On the interaction between tripod-type four-level atom and one-mode field in the presence of a classical homogeneous gravitational field

AHMED SALAH$^{1,\ast}$, L E THABET$^2$, T M EL-SHAHAT$^3$ and N H ABD EL-WAHAB$^4$

$^1$Department of Mathematics and Theoretical Physics, Nuclear Research Center, Atomic Energy Authority, Cairo, Egypt
$^2$Department of Business Administration, Community College Khaybar, Taibah University, Madina, Kingdom of Saudi Arabia
$^3$Department of Mathematics, Faculty of Science, Al-Azhar University, Assiut 71524, Egypt
$^4$Department of Mathematics, Faculty of Science, Minia University, Minia, Egypt

$\ast$Corresponding author. E-mail: asalah3020@gmail.com

MS received 19 December 2019; revised 22 June 2020; accepted 6 July 2020

Abstract. In this paper, we investigate the effects of gravitational field on the dynamical behaviour of nonlinear atom–field interaction. We consider a moving tripod-type four-level atom interacting with a single mode field in the presence of a classical homogeneous gravitational field. The analytical solution of the model is calculated by using the Schrödinger equation for a coherent electromagnetic field and when the atom is in its excited state. The influence of the detuning parameter and the classical homogeneous gravitational field on the temporal behaviour of atomic inversion, Mandel Q-parameters, purity of the atomic system and concurrence is studied. Our proposal has many advantages over the previous optical schemes and can be realised in several multiple experiments, such as quantum gravity. The results show that the gravity parameter has an important effect on the properties of these phenomena. Finally, conclusions and some features and comments are given.

Keywords. Four-level atom; classical homogeneous gravitational field; atomic inversion; Mandel Q-parameters; purity; concurrence.

PACS Nos 03.65.Ge; 32.80.–t; 42.50.Vk

1. Introduction

The interaction between the quantised electromagnetic field and a two-level atom is one of the important studies in quantum optics. Under the rotating wave approximation (RWA), the model, which is called the Jaynes-Cummings model (JCM), of a two-level atom and a single mode of the field is solvable [1]. This model has been studied theoretically and experimentally by many researchers. For example, a single two-level atom or N-level atom and one mode or $N$ modes were studied theoretically [2,3]. Experimentally, the JCM has been realised through a superconducting cavity and Rydberg atoms [4–6]. The fluctuations of the driven JCM by an external classical field incoherent state have been studied in [7]. A system of $N$ two-level atoms in a cavity driven by a strong classical field have been demonstrated in [8].

Another significant dressed state for the dynamics of a moving two-level atom interacting with a quantised cavity field by taking into account the Roentgen interaction is discussed in [9,10]. In all these studies devoted to the JC model with atomic motion, the influence of the gravitational field is not taken into account. However, it has often been argued that the achievement of the cooled atoms would be a formidable task as the atoms move so slowly (with a velocity of a few millimetres or centimeters per second for a time period of several milliseconds or more) that they start getting extremely sensitive to the Earth’s gravity. It is obvious that for atoms moving with very low velocity, the influence of Earth’s acceleration becomes important and cannot be neglected [11]. So, it is important to study the temporal evolution of a moving atom simultaneously exposed to the gravitational field. Experimentally, atomic beams with very low velocities are
generated in laser cooling and atomic interferometry [12].

A semi-classical description of a two-level atom interacting with a running laser wave in a gravitational field has been studied in [13]. However, the semiclassical treatment is for studying pure quantum effects occurring in the course of the atom–field interaction [14]. The influence of a classical homogeneous gravitational field on the quantum dynamics of some aspects of the JC model has been analysed. It is interesting to investigate the influence of a classical homogeneous gravitational field on the quantum non-demolition measurement (QND) of atomic momentum in the dispersive JCM. Also, the effects of the gravitational field on quantum statistical properties of the lossless [15] as well as the phase-damped JCMs are investigated in [16]. They found that the gravitational field seriously suppresses non-classical properties of both the cavity-field and the moving atom. Also, the model describing a double Λ five-level atom interacting with a single-mode electromagnetic cavity field in the (off) non-resonate case have been investigated in [17].

