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Integrability and exact solutions of deformed fifth-order Korteweg–de Vries equation

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Abstract. We consider a deformed fifth-order Korteweg–de Vries (D5oKdV) equation and investigated its integrability and group theoretical aspects. By extending the well-known Lax pair technique, we show that the D5oKdV equation admits a Lax representation provided that the deformed function satisfies certain differential constraint. It is observed that the D5oKdV equation admits the same differential constraint (on the deforming function) as that of the deformed Korteweg–de Vries (DKdV) equation. Using the Lax representation, we show that the D5oKdV equation admits infinitely many conservation laws, which guarantee its integrability. Finally, we apply the Lie symmetry analysis to the D5oKdV equation and derive its Lie point symmetries, the associated similarity reductions and the exact solutions.

Keywords. Integrability; deformed fifth-order Korteweg–de Vries equation; conservation laws; Lie symmetry analysis.

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1. Introduction

It is well known that, the study of integrability of nonlinear partial differential equations (NPDEs) is important, because it helps to understand the nature of the phenomena concerned [1–7]. NPDEs are not integrable in general. The concept of integrability of NPDE is not well defined and there exists no unique definition as yet. However, there exists several mathematical characteristics such as the existence of Lax pair, possessing Painlevé property, the existence of multisoliton solutions, the existence of infinitely many generalised symmetries, the existence of infinitely many conservation laws, etc., to determine the integrability of NPDEs. In the past decades, many integrable NPDEs [4–16] have been identified through these mathematical characteristics.

It is widely believed that, if an integrable NPDE gets perturbed or deformed, then its integrability is lost in general. However, recent investigations [12–18] have shown that the deformed NPDE with two independent variables maintains the integrability of the original NPDE. For example, Kundu *et al* [16] have considered the deformed Korteweg–de Vries (DKdV) equation

$$u_t + u_{xxx} + 6uu_x = g_x \tag{1a}$$

and shown that it admits a Lax representation, infinitely many generalised symmetries, etc., provided that the deforming function $g(x,t)$ satisfies a differential constraint given by

$$g_{xxx} + 2u_x g + 4ug_x = 0, \tag{1b}$$

where $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, $g_x = \partial g/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, etc.

In this paper, we consider a deformed fifth-order Korteweg–de Vries (D5oKdV) equation given by

$$u_t + \alpha(u_{xxx} + 6uu_x) + \beta(u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x) = g_x, \tag{2a}$$

where α and β are real constants and we can see that it admits a Lax representation provided the deformed function $g(x,t)$ satisfies a differential constraint

$$g_{xxx} + 2u_x g + 4ug_x = 0. \tag{2b}$$

We also show that the D5oKdV equation (2) admits infinitely many conservation laws. It is appropriate to

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mention here that the differential constraint (2b) is exactly the same as given in (1b). Note that, (i) when $g = 0$, eq. (2) is referred to as the fifth-order KdV (5oKdV) equation, (ii) when $g = 0$, $\alpha = 1$, $\beta = 0$, eq. (2) is the KdV equation, (iii) when $g = 0$, $\alpha = 0$, $\beta = 1$, eq. (2) is the fifth-order Lax equation and (iv) when $\alpha = 1$, $\beta = 0$, eq. (2) coincides with the DKdV equation (1).

The plan of this paper is as follows: In §2, we show that the D5oKdV equation (2) admits a Lax representation. In §3, using the obtained Lax representation, we derive its infinitely many conservation laws. In §4, we apply the Lie symmetry analysis to the D5oKdV equation (2) and we derive its Lie point symmetries, similarity reductions and exact solutions. In §5, we summarise our results.

2. Lax representation of the D5oKdV equation

Recall that, if a NPDE involving two independent variables t and x arises from the compatibility condition of a system of linear equations, then it is called integrable in the sense of Lax. This procedure is known as Lax approach originally formulated by Lax [6] and later on generalised by Ablowitz, Kaup, Newell and Segur (AKNS) [1,2]. By employing the procedure given by AKNS, one can obtain the linear eigenvalue problem

$$\Phi_x = L(x,t,\lambda)\ \Phi = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3a}$$

$$\Phi_t = M(x,t,\lambda)\ \Phi = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{3b}$$

where λ is the spectral parameter and $\phi_1,\ \phi_2$ are functions of $x,\ t$ with the compatibility condition $\Phi_{xt} = \Phi_{tx}$ which yields

$$L_t - M_x + [L, M] = 0$$

or

$$L_t - M_x + LM - ML = 0. \tag{4}$$

Equation (4) is referred to as the zero curvature equation or the Lax equation and the matrices L and M are known as the Lax pair.

