is noticed that although our constructed local potential, via the

irregular solution, produces correct trends of the phase shifts, it cannot generate proper numerical values in the low energy range. However, we have treated nucleon–nucleon and α –nucleon systems within this equivalent local potential model with rank-one separable potential and achieved good agreement in phase shift values [49]. For α – α system they are in reasonable agreement in the high-energy range which is not shown in figure 1. This may be due to the fact that this equivalent potential model is not suitable enough for higher-rank potential at low energies. The compact expressions for the Laplace transforms of Coulomb Green's function to deal with the scattering by Coulomb plus non-local separable potential for all partial waves have earlier been published by one of us [39,40]. In this paper, we present the result of a similar investigation by using a Hulthén potential instead of pure Coulomb interaction for $\ell = 0$ as the Hulthén potential is exactly solvable for s-wave only. The extension of the present work for $\ell > 0$ requires higher partial wave solutions of the Hulthén potential. Thus, one may conclude by noting that the α – α scattering can also be described phenomenologically by a soft core energy-dependent local potential equivalent to a non-local interaction. As the fundamental studies of the α – α interactions provide a useful platform for understanding the interaction between complex nuclei, our closed form expression for $D^{(+)}(k)$ and the local potential, constructed via the Taylor expansion, may turn out to be helpful for calculating complex nucleus–nucleus or nucleon–nucleus scattering phases.

References

- [1] E Rutherford and J Chadwick, *Phil. Mag.* **4**, 605 (1927)
- [2] N P Heydenberg and G M Temmer, *Phys. Rev.* **104**, 123 (1956)
- [3] J L Russell, G C Phillips and C W Reich, *Phys. Rev.* **104**, 135 (1956)
- [4] C M Jones, G C Phillips and P D Miller, *Phys. Rev.* **117**, 525 (1960)
- [5] N Berk, F E Steigert and G L Salinger, *Phys. Rev.* **117**, 531 (1960)
- [6] J R Dunning, A M Smith and F E Steigert, *Phys. Rev.* **121**, 580 (1961)
- [7] R Chiba, H E Conzett, H Morinaga, N Mutsuro, K Shoda and M Kimura, *J. Phys. Soc. Jpn.* **16**, 1077 (1961)
- [8] T A Tombrello and L S Senhouse, *Phys. Rev.* **129**, 2252 (1963)
- [9] P Darriulat, G Igo, H G Pugh and H D Holmgren, *Phys. Rev.* **137**, B315 (1965)
- [10] R R Haefner, *Rev. Mod. Phys.* **23**, 228 (1951)
- [11] E van der Spuy and H J Pienaar, *Nucl. Phys.* **7**, 397 (1958)
- [12] G Igo, *Phys. Rev.* **117**, 1079 (1960)
- [13] O Endo, I Shimodaya and J Hiura, *Prog. Theor. Phys. (Kyoto)* **31**, 1157 (1964)
- [14] S Ali and A R Bodmer, *Nucl. Phys.* **80**, 99 (1966)
- [15] S Ali and S A Afzal, *Nuovo Cimento* **50**, 355 (1967)
- [16] V I Kukulin, V G Neudatchin and Y F Smirnov, *Nucl. Phys. A* **245**, 429 (1975)
- [17] B Buck, H Friedrich and C Wheatley, *Nucl. Phys. A* **275**, 246 (1977)
- [18] H Friedrich, *Phys. Rep.* **74**, 209 (1981); *Phys. Rev. C* **30**, 1102 (1984)
- [19] L Marquez, *Phys. Rev. C* **28**, 2525 (1983)
- [20] P Mohr *et al.*, *Z. Phys. A* **349**, 339 (1994)
- [21] S Elhatisari, D Lee, G Rupak, E Epelbaum, H Krebs, T A Lähde, T Luu and Ulf-G Meißner, *Nature* **528**, 111 (2015)
- [22] E Caurier, P Navrátil, W E Ormand and J P Vary, *Phys. Rev. C* **64**, 051301 (2001)
- [23] T Yoshida, N Shimizu, T Abe and T Otsuka, *J. Phys. Conf. Ser.* **454**, 012050 (2013)
- [24] T Myo, A Umeya, K Horii, H Toki and K Ikeda, *Prog. Theor. Exp. Phys.* **2014**, 033D01 (2014)
- [25] V M Datar *et al.*, *Phys. Rev. Lett.* **111**, 062502 (2013)
- [26] J Bhoi and U Laha, *Phys. At. Nucl.* **79**, 370 (2016)
- [27] J Bhoi and U Laha, *Pramana – J. Phys.* **88**: 42 (2017)
- [28] M Rahaman, D Husain and S Ali, *Phys. Rev. C* **10**, 1 (1974)
- [29] U Laha, N Haque, T Nandi and G C Sett, *Z. Phys. A* **332**, 305309 (1989)
- [30] U Laha, M Majumder and J Bhoi, *Pramana – J. Phys.* **90**: 48 (2018)
- [31] M Coz, L G Arnold and A D Mac Kellar, *Ann. Phys. (N. Y.)* **59**, 219 (1970)
- [32] L G Arnold and A D MacKellar, *Phys. Rev. C* **3**, 1095 (1971)
- [33] F Calogero, *Variable phase approach to potential scattering* (Academic Press, New York, 1967)
- [34] B Talukdar, G C Sett and S R Bhattaru, *J. Phys. G* **11**, 591 (1985)
- [35] R G Newton, *Scattering theory of waves and particles* (McGraw-Hill, New York, 1982)
- [36] H van Haeringen, *Charged particle interactions – Theory and formulas* (The Coulomb Press, Leyden, 1985)
- [37] U Laha, C Bhattacharyya, K Roy and B Talukdar, *Phys. Rev. C* **38**, 558 (1988)
- [38] C S Warke and R K Bhaduri, *Nucl. Phys. A* **162**, 289 (1971)
- [39] B Talukdar, U Laha and T Sasakawa, *J. Math. Phys.* **27**, 2080 (1986)
- [40] U Laha, B J Roy and B Talukdar, *J. Phys. A* **22**, 3597 (1989)
- [41] L J Slater, *Generalized hypergeometric functions* (Cambridge University Press, London, 1966)
- [42] A Erdelyi, *Higher transcendental functions* (McGraw-Hill, New York, 1953) Vol. 1

- [43] W Magnus and F Oberhettinger, *Formulae and theorems for the special functions of mathematical physics* (Chelsea, New York, 1949)
- [44] A W Babister, *Transcendental functions satisfying non-homogeneous linear differential equations* (MacMillan, New York, 1967)
- [45] W N Balley, *Generalised hypergeometric series* (Cambridge University Press, London, 1935)
- [46] U Laha, S Ray, S Panda and J Bhoi, *Theor. Math. Phys.* **193**, 1498 (2017)
- [47] B Talukdar, D Chatterjee and P Banerjee, *J. Phys. G* **3**, 813 (1977)
- [48] G C Sett, U Laha and B Talukdar, *J. Phys. A* **21**, 3643 (1988)
- [49] U Laha, S Das and J Bhoi, *Turk. J. Phys.* **41**, 447 (2017)