On the other hand, a semiclassical description of a two-level atom interacting with a running laser wave in a gravitational field has been given in [13,18]. Furthermore, the interaction between a single-mode electromagnetic field and a three-level atom in the presence of a classical homogeneous gravitational field when the atom is prepared initially in the momentum eigenstate has been studied in [19,20]. More recently, the interaction of a three-level Λ-type atom with a one-mode cavity field including nonlinearity of both the field and the intensity-dependent atom–field coupling is studied by using the modified homotopy analysis method [19]. The eigenvalues and eigenstates of the effective Hamiltonian in the interaction of the atom with both classical gravity and quantum radiation within the framework of the JCM has been studied in [23]. A model which exhibits two identical Ξ-type three-level atoms interacting with a single-mode field with the multiphoton transition in an optical cavity enclosed by a Kerr medium has been presented in [24]. Also, the interaction between a single-mode electromagnetic field and a three-level atom in the presence of a classical homogeneous gravitational field when the atom is prepared initially in the momentum eigenstates has been studied in [20]. Furthermore, atomic beams with very low velocities are generated experimentally in laser cooling and atomic interferometry [12]. It is obvious that for atoms moving with a velocity of a few millimetres or centimetres per second for a time period of several milliseconds or more, the influence of the Earth’s acceleration becomes important and cannot be neglected [21]. For this reason, it is interesting to study the temporal evolution of a moving atom simultaneously exposed to the gravitational field and a single-mode travelling-wave field. Since any quantum optical experiment in the laboratory is actually made in a non-inertial frame, it is important to estimate the influence of the Earth’s acceleration on the outcome of the experiment. A semiclassical description of a two-level atom interacting with a running laser wave in a gravitational field has been given in [13,18]. Furthermore, a complementary scheme based on SU(2) dynamical algebraic structures to investigate the influence of gravity on the quantum non-demolition measurements of atomic momentum in the dispersive JCM has been demonstrated in [16,22].

The non-classical properties of the state and dynamics of entropy of a Λ-type three-level atom interacting with a single-mode cavity field with intensity-dependent coupling in a Kerr medium is explored in [25]. A model of a three-level atom in the Λ-configuration interacting with a two-mode field under a multiphoton process is considered in [26]. A semiclassical vs. the quantum description of the ground state of three-level atoms interacting with a one-mode electromagnetic field is studied in [27]. The quantum entanglement and position-momentum of the entropic squeezing of a moving Λ-type three-level atom interacting with a single-mode quantised field with intensity-dependent coupling is investigated in [28]. The interaction between a Λ-type three-level atom and two quantised electromagnetic fields which are simultaneously injected in a bichromatic cavity surrounded by a Kerr medium in the presence of field interaction (parametric down-conversion) and detuning parameters is studied in [29].

In this paper, we study the system of a moving tripod-type four-level atom interacting with a single mode quantised cavity-field in the presence of a classical homogeneous gravitational field. We obtain the wave function when the atom is initially prepared in the excited state and the field is assumed in a coherent state. The non-classical statistical aspects of the considered model are calculated. The purpose of the present contribution is to examine the effects of the detuning parameter and the classical homogeneous gravitational field on the behaviour of atomic inversion of the atomic system, Mandel Q-parameters, purity of the atomic system and concurrence as a measure of entanglement degree between the considered atomic systems.

The paper is organised as follows. In §2, we present the Hamiltonian describing the interaction between a four-level atom and one-mode field in the presence of the classical homogeneous gravitational field, the detuning parameter and the function of the photon number. In §3, the solution of the considered system using the probability amplitude method has been obtained. In §4–6, we
calculate the expressions of atomic inversion, Mandel Q-parameters, purity of the atomic system and concurrence. In §7, conclusions and some features are given.

2. Description of the model

The system we consider here is a moving tripod-type four-level atom of mass \( M \) exposed simultaneously to a single-mode travelling wave field and a classical homogeneous gravitational field. The considered model indicated by atomic levels \(| j \rangle\), \((j = 1, \ldots, 4)\) that possess energies \( \omega_j \), interacts with a single-mode quantised electromagnetic field with the frequency \( \Omega \), in the presence of a homogeneous classical gravitational field. The atom has upper state \(| 1 \rangle\), bottom state \(| 4 \rangle\) and two intermediate states \(| 2 \rangle\) and \(| 3 \rangle\) with allowed transitions \(| 1 \rangle \leftrightarrow (| 2 \rangle, | 3 \rangle, | 4 \rangle)\), respectively, as shown in figure 1.

In RWA, with the atomic motion along the position vector \( \vec{x} \), the total Hamiltonian for the atom–field system in the presence of a classical homogeneous gravitational field is given by

\[
\hat{H} = \hat{H}_{\text{free}} + \hat{H}_{\text{RWA}},
\]

where \( \hat{H}_{\text{free}}(\hat{H}_{\text{RWA}}) \) is the free(interaction) part of the Hamiltonian. The free part of the Hamiltonian can be written as

\[
\hat{H}_{\text{free}} = \frac{\hat{p}^2}{2M} - M\vec{g} \cdot \vec{x} + \sum_{j=1}^{4} \omega_j \hat{\sigma}_{jj} + \Omega \hat{a}^\dagger \hat{a},
\]

where \( \hat{p} \) and \( \vec{x} \) are the momentum and position operators of the atomic centre of mass motion \( M \), \( \vec{g} \) is the constant gravitational acceleration acting on the atom.