The Lax matrices for the deformed equations are constructed in the following way:

(i) Keep the Lax matrix $L(x,t,\lambda)$ of the unperturbed NPDE unchanged which implies that the scattering problem for the deformed NPDE remains the same.

(ii) Modify the Lax matrix $M(x,t,\lambda)$ suitably from the unperturbed NPDE which implies that the time evolution of the spectral data becomes different in the perturbed models. From a straightforward calculation, we find that the D5oKdV equation (2) admits a Lax representation provided the Lax matrices L and M have the following form:

$$L(x,t,\lambda) = \begin{pmatrix} \lambda & u \\ -1 & -\lambda \end{pmatrix}, \tag{5a}$$

$$M(x,t,\lambda) = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & -M_{11} \end{pmatrix}, \tag{5b}$$

where

$$\begin{aligned} M_{11} = \sum_{j=-2}^{5} A_j\lambda^j &= -\frac{g_x}{4\lambda^2} - \frac{g}{2\lambda} - \beta(u_{xxx} + 6uu_x) \\ &\quad -\alpha u_x - 2(\beta u_{xx} + 3\beta u^2 + \alpha u)\lambda - 4\beta u_x\lambda^2 \\ &\quad -4(2\beta u + \alpha)\lambda^3 - 16\beta\lambda^5, \\ M_{12} = \sum_{j=-2}^{4} B_j\lambda^j &= -\frac{(g_{xx} + 2ug)}{4\lambda^2} - \frac{g_x}{2\lambda} \\ &\quad -\beta(u_{xxxx} + 8uu_{xx} + 6u_x^2 + 6u^3) \\ &\quad -\alpha(u_{xx}+2u^2) - 2(\beta(u_{xxx}+6uu_x) + \alpha u_x)\lambda \\ &\quad -4(\beta u_{xx}+2\beta u^2+\alpha u)\lambda^2 - 8\beta u_x\lambda^3 - 16\beta u\lambda^4, \\ M_{21} = \sum_{j=-2}^{4} C_j\lambda^j &= \frac{g}{2\lambda^2} + 2\beta u_{xx} + 6\beta u^2 + 2\alpha u \\ &\quad +4(2\beta u + \alpha)\lambda^2 + 16\beta\lambda^4 \end{aligned}$$

and satisfies the Lax equation (4) and the deformation function $g(x,t)$ satisfies a differential constraint

$$g_{xxx} + 2u_x g + 4ug_x = 0.$$

Thus, the D5oKdV equation (2) is expected to be integrable in the sense of Lax.

3. Conservation laws of the D5oKdV equation

Consider a scalar NPDE involving two independent variables t and x,

$$\Delta\left(t, x, u(x,t), \frac{\partial u}{\partial x}, \ldots\right) = 0. \tag{6}$$

A conservation law of (6) is an equation of the form

$$\frac{\partial \rho}{\partial t} = \frac{\partial F}{\partial x}, \tag{7}$$

which is satisfied for all solutions of (6), where $\rho(x,t)$, the conserved density and $F(x,t)$, the associated flux, are in general functions of x, t, u and the partial derivatives of u. In this section, we explain how to construct conservation laws of D5oKdV equation (2) through the Lax representation given in the previous section. For this, we use the Lax matrices L, M given in (5) into eq. (3) and we get

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_x=\begin{pmatrix}\lambda & u\\ -1 & -\lambda\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}, \tag{8a}$$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}_t=\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & -M_{11}\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}. \tag{8b}$$

To start with, we assume that

$$\phi_2(x,t)=T(x,t)\phi_1(x,t).$$

As a consequence, eq. (8a) reduces to

$$\phi_{1x}=(\lambda+u\,T)\phi_1, \tag{9a}$$

$$T\phi_{1x}+T_x\phi_1=-(1+\lambda\,T)\phi_1$$

or

$$\phi_{1x}=\frac{-T_x\phi_1-(1+\lambda\,T)\phi_1}{T} \tag{9b}$$

and eq. (8b) reduces to

$$\phi_{1t}=(M_{11}+M_{12}T)\,\phi_1, \tag{10a}$$

$$T\phi_{1t}+T_t\phi_1=(M_{21}-M_{11}T)\,\phi_1. \tag{10b}$$

Obviously, eq. (9) is the well-known Riccati equation given by

$$T_x=-(1+2\lambda T+uT^2). \tag{11}$$

Now, eqs (9a) and (10a) can be rewritten respectively in the following form:

$$(\ln\phi_1)_x=\lambda+uT, \tag{12a}$$

$$(\ln\phi_1)_t=M_{11}+M_{12}T. \tag{12b}$$

The compatibility condition $(\ln\phi_1)_{xt}=(\ln\phi_1)_{tx}$ reads as

$$\frac{\partial}{\partial t}[\lambda+uT]=\frac{\partial}{\partial x}[M_{11}+M_{12}T]. \tag{13}$$

In order to derive conservation laws, we expand $T(x,t)$ as a series in negative powers of λ,

$$T(x,t)=\sum_{j=1}^{\infty}\frac{-w_j\lambda^{-j}}{2u}, \tag{14}$$

where w_j is a function of x,t, for all $j=1,2,3,\ldots$. Making use of the above expansion for $T(x,t)$ in Riccati equation (11) and equating the different powers of λ, we obtain a system of linear equations in w_j. Solving them we get

$$w_1=u, \tag{15a}$$

$$w_2=\frac{u_x}{2u}w_1-\frac{w_{1x}}{2}=0, \tag{15b}$$

$$w_3=\frac{u_x}{2u}w_2-\frac{w_{2x}}{2}+\frac{w_1^2}{4}=\frac{u^2}{4}, \tag{15c}$$

$$w_4=\frac{u_x}{2u}w_3-\frac{w_{3x}}{2}+\frac{w_1w_2}{2}=-\frac{uu_x}{8}, \tag{15d}$$

$$w_5=\frac{u_x}{2u}w_4-\frac{w_{4x}}{2}+\frac{w_2^2}{4}+\frac{w_1w_3}{2}=\frac{u(u_{xx}+2u^2)}{16}, \tag{15e}$$

$$w_6=\frac{u_x}{2u}w_5-\frac{w_{5x}}{2}+\frac{w_1w_4}{2}+\frac{w_2w_3}{2}=-\frac{u(u_{xxx}+6uu_x)}{32}, \tag{15f}$$

$$\vdots$$

$$w_j=\frac{u_x}{2u}w_{j-1}-\frac{w_{(j-1)x}}{2}+\frac{1}{4}\sum_{k=1}^{j-2}w_kw_{j-k-1},\quad j\geq 3. \tag{15g}$$

Substituting the expression for $T(x,t)$ given in (14) into eq. (13) and equating the different powers of λ, we can obtain infinitely many conservation laws of the D5oKdV equation (2). For example, the coefficient of λ^{-1} reads as

$$\frac{\partial}{\partial t}\left[-\frac{w_1}{2}\right]=\frac{\partial}{\partial x}\Bigg[8\beta w_5+\frac{4\beta u_x}{u}w_4+\left(4\beta u+2\alpha+\frac{2\beta u_{xx}}{u}\right)w_3+\left(6\beta u_x+\frac{\beta u_{xxx}}{u}+\frac{\alpha u_x}{u}\right)w_2+\left(4\beta u_{xx}+3\beta u^2+\frac{\beta u_{xxxx}}{2u}+\frac{3\beta u_x^2}{u}+\alpha u+\frac{\alpha u_{xx}}{2u}\right)w_1-\frac{g}{2}\Bigg].$$

Using the values of w_j, $j=1,2,\ldots,5$ given in (15) and after simplification we obtain the first member of the infinitely many conservation laws of (2)

$$\frac{\partial}{\partial t}\left[-\frac{u}{2}\right]=\frac{\partial}{\partial x}\left[\beta\left(\frac{u_{xxxx}}{2}+5uu_{xx}+\frac{5u_x^2}{2}+5u^3\right)+\alpha\left(\frac{u_{xx}}{2}+\frac{3u^2}{2}\right)-\frac{g}{2}\right].$$