The operators \( \hat{a}^\dagger(\hat{a}) \), are the creation(annihilation) operators of the field mode under consideration with frequency \( \Omega \), and they satisfy the commutation relations

\[
[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{a}, \hat{a}^\dagger] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0,
\]

\[
[\hat{a}, \hat{n}] = \hat{a}, \quad [\hat{a}^\dagger, \hat{n}] = -\hat{a}^\dagger,
\]

while \( \hat{\sigma}_{ij} = |i\rangle\langle j| \) denotes the atomic operators for \( i = j \) and polarisation operators for \( i \neq j \) and they satisfy the commutation relation

\[
[\hat{\sigma}_{ab}, \hat{\sigma}_{cd}] = \hat{\sigma}_{ad}\delta_{bc} - \hat{\sigma}_{ac}\delta_{bd}, \quad \hat{\sigma}_{ab}|b\rangle = |a\rangle,
\]

where \( \delta_{dd} \) is the Kronecker delta. It is important to mention that the operators \( \hat{\sigma}_{ij} \) are \( 4 \times 4 \) matrices, the generators of the unitary group \( SU(4) \) [30]. On the other hand, \( \hat{H}_{\text{RWA}} \) is given as

\[
\hat{H}_{\text{RWA}} = \sum_{s=1}^{3} \lambda_s (\hat{R}\hat{\sigma}_{1,s+1} + \hat{R}^\dagger\hat{\sigma}_{s+1,1}),
\]

where \( \lambda_s \) \((s = 1, 2, 3)\) is the coupling constant between the atom and the field mode. Also, the operators \( \hat{R} \) and \( \hat{R}^\dagger \) are defined as

\[
\hat{R} = \hat{a} e^{i\vec{k} \cdot \vec{x}}, \quad \hat{R}^\dagger = \hat{a}^\dagger e^{-i\vec{k} \cdot \vec{x}},
\]

and also they satisfy the bosonic oscillator commutation relations

\[
[\hat{R}, \hat{R}^\dagger] = 1, \quad [\hat{R}, \hat{n}] = \hat{R}, \quad [\hat{R}^\dagger, \hat{n}] = -\hat{R}^\dagger,
\]

where \( \vec{k} \) is the wave vector of the running wave. We consider that the atom moving in the Earth’s gravitational field is equivalent to a free atom moving in a uniformly accelerated reference frame. Under this consideration, the momentum operator satisfies the following relations:

\[
e^{\pm i\vec{k} \cdot \vec{x}}, \quad \hat{p}|p_0\rangle = p_0|p_0\rangle,
\]

\[
e^{\pm i\vec{k} \cdot \vec{x}}|p_0\rangle = |p_0 \pm \vec{k}\rangle, \quad \vec{p} = \vec{p}_0 + M\vec{g}t
\]

and the position operator satisfies

\[
\hat{x} = \frac{\vec{p}_0 \cdot t}{M} - \frac{1}{2} \vec{g}t^2, \quad \hat{x}|p_0\rangle = \frac{\vec{p}_0 \cdot t}{M}|p_0\rangle.
\]
\[ \hat{\sigma}_{11} + \hat{\sigma}_{22} + \hat{\sigma}_{33} + \hat{\sigma}_{44} = \hat{I}, \]
\[ \hat{a}^\dagger \hat{a} - \sum_{s=1}^{3} \hat{\sigma}_{s+1,s+1} = \hat{N}, \]  
(11)

where \( \hat{I} \) and \( \hat{N} \) are the unity operator and the number of excitation, respectively. Furthermore, the conservation of atomic momentum plus photon momentum \([31]\) of this model can be written as
\[ \hat{p} + \vec{k} \hat{a}^\dagger \hat{a} = \hat{N}_0. \]  
(12)

Using these conservations, we can rewrite the free part (2) in the form
\[ \hat{H}_{\text{free}} = \frac{\hat{p}^2}{2M} + \frac{1}{2} Mg^2 t^2 - \hat{p}_0 \cdot \vec{g} t + \omega_i \hat{I} + \Omega \hat{N} 
+ \sum_{s=1}^{3} \Delta_s (\hat{p}_0, t) \hat{a}^\dagger \hat{a} + \Omega \hat{a}^\dagger \hat{a}, \]  
(13)

where
\[ \Delta_s (\hat{p}_0, t) = \omega_s - \omega_1 + \Omega - \frac{\hat{p}_0 \cdot \vec{k}}{M} + \frac{k^2}{2M} + \vec{k} \cdot \vec{g} t. \]  
(14)

\( \Delta_s (\hat{p}_0, t) \) is introduced as the Doppler shift detuning at time \( t \). In the interaction picture, the transformed Hamiltonian (1) takes the following form:
\[ \hat{H}_{\text{int}} = \exp \left\{ -i \int (\hat{H}_0 (\tau) d\tau) \right\} \hat{H}_{\text{RWA}} \times \exp \left\{ i \int (\hat{H}_0 (\tau) d\tau) \right\}. \]  
(15)

After some algebraic calculations, the interaction effective Hamiltonian take the following form:
\[ \hat{H}_{\text{int}} = \sum_{s=1}^{3} \lambda_s [\hat{R} e^{i \Delta_s (t) t} \hat{a}^\dagger_{s+1} + \hat{R}^\dagger e^{-i \Delta_s (t) t} \hat{a}_{s+1}]. \]  
(16)

In what follows, we use the above calculations to calculate the wave function as well as its solution using the probability amplitude method. This is seen in the next section.

### 3. The solution of the considered system

To discuss the statistical properties of the present system, we have to obtain the solution of the time-dependent Schrödinger equation
\[ i \frac{d}{dt} |\Psi(t)\rangle = \hat{H}_{\text{int}} |\Psi(t)\rangle, \]  
(17)

where \( \hat{H}_{\text{int}} \) is given by eq. (16), and the wave function \( |\Psi(t)\rangle \) at time \( t > 0 \) takes the form
\[ |\Psi(t)\rangle = \sum_{n=0}^{\infty} q_n \left[ \Psi_1 (n, t) |1, \hat{p}_0, n\rangle 
+ \sum_{s=2}^{4} \Psi_s (n + 1, t) |s, \hat{p}_0 - \vec{k}, n + 1\rangle \right]. \]  
(18)

The coefficients \( \Psi_j (t) \) are the probability amplitude (17) and \( q_n \) is the probability distribution of the initial field state of the cavity field. We now assume that the field is initially in the coherent state and the atoms enter the cavity in the upper state. In this case, the wave function of the atoms–field system at \( t = 0 \) can be written as
\[ |\Psi(0)\rangle = |1\rangle \otimes |\alpha\rangle, \]  
(19)

where \( |\alpha\rangle \) is the coherent state given by
\[ |\alpha\rangle = \sum_{n=0}^{\infty} q_n |n\rangle, \quad q_n = \frac{\alpha^n}{\sqrt{n!}} \exp \left( -|\alpha|^2 \right). \]  
(20)

After some calculations, we obtain the following system of differential equations for the atomic probability amplitudes:
\[ i \frac{d}{dt} \Psi_1 (n, t) = \sum_{s=1}^{3} f_s e^{i \Delta_s (t) t} \Psi_s (n + 1, t), \]  
\[ i \frac{d}{dt} \Psi_\ell (n + 1, t) = f_{\ell - 1} e^{-i \Delta_{\ell - 1} (t) t} \Psi_1 (n + 1), \]  
(21)

where
\[ f_s = \lambda_s \sqrt{n + 1}. \]  
(22)

Now, we shall resolve the above system in different ways as follows: Let us first start with the solution for this system in the non-resonant case \( \Delta = \Delta_\ell \neq \Delta \), and also in the absence of the classical homogeneous gravitational field \( \vec{k} \cdot \vec{g} = 0 \). According to Laplace transform, and after some lengthy calculations, we find that the time-dependent coefficients \( \Psi_j (t) \) are given by
\[ \Psi_1 (n, t) = a_{11} e^{-\mu_+ t} + a_{12} e^{-\mu_- t} + a_{13} e^{-v_+ t} + a_{14} e^{-v_- t}, \]
\[ \Psi_\ell (n + 1, t) = a_{\ell 1} e^{-(\mu_+ + i \Delta) t} + a_{\ell 2} e^{-(\mu_- + i \Delta) t} + a_{\ell 3} e^{-(v_+ + i \Delta) t} + a_{\ell 4} e^{-(v_- + i \Delta) t}, \quad \ell = 2, 3, 4 \]  
(23)

with
\[ \mu_\pm = A_+ \pm B_-, \quad v_\pm = A_- \pm B_+, \]
\[ A_\pm = \frac{\gamma_3}{4} \pm \frac{\gamma}{2}, \quad B_\pm = \frac{Z_\pm}{2}, \]  
(24)
Also, the coefficients and are given by the following expressions:

\[
\begin{align*}
\omega_1 &= v_1/(3v_2) + u_2, \\
\omega_2 &= u_1/(3v_2) + (v_2/3), \\
\omega_3 &= -8\sigma_1 + 4\sigma_2\sigma_3 - \sigma_3^3, \\
u_1 &= 12\sigma_0 + \sigma_2^2 - 3\sigma_1\sigma_3, \\
u_2 &= (-2\sigma_2^2 + (\sigma_3^2)/4, \\
v_2 &= 27\sigma_1^2 - 72\sigma_0\sigma_2 + 2\sigma_3^2 - 9\sigma_1\sigma_2\sigma_3 \\
&+ 27\sigma_0\sigma_3^2, \\
v_2 &= \left( v_1 + \sqrt{-4u_1^2 + v_1^2} \right)/2 \right)^{1/3}
\end{align*}
\]

and

\[
\begin{align*}
\sigma_1 &= -i(\Delta_1 + \Delta_2 + \Delta_3), \\
\sigma_2 &= (f_1^2 + f_2^2 + f_3^2 - \Delta_1\Delta_2 - \Delta_1\Delta_3 - \Delta_2\Delta_3), \\
\sigma_3 &= i(\Delta_1\Delta_2\Delta_3 - (\Delta_2 + \Delta_3) f_1^2 - (\Delta_1 + \Delta_3) f_2^2 \\
&- (\Delta_1 + \Delta_2) f_3^2), \\
\sigma_4 &= -(\Delta_2\Delta_3 f_1^2 + \Delta_1\Delta_3 f_2^2 + \Delta_1\Delta_2 f_3^2).
\end{align*}
\]