In other words, the conserved density and its associated flux read as

$$\rho_1 = -\frac{w_1}{2} = -\frac{u}{2}, \tag{16a}$$

$$\begin{aligned} F_1 &= 8\beta w_5 + \frac{4\beta u_x}{u} w_4 \\ &\quad + \left(4\beta u + 2\alpha + \frac{2\beta u_{xx}}{u}\right) w_3 \\ &\quad + \left(6\beta u_x + \frac{\beta u_{xxx}}{u} + \frac{\alpha u_x}{u}\right) w_2 \\ &\quad + \left(4\beta u_{xx} + 3\beta u^2 + \frac{\beta u_{xxxx}}{2u} + \frac{3\beta u_x^2}{u} \right. \\ &\quad \left. + \alpha u + \frac{\alpha u_{xx}}{2u}\right) w_1 - \frac{g}{2} \\ &= \beta\left(\frac{u_{xxxx}}{2} + 5uu_{xx} + \frac{5u_x^2}{2} + 5u^3\right) \\ &\quad + \alpha\left(\frac{u_{xx}}{2} + \frac{3u^2}{2}\right) - \frac{g}{2}. \end{aligned} \tag{16b}$$

Similarly, the coefficient of λ^{-2} (that is, the second member of the infinitely many conservation laws of (2)) reads as

$$\begin{aligned} \frac{\partial}{\partial t}\left[-\frac{w_2}{2}\right] = \frac{\partial}{\partial x}\Bigg[& 8\beta w_6 + \frac{4\beta u_x}{u} w_5 \\ & + \left(4\beta u + 2\alpha + \frac{2\beta u_{xx}}{u}\right) w_4 \\ & + \left(6\beta u_x + \frac{\beta u_{xxx}}{u} + \frac{\alpha u_x}{u}\right) w_3 \\ & + \left(4\beta u_{xx} + 3\beta u^2 + \frac{\beta u_{xxxx}}{2u} + \frac{3\beta u_x^2}{u} \right. \\ & \left. + \alpha u + \frac{\alpha u_{xx}}{2u}\right) w_2 + \frac{g_x}{4u} w_1 - \frac{g_x}{4}\Bigg] \end{aligned} \tag{17a}$$

or

$$\rho_2 = -\frac{w_2}{2} = 0,$$

$$\begin{aligned} F_2 &= 8\beta w_6 + \frac{4\beta u_x}{u} w_5 \\ &\quad + \left(4\beta u + 2\alpha + \frac{2\beta u_{xx}}{u}\right) w_4 \\ &\quad + \left(6\beta u_x + \frac{\beta u_{xxx}}{u} + \frac{\alpha u_x}{u}\right) w_3 \\ &\quad + \left(4\beta u_{xx} + 3\beta u^2 + \frac{\beta u_{xxxx}}{2u} + \frac{3\beta u_x^2}{u} \right. \\ &\quad \left. + \alpha u + \frac{\alpha u_{xx}}{2u}\right) w_2 + \frac{g_x}{4u} w_1 - \frac{g_x}{4} \\ &= 0. \end{aligned} \tag{17b}$$

Proceeding in a similar manner, we can obtain infinitely many conservation laws of the D5oKdV equation (2). For $j \geq 3$, the conserved densities and the associated fluxes are

$$\rho_j = -\frac{w_j}{2}, \tag{18a}$$

$$\begin{aligned} F_j &= 8\beta w_{j+4} + \frac{4\beta u_x}{u} w_{j+3} \\ &\quad + \left(4\beta u + 2\alpha + \frac{2\beta u_{xx}}{u}\right) w_{j+2} \\ &\quad + \left(6\beta u_x + \frac{\beta u_{xxx}}{u} + \frac{\alpha u_x}{u}\right) w_{j+1} \\ &\quad + \left(4\beta u_{xx} + 3\beta u^2 + \frac{\beta u_{xxxx}}{2u} + \frac{3\beta u_x^2}{u} + \alpha u \right. \\ &\quad \left. + \frac{\alpha u_{xx}}{2u}\right) w_j + \frac{g_x}{4u} w_{j-1} + \left(\frac{g}{4} + \frac{g_{xx}}{8u}\right) w_{j-2}, \end{aligned} \tag{18b}$$

for all $j \geq 3$. Here, we note that (18b) is valid only for $j \geq 3$.

Therefore, the D5oKdV equation (2) admits infinitely many conservation laws and hence it is integrable. From the above analysis, we observe that, the deformed function g occurs only on the fluxes, not on the conserved densities of the D5oKdV equation (2). It is appropriate to mention here that the conserved densities of the DKdV equation (1) and the D5oKdV equation (2) remain the same but fluxes vary.

4. Lie symmetry analysis of the D5oKdV equation

In this section, we apply the well-known Lie symmetry analysis [19–24] to the D5oKdV equation (2) and report the results, namely, the Lie point symmetries, symmetry algebra, similarity reductions. We also make an attempt to obtain exact solutions of the D5oKdV equation (2).