Also, the coefficients \(a_{j1}, a_{j2}, a_{j3}, a_{j4}, j = 1, 2, 3, 4\) are given by the following expressions:

\[
\begin{align*}
a_{11} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{12} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{13} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{14} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{21} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{22} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{23} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{24} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{31} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{32} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{33} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{34} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{41} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{42} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{43} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}, \\
a_{44} &= \frac{-3\sigma_3}{(\sigma_2 - \sigma_3)}.
\end{align*}
\]

Secondly, we consider that the classical homogeneous gravitational field is absent (\(\vec{k} \cdot \vec{g} = 0\)), and also the system is in the resonance case \(\Delta_j = 0\) (\(j = 1, 2, 3\)). Hence, the probability amplitudes \(\Psi_j(t), j = 1, 2...4\), under these initial conditions have the form

\[
\begin{align*}
\Psi_1(n, t) &= \cos(\mu t), \\
\Psi_\ell(n + 1, t) &= -i f_{\ell-1} \sin(\mu t) \mu, \quad \ell = 2, 3, 4,
\end{align*}
\]

where

\[
\mu = \sqrt{f_1^2 + f_2^2 + f_3^2}.
\]

Finally, in the off-resonant case \(\Delta_j = \Delta\) when the classical homogeneous gravitational field is absent (\(\vec{k} \cdot \vec{g} = 0\)). After straightforward calculations, we can obtain the probability amplitudes \(\Psi_j(t), j = 1, 2...4\) in the following form:

\[
\begin{align*}
\Psi_1(n, t) &= \exp \left( \frac{i\Delta t}{2} \right) \left( \cos \sqrt{\mu^2 + \left( \frac{\Delta}{2} \right)^2} \\
&\quad - i \Delta \sin \sqrt{\mu^2 + \left( \frac{\Delta}{2} \right)^2} \right), \\
\Psi_\ell(n + 1, t) &= 2f_{\ell-1} \exp \left( \frac{-i\Delta t}{2} \right) \frac{\sin \sqrt{\mu^2 + \left( \frac{\Delta}{2} \right)^2}}{\sqrt{\mu^2 + \left( \frac{\Delta}{2} \right)^2}}.
\end{align*}
\]
Moreover, in the presence of the classical homogeneous gravitational field $\vec{k} \cdot \vec{g} \neq 0$, the probability amplitudes $\Psi_j(t)$ is in the following form:

$$\Psi_1(n, t) = \exp\{i \Delta(t) \cdot t\} \left\{ C_1 H[-\gamma - 1, \eta(t)] + C_2 \sum_{\ell=2,3,4} F_1 \left[ \frac{\gamma + 1}{2}, \frac{1}{2}, \eta^2(t) \right] \right\},$$

and

$$\Psi_\ell(n + 1, t) = \frac{f_{\ell-1}}{f_1} \left\{ C_1 H[-\gamma, \eta(t)] + C_2 \sum_{\ell=2,3,4} F_1 \left[ \frac{\gamma + 1}{2}, \frac{1}{2}, \eta^2(t) \right] \right\}, \quad \ell = 2, 3, 4,$$

where

$$C_1 = \frac{1 \cdot 1}{H[-\gamma - 1, \eta(0)] \cdot 1 \cdot 1 \cdot 1 \cdot 1}$$

and

$$C_2 = \frac{-H[-\gamma, \eta(0)]}{H[-\gamma - 1, \eta(0)] \cdot 1 \cdot 1 \cdot 1 \cdot 1}$$

and

$$\gamma = \frac{i \mu^2}{\vec{k} \cdot \vec{g}}$$

and

$$\eta(t) = \frac{1 + i (\vec{k} \cdot \vec{g} \Delta t)}{2\sqrt{\vec{k} \cdot \vec{g}}}$$

where $H(n, t)$ and $F_1(\alpha, \beta, t)$ are the Hermite and the confluent hypergeometric functions, respectively. In what follows, using the above results we are able to discuss some statistical properties of the present system such as atomic inversion, Mandel Q-parameter, purity and concurrence (see the forthcoming sections).

4. Atomic inversion

We shall start with the atomic inversion from which we can discuss the collapse and revival phenomena. Add to that, the atomic inversion measures the difference in the populations of the two levels of the atom and plays a fundamental role in the laser theory [32]. We can calculate the time evolution of the atomic inversion $W(t)$ by using the following formula:

$$W(t) = \rho_{11} - \rho_{44},$$

where $\rho_{11}$ and $\rho_{44}$ describe the atomic occupation probabilities of the excited and ground states, respectively, which are given by

$$\rho_{11} = \sum_{n=0}^{\infty} P_n \psi_1(n, t) \psi_1^*(n, t),$$

and

$$\rho_{44} = \sum_{n=0}^{\infty} P_n \psi_4(n + 1, t) \psi_4^*(n + 1, t),$$

where $\psi_1(n, t)$ and $\psi_4(n + 1, t)$ are given in (36). After straightforward calculations, the atomic inversion (39), takes the form

$$W(t) = \sum_{n=0}^{\infty} P_n \left[ |C_1|^2 \left( |H[-\gamma - 1, \eta(t)]|^2 - \frac{f_3}{f_1}^2 |H[-\gamma, \eta(t)]|^2 \right) \right]$$

In the atomic system vs. the scaled time $\lambda t$, we assume that the atom is initially prepared in an excited state and the field is initially prepared in a coherent state. Also, we set six different values of the gravity parameters $\vec{k} \cdot \vec{g}$ where $\vec{k} \cdot \vec{g} = 0, 0.02, 0.2, 0.4, 0.6$ and 0.8 in figures 2a–2f, respectively. Also, the initial mean photon number $|\alpha| = 5$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$. In figure 2a, we set all parameters equal to zero. We note that the atomic inversion baseline oscillates around 0.3 and the fluctuations always have a positive value. As one can see, the phenomenon of revival occurs once after the onset of the interaction for a short period of time. In figures 2b–2f, we illustrate the case when the gravitational field is taken into account, and we observe
that the revival and collapse phenomena occur periodically. With increasing values of gravity parameter, the Rabi oscillations of the atomic inversion disappear.

Now, we focus on investigating the influence of detuning parameters with the weak gravitational field on the time evolution of the atomic inversion. In figure 3, we plot $W(t)$ against the scaled time for different values of detuning parameters where the off-resonant case ($\Delta = \Delta_r$) is considered by taking $\Delta = 5, 10$ and 15 in figures 3a–3c, respectively. Also, the non-resonant case is examined with $\Delta_1 = 10$ (fixed value), $\Delta_2 = \Delta_3 = 5, 15$ and $\Delta_1 = 5, \Delta_2 = 10$ and $\Delta_3 = 15$ in figures 3d–3f, respectively. From these figures, we notice that the atomic inversion function is shifted upward above 0.3. Moreover, it fluctuates around 0.7 with fast oscillations and in the off-resonant case leads to increasing the oscillations of the atomic inversion, which means that the energy is stored more in the atom. From this figure, it is clear that the collapse revival phenomenon occurs periodically. Also, we observed that as the detuning increases, the collapse time increases while the amplitude of the oscillations decreases.

In figure 4, we demonstrated the effect of gravitational field when $\vec{k} \cdot \vec{g} = 0.4$ and the detuning parameters the same as in figure 3. It is remarkable that fluctuations are shifted downward and the oscillations decrease with increase in $\vec{k} \cdot \vec{g}$. It is obvious from the figures that an elongation in the revival time occurs. Also, we see that the atomic inversion evolves periodically and the oscillations increase whereas the amplitude decreases with increase in scaled time. Also, the collapse and revival phenomena are very obvious, and the oscillations disappear very fast in a short time. In the next section, we shall focus on the Mandel Q-function and the effect of detuning parameters and classical homogeneous gravitational field on the Poissonian behaviour of the photon statistics will be investigated.

5. Mandel Q-parameter

Mandel introduced Mandel Q-parameter in quantum optics in 1979 for measuring the departure of the occupation number distribution from Poissonian statistics to understand the non-classical behaviour of the system [33] better. In fact, Mandel Q-parameter is usually
Figure 4. The evolution of atomic inversion of the atomic system vs. the scaled time $\lambda t$ for the same data as in figure 3 but for $k \cdot g = 0.4$.

Figure 5. The Mandel Q-parameter vs. the scaled time. Other values are the same as in figure 2.

used to discuss the sub-Poissonian and super-Poissonian behaviour of the photon statistics from which we can distinguish between the classical and non-classical behaviour. This parameter is defined as follows:

$$Q(t) = \frac{\langle (\hat{a}\dagger(t)\hat{a}(t))^2 \rangle - \langle \hat{a}\dagger(t)\hat{a}(t) \rangle^2}{\langle \hat{a}\dagger(t)\hat{a}(t) \rangle} - 1,$$  \hspace{1cm} (42)

where $\langle \hat{a}\dagger(t)\hat{a}(t) \rangle$ is the photon number operator which is given by

$$\langle \hat{a}\dagger(t)\hat{a}(t) \rangle = \bar{n} + \sum_{n=0}^{\infty} P_n(|\Psi_2(n+1,t)|^2 + |\Psi_3(n+,t)|^2 + |\Psi_4(n+,t)|^2).$$  \hspace{1cm} (43)

In particular, when $Q(t) = 0$, $Q(t) < 0$ and $Q(t) > 0$, then the state is called Poissonian, sub-Poissonian and super-Poissonian, respectively. Here, the effect of detuning parameters and gravity parameter on the temporal evolution of $Q(t)$, taking the same data as in figures 2–4, is visualised. In figure 5, the Mandel Q-parameter is plotted vs. the scaled time for different values of gravity parameter. We notice that the system oscillates between super-Poissonian and sub-Poissonian statistics in the absence of both gravity and detuning parameters as seen in figure 5a. As the gravity parameter increases, the system is inclined to super-Poissonian statistics.