4.1 *Lie point symmetries*

Let us assume that the D5oKdV equation (2) is invariant under one-parameter (ϵ) continuous point transformations,

$$\begin{aligned} x^* &= x + \epsilon\xi + O(\epsilon^2), \\ t^* &= t + \epsilon\tau + O(\epsilon^2), \\ u^* &= u + \epsilon\eta + O(\epsilon^2), \\ g^* &= g + \epsilon\psi + O(\epsilon^2), \end{aligned} \tag{19}$$

where ξ, τ, η and ψ are functions of (x, t, u, g) provided any solution $u(x, t)$, $g(x, t)$ satisfies system (2). Then the infinitesimal generator reads as

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u} + \psi\frac{\partial}{\partial g}. \quad (20)$$

The invariant equations of (2) are given by

$$[\eta_t] + \alpha([\eta_{xxx}] + 6u[\eta_x] + 6u_x\eta) + \beta([\eta_{xxxxx}] + 10u[\eta_{xxx}] + 10u_{xxx}\eta + 20u_x[\eta_{xx}] + 20u_{xx}[\eta_x] + 30u^2[\eta_x] + 60u_x u\eta) = \psi_x, \quad (21a)$$

$$[\psi_{xxx}] + 4u[\psi_x] + 4g_x\eta + 2g[\eta_x] + 2u_x\psi = 0. \quad (21b)$$

Here

$$\begin{aligned}
[\eta_x] &= \frac{D\eta}{Dx} - \frac{D\xi}{Dx}u_x - \frac{D\tau}{Dx}u_t,\\
[\eta_t] &= \frac{D\eta}{Dt} - \frac{D\xi}{Dt}u_x - \frac{D\tau}{Dt}u_t,\\
[\psi_x] &= \frac{D\psi}{Dx} - \frac{D\xi}{Dx}g_x - \frac{D\tau}{Dx}g_t,\\
[\eta_{xx}] &= \frac{D[\eta_x]}{Dx} - \frac{D\xi}{Dx}u_{xx} - \frac{D\tau}{Dx}u_{xt},\\
[\eta_{xxx}] &= \frac{D[\eta_{xx}]}{Dx} - \frac{D\xi}{Dx}u_{xxx} - \frac{D\tau}{Dx}u_{xxt},\\
[\psi_{xxx}] &= \frac{D[\psi_{xx}]}{Dx} - \frac{D\xi}{Dx}g_{xxx} - \frac{D\tau}{Dx}g_{xxt},\\
[\eta_{xxxxx}] &= \frac{D[\eta_{xxxx}]}{Dx} - \frac{D\xi}{Dx}u_{xxxxx} - \frac{D\tau}{Dx}u_{xxxxt},
\end{aligned} \quad (22)$$

where

$$\begin{aligned}
\frac{D}{Dx} &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + g_x\frac{\partial}{\partial g} + u_{xx}\frac{\partial}{\partial u_x} + g_{xx}\frac{\partial}{\partial g_x} + u_{xt}\frac{\partial}{\partial u_t} + g_{xt}\frac{\partial}{\partial g_t} + \cdots,\\
\frac{D}{Dt} &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + g_t\frac{\partial}{\partial g} + u_{xt}\frac{\partial}{\partial u_t} + g_{xt}\frac{\partial}{\partial g_t} + u_{tt}\frac{\partial}{\partial u_t} + g_{tt}\frac{\partial}{\partial g_t} + \cdots.
\end{aligned}$$

Making use of the above expressions in invariant equations (21) we get a system of overdetermined PDEs and solving them consistently yield the following infinitesimals:

Case 1: When $\alpha \neq 0$, $\beta = 0$,

$$\xi = c_1 x + c_2, \quad \tau = 3c_1 t + c_3, \quad \eta = -2c_1 u, \quad \psi = -4c_1 g. \quad (23)$$

Case 2: When $\alpha = 0$, $\beta \neq 0$,

$$\xi = c_4 x + c_5, \quad \tau = 5c_4 t + c_6, \quad \eta = -2c_4 u, \quad \psi = -6c_4 g. \quad (24)$$

Table 1. Commutator table for the infinitesimal generators X_1, X_2 and X_3.

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	0	$3X_1$
X_2	0	0	X_2
X_3	$-3X_1$	$-X_2$	0

Case 3: When $\alpha \neq 0$, $\beta \neq 0$, there is no non-trivial infinitesimals

$$\xi = c_7, \quad \tau = c_8, \quad \eta = 0, \quad \psi = 0, \quad (25)$$

where c_k's are arbitrary constants, $k = 1, 2, ..., 8$.

It is appropriate to mention here that, when $c_1 = 0$, infinitesimals (23) is similar to infinitesimals (25). Similarly, when $c_4 = 0$, infinitesimals (24) is similar to infinitesimals (25). Therefore, we omitted cases $c_1 = 0$ and $c_4 = 0$ in Case 1 and Case 2 respectively, for further discussion.

4.2 *Symmetry algebra*

Case 1: When $\alpha \neq 0$, $\beta = 0$, the infinitesimal generator X reads as

$$X = (3c_1 t + c_3)\frac{\partial}{\partial t} + (c_1 x + c_2)\frac{\partial}{\partial x} - 2c_1 u\frac{\partial}{\partial u} - 4\, c_1 g\frac{\partial}{\partial g} \quad (26)$$

which yields the following generators:

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t},\\
X_2 &= \frac{\partial}{\partial x},\\
X_3 &= 3t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2\, u\frac{\partial}{\partial u} - 4g\frac{\partial}{\partial g}.
\end{aligned}$$

From a straightforward calculation, we find commutator table 1 and so the infinitesimal generators X_1, X_2 and X_3 are closed under the Lie bracket and hence they form a three-dimensional symmetry algebra $L^3 = \{X_1, X_2, X_3\}$.

It is appropriate to mention here that from ref. [25], it is known that the symmetry algebra of the KdV equation is four-dimensional. From the above analysis, we note that, the dimension of the symmetry algebra admitted by the KdV equation and the dimension of the symmetry algebra admitted by the DKdV equation (1) are different.

Table 2. Commutator table for the infinitesimal generators X_4, X_5 and X_6.

$[X_i, X_j]$	X_4	X_5	X_6
X_4	0	0	$5X_4$
X_5	0	0	X_5
X_6	$-5X_4$	$-X_5$	0

In this case, the introduction of deformation function reduces the dimension of the symmetry algebra.

Case 2: When $\alpha = 0, \beta \neq 0$, the infinitesimal generator X reads as

$$X = (5c_4 t + c_6)\frac{\partial}{\partial t} + (c_4 x + c_5)\frac{\partial}{\partial x} - 2c_4 u\frac{\partial}{\partial u} - 6c_4 g\frac{\partial}{\partial g} \quad (27)$$

which yields the following generators:

$$X_4 = \frac{\partial}{\partial t},$$

$$X_5 = \frac{\partial}{\partial x},$$

$$X_6 = 5t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u} - 6g\frac{\partial}{\partial g}.$$

From a straightforward calculation, we find commutator table 2 and so the infinitesimal generators X_4, X_5 and X_6 are closed under the Lie bracket and hence they form a three-dimensional symmetry algebra.

From refs [26–28], it is known that, when $\alpha = 0$, $g = 0$, $\beta = 1$, the D5oKdV (that is, the fifth-order Lax equation) equation (2) admits three-dimensional symmetry algebra. From the above analysis, we note that, when $\alpha = 0$, $\beta = 1$, the fifth-order Lax equation and its deformed version (2) admit symmetry algebras with same dimension. In this case, the deformation function does not make any impact on the dimension of the symmetry algebra.

Case 3: When $\alpha \neq 0, \beta \neq 0$, the D5oKdV equation (2) admits two-dimensional symmetry algebra (which is an Abelian one) with infinitesimal generators given by

$$X_7 = \frac{\partial}{\partial t},$$

$$X_8 = \frac{\partial}{\partial x}.$$

4.3 *Similarity reduction*

Case 1: When $\alpha \neq 0, \beta = 0$, the characteristic equation corresponding to the infinitesimal generator X given in (26) with $c_1 \neq 0$ reads as

$$\frac{\mathrm{d}x}{c_1 x + c_2} = \frac{\mathrm{d}t}{3c_1 t + c_3} = \frac{\mathrm{d}u}{-2c_1 u} = \frac{\mathrm{d}g}{-4c_1 g}.$$