To see the effect of detuning parameter in the weak gravity parameter $k \cdot g = 0.04$, we plotted the quantity $Q(t)$ in figure 6. We notice that the system changes between super-Poissonian and sub-Poissonian. Figure 7 shows the effect of detuning parameters on the time evolution of the Mandel Q-parameter when $k \cdot g = 0.4$. It is clear that collapses and revivals occur. In figure 7a, we see that the system changes between super-Poissonian and sub-Poissonian and become super-Poissonian at large time. As detuning parameter increases, the system becomes sub-Poissonian. It means that the classical homogeneous gravitational field converts some parts of the super-Poissonian to sub-Poissonian statistic especially at large times. It can be observed that both detuning parameters and gravity parameter affect the evolution of the Mandel Q-parameter and the
sub-Poissonian behaviour increases when \( \vec{k} \cdot \vec{g} \) is taken into account. In the next two sections, we are going to discuss the entanglement degree due to purity and concurrence for the system under consideration.

6. Purity

The entanglement is one of the important non-classical parameters in quantum optics, and it is the cornerstone of the quantum information theory [34]. In fact, much attention has been given on the properties of entanglement between the atom and the field, in particular, the purity of the system. The purity is a good tool to give information about the degree of entanglement (not to measure the entanglement) of the system. For this reason, we devote the present section to discuss the purity of the system under consideration. The purity of the field state can be determined from the quantity

\[
P(t) = \text{Tr}[\hat{\rho}_A^2],
\]

where \( \rho_A(t) \) is the atomic density operator which is given by

\[
\rho_A = \text{Tr}_F [\Psi(t) \Psi(t)^*] = \begin{pmatrix}
\rho_{33} & \rho_{34} & \rho_{32} & \rho_{31} \\
\rho_{43} & \rho_{44} & \rho_{42} & \rho_{41} \\
\rho_{23} & \rho_{24} & \rho_{22} & \rho_{21} \\
\rho_{13} & \rho_{14} & \rho_{12} & \rho_{11}
\end{pmatrix}
\]

with

\[
\rho_{11} = \sum_{n=0}^{\infty} P_n |\Psi_1(n, t)|^2,
\]

\[
\rho_{\ell\ell} = \sum_{n=0}^{\infty} P_n |\Psi_\ell(n + 1, t)|^2, \quad \ell = 2, 3, 4,
\]

\[
\rho_{12} = \sum_{n=0}^{\infty} q_{n+1} q_n^* \Psi_1(n + 1, t) \psi_2^*(n + 1, t) = \rho_{12}^*,
\]

\[
\rho_{13} = \sum_{n=0}^{\infty} q_{n+1} q_n^* \Psi_1(n + 1, t) \psi_3^*(n + 1, t) = \rho_{31}^*,
\]

\[
\rho_{14} = \sum_{n=0}^{\infty} q_{n+1} q_n^* \Psi_1(n + 1, t) \psi_4^*(n + 1, t) = \rho_{41}^*,
\]

\[
\rho_{23} = \sum_{n=0}^{\infty} P_n \Psi_2(n + 1, t) \psi_3^*(n + 1, t) = \rho_{32}^*,
\]

\[
\rho_{24} = \sum_{n=0}^{\infty} P_n \Psi_2(n + 1, t) \psi_4^*(n + 1, t) = \rho_{42}^*,
\]

\[
\rho_{34} = \sum_{n=0}^{\infty} P_n \Psi_3(n + 1, t) \psi_4^*(n + 1, t) = \rho_{43}^*.
\]

From eq. (46), it is easy to show that
The purity occurs in both subsystems at the same time and precisely at the same rate. To analyse and discuss the purity, we use eq. (36). However, due to the complicated expression, we have plotted figures 8–10 as a function $P(t)$ against the scaled time $\lambda t$, where we used the same values of parameters as in the atomic inversion. From these figures, we see that the collapse and revival time occurs at the same time in the corresponding figures of atomic population inversion and the purity varies between two ranges, from 0 for pure states and (or disentangled states) to 1.0 for maximally mixed states. We can use this fact to discuss the behaviour of purity. For example, when we set all parameters to zero, the function shows maximum entanglement after the onset of interaction. This behaviour is consistent with the revival period which is seen in the atomic inversion, see figure 8(a), purity becomes stable and less than 0.8. In figures 8(b) and 8(c), the presence of gravity parameter pulled the purity down sharply and in figures 8(d)–8(f), when the gravity parameter increases, it is obvious that the purity of the system takes longer time to be pulled down compared to figures 8(a)–8(c).