Solving this characteristic equation, we can obtain the similarity variable $z_1(x,t)$ and the similarity transformations $U(z_1)$ and $G(z_1)$ given by

$$z_1(x,t) = \frac{x + a_1}{(3t + a_2)^{1/3}}, \quad u(x,t) = \frac{U(z_1)}{(3t + a_2)^{2/3}},$$

$$g(x,t) = \frac{G(z_1)}{(3t + a_2)^{4/3}},$$

where

$$a_1 = \frac{c_2}{c_1}, \quad a_2 = \frac{c_3}{c_1}.$$

Using these transformations along with the similarity variable $z_1(x,t)$ into eq. (2), we find that it reduces into a system of two coupled third-order non-autonomous ordinary differential equations (ODEs)

$$-z_1 U' - 2U + \alpha(U''' + 6UU') = G', \quad (28a)$$

$$G''' + 2U'G + 4UG' = 0. \quad (28b)$$

Here $U' = \mathrm{d}U/\mathrm{d}z_1$, etc.

Case 2: When $\alpha = 0$, $\beta \neq 0$, solving the following characteristic equation corresponding to the infinitesimal generator X given in (27) with $c_4 \neq 0$,

$$\frac{\mathrm{d}x}{c_4 x + c_5} = \frac{\mathrm{d}t}{5c_4 t + c_6} = \frac{\mathrm{d}u}{-2c_4 u} = \frac{\mathrm{d}g}{-6c_4 g}$$

we get the following similarity variable $z_2(x,t)$ and the similarity transformations $U(z_2)$ and $G(z_2)$ given by

$$z_2(x,t) = \frac{x + a_3}{(5t + a_4)^{1/5}}, \quad u(x,t) = \frac{U(z_2)}{(5t + a_4)^{2/5}},$$

$$g(x,t) = \frac{G(z_2)}{(5t + a_4)^{6/5}},$$

where

$$a_3 = \frac{c_5}{c_4}, \quad a_4 = \frac{c_6}{c_4}.$$

Using these transformations along with the similarity variable $z_2(x,t)$ into eq. (2), we find that it reduces into a system of two coupled fifth-order non-autonomous ODEs

$$-z_2 U' - 2U + \beta(U''''' + 10UU''' + 20U'U'' + 30U^2U') = G', \quad (29a)$$

$$G''' + 2U'G + 4UG' = 0. \quad (29b)$$

Here $U' = \mathrm{d}U/\mathrm{d}z_2$, etc.

Case 3: When $\alpha \neq 0, \beta \neq 0$, the characteristic equation reads as

$$\frac{\mathrm{d}x}{c_7} = \frac{\mathrm{d}t}{c_8} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}g}{0}.$$

Solving the above characteristic equation we can obtain the similarity variable $z_3(x,t) = x - \mu t$, where $\mu = c_7/c_8$ and the similarity transformations $U(z_3)$ and $G(z_3)$ which are given below

$$u(x,t) = U(z_3), \quad g(x,t) = G(z_3). \tag{30}$$

Using similarity transformations (30) along with the similarity variable $z_3(x,t)$ into eq. (2), we find that it reduces into a system of two coupled fifth-order ODEs

$$-\mu U' + \alpha(6UU' + U''') + \beta(U''''' + 10UU''' + 20U'U'' + 30U^2U') = G', \tag{31a}$$

$$G''' + 2U'G + 4UG' = 0, \tag{31b}$$

where $U' = \mathrm{d}U/\mathrm{d}z_3$, etc.

It is appropriate to mention here that, from ref. [25], it is known that when $G = 0$, the similarity reduction (28) is of Painlevé type II. To the best of our knowledge similarity reduction (28) is not solvable, in general. However, we checked that the similarity reduction (28) possesses the Painlevé property of ODEs and hence the Ablowitz *et al* [29] conjecture holds and it is expected to be integrable. We also checked that the similarity reductions (29) and (31) possess the Painlevé property of ODEs and hence the ARS conjecture holds and they are expected to be integrable.

4.4 *Exact solutions*

In this subsection, we explore the possibility of constructing solutions of similarity reductions (28), (29) and (31).

Case 1: When $\alpha \neq 0$, $\beta = 0$, the similarity reduction (28) admits the following exact solution:

$$U(z_1) = -\frac{2}{(z_1 - b_1)^2}, \quad G(z_1) = \frac{2b_1}{(z_1 - b_1)^2} \tag{32}$$

so the D5oKdV equation (2) admits the following exact solution:

$$u(x,t) = -\frac{2}{(z_1 - b_1)^2(3t + a_2)^{2/3}}, \tag{33a}$$

$$g(x,t) = \frac{2b_1}{(z_1 - b_1)^2(3t + a_2)^{4/3}}, \tag{33b}$$

where

$$z_1(x,t) = \frac{x + a_1}{(3t + a_2)^{1/3}}, \quad a_1 = \frac{c_2}{c_1}, \quad a_2 = \frac{c_3}{c_1}$$

and b_1 is an arbitrary constant.