To visualise the influence of detuning parameter on purity, we plotted purity in figure 9 for weak gravity where we used the same data as in figure 3 of the involved parameters as in atomic inversion. We noticed that the presence of detuning parameters pulled down purity. We see that purity has increased and many sharp peaks of high values appear with some kind of periodicity and more revivals have appeared. Also, by increasing the detuning parameters, we noticed that the oscillations are elongated and the amplitude of purity decreases as the detuning increases.

Now we shed some light on the effects of detuning parameters in the presence gravitational field as plotted in figure 10 by taking the same data as in figure 4. We noticed that as time increases, the function shows rapid fluctuations with a decrease in its maximum value and purity reduces its maximum as well as its minimum. This means that an increase in gravity parameter leads to the reduction in the degree of entanglement and it gets far from the pure state of the field.

7. Concurrence

Hill and Wootters have presented concurrence as an appropriate scale of entanglement of any state of two qubits, mixed or pure [35,36]. The concurrence is known

$$P(t) = \rho_{11}^2 + \rho_{22}^2 + \rho_{33}^2 + \rho_{44}^2 + 2|\rho_{12}|^2 + 2|\rho_{13}|^2 + 2|\rho_{14}|^2 + 2|\rho_{23}|^2 + 2|\rho_{24}|^2 + 2|\rho_{34}|^2$$ (48)
in the pure state $|\Psi(t)\rangle$ on $N \times M$-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_M$ as follows [37]:

$$C(|\Psi(t)\rangle) = \sqrt{2|\langle \Psi | \Psi \rangle|^2 - \text{Tr}[\text{Tr}_M |\Psi(t)\rangle \langle |\Psi(t)\rangle]|^2},$$

(49)

where $\text{Tr}_M$ is the partial trace over $\mathcal{H}_M$. The concurrence is a measure of entanglement degree between the atoms and the field, the value for which varies between 0 for separable state and $\sqrt{2(N-1)/N}$ for the maximally entangled state. To investigate concurrence, we calculated the atomic reduced density matrix in (46). So we can rewrite concurrence in the following form:

$$C(t) = \sqrt{2 \sum_{i,j=1,2,3,4, i \neq j} (\rho_{ii} \rho_{jj} - \rho_{ij} \rho_{ji})}.$$  

(50)

We turn to examine the temporal evolutions of the entropy field of the present model. For this reason, we plot several figures to display the behaviour of the field entropy against the scaled time for different values of the given parameters. For example, in figure 11, we plot the evolution of concurrence $C(t)$ vs. scaled time $\lambda t$ with the same data as in figure 2. We notice that the maximum value of concurrence is approximately 0.7, and this value is affected by the gravity parameter. Furthermore, it can be seen that, maximum concurrence is achieved in the collapse time, while at one-half of the revival time concurrence reaches its local minimum. As the value of the gravity parameter increases, the maximum value is decreased as shown in figures 10b–10f. In addition, fluctuations start after a long interval of time by comparison with the case $\vec{k} \cdot \vec{g} = 0$ as in figure 10a. Also, we notice that the time delaying more fluctuations by increasing the parameter.

In figure 12, we investigate the effect of detuning parameter with the same data in figure 3 when the gravity parameter is small. We see that the entanglement is affected by the detuning parameters when the maximum value of entropy is approximately 0.6. In this case, a drastic change occurs in concurrence function behaviour. It is remarkable that concurrence evolves periodically and shows off the disentangled between the field and the atom. Furthermore, we notice that the amplitude decreases by increasing the time and detuning parameters.

Also, we shall discuss the influence of detuning parameters in the presence of gravity parameter on the dynamical behaviour of concurrence. This is plotted in figure 13 using the same data as in figure 4. It is remarkable that the field entropy evolves periodically and shows off the disentanglement between the field.
and the atom. Furthermore, we notice that the amplitude decreases by increasing the time and the gravity parameter. On the other hand, when the gravity parameter is taken into account, concurrence decreases and fluctuations increase.

8. Conclusion

In this paper, we have studied the interaction between a four-level tripod-type atom and one-mode electromagnetic cavity field in the presence of classical homogeneous gravitational field. We have extended the previous studies in this context [19,20]. The quantum treatment of the internal and external dynamics of the atom has been presented. The effective Hamiltonian has been obtained based on an alternative $SU(4)$ dynamical algebraic structure. In the framework of the Schrödinger equation, the wave function was calculated exactly when the atom and the field are initially prepared in an excited and coherent state. The influence of gravity parameter and detuning parameters on the collapse-revival and Poissonian statistics phenomena were investigated. Furthermore, the degree of entanglement using purity and concurrence was explored. The results show that the detuning parameter and gravity parameters are very useful in generating a high amount of entanglement, leading to a change in the behaviour of these quantities. The problem proposed in this article can be extended to the multimode system by adding the Kerr-like medium and we can discuss phenomena when the field is in another state such as a thermal or a squeezed state.

References


[37] R Horodecki, P Horodecki, M Horodecki and K Horodecki, Rev. Mod. Phys. 81, 865 (2009)