Case 2: When $\alpha = 0$, $\beta \neq 0$, the similarity reduction (29) admits the following exact solution:

$$U(z_2) = -\frac{2}{(z_2 - b_2)^2}, \quad G(z_2) = \frac{2b_2}{(z_2 - b_2)^2} \tag{34}$$

and so the D5oKdV equation (2) admits the following exact solution:

$$u(x,t) = -\frac{2}{(z_2 - b_2)^2(5t + a_4)^{2/5}}, \tag{35a}$$

$$g(x,t) = \frac{2\, b_2}{(z_2 - b_2)^2(5t + a_4)^{6/5}}, \tag{35b}$$

where

$$z_2(x,t) = \frac{x + a_3}{(5t + a_4)^{1/5}}, \quad a_3 = \frac{c_5}{c_4}, \quad a_4 = \frac{c_6}{c_4}$$

and b_2 is an arbitrary constant.

Case 3: When $\alpha \neq 0$, $\beta \neq 0$, the similarity reduction (31) admits the following exact solutions:

(i)

$$U(z_3) = \frac{2a_5 + 1}{a_5} - 2\, \tanh^2(z_3), \tag{36a}$$

$$G(z_3) = b_3[1 + a_5\, \tanh^2(z_3)], \tag{36b}$$

where

$$\mu = \frac{2\alpha(2a_5 + 3)}{a_5} + \frac{\beta(16a_5^2 + 40a_5 + 30)}{a_5^2} + \frac{a_5 b_3}{2}.$$

(ii)

$$U(z_3) = a_6 - \frac{2}{z_3^2}, \tag{37a}$$

$$G(z_3) = b_4\left[1 + \frac{1}{a_6 {z_3}^2}\right], \tag{37b}$$

where

$$\mu = \frac{b_4 + 12\alpha a_6^2 + 60\, \beta a_6^3}{2a_6}.$$

So the D5oKdV equation (2) admits the following exact solutions:

(i)

$$u(x,t) = \frac{2a_5 + 1}{a_5} - 2\tanh^2(x - \mu t), \tag{38a}$$

$$g(x,t) = b_3[1 + a_5 \tanh^2(x - \mu t)], \tag{38b}$$

where

$$\mu = \frac{2\alpha(2a_5 + 3)}{a_5} + \frac{\beta(16a_5^2 + 40a_5 + 30)}{a_5^2} + \frac{a_5 b_3}{2}$$

and a_5, b_3 are arbitrary constants.

(ii)

$$u(x,t) = a_6 - \frac{2}{(x - \mu t)^2}, \tag{39a}$$

$$g(x,t) = b_4\left[1 + \frac{1}{a_6\,(x - \mu t)^2}\right], \tag{39b}$$

where

$$\mu = \frac{b_4 + 12\alpha a_6^2 + 60\beta a_6^3}{2a_6}$$

and a_6, b_4 are arbitrary constants.

We can easily see that the exact solutions (38) and (39) are exact solutions of the D5oKdV equation (2), for any arbitrary α, β. When $g = 0$ or $b_1 = 0$, $b_2 = 0$, $b_3 = 0$, $b_4 = 0$, the solutions (33), (35), (38) and (39) respectively, are the exact solutions of 5oKdV equation.

5. Summary and concluding remarks

In this paper, we considered the D5oKdV equation (2) and showed that it admits a Lax representation provided that the deformed function $g(x, t)$ satisfies certain differential constraint, which leads to its integrability. From the obtained Lax representation, we showed that the D5oKdV equation (2) admits infinitely many conservation laws, which guarantees its integrability. We applied the Lie symmetry analysis to the D5oKdV equation (2) and derived its Lie point symmetries, the associated similarity reductions and the exact solutions.

From our analysis, we observed the following:

(i) The DKdV equation and the D5oKdV equation admit the same differential constraint on the deforming function $g(x, t)$.
(ii) The deformed function g occurs only on the fluxes, not on the conserved densities of the D5oKdV equation.
(iii) The conserved densities of the 5oKdV equation and the D5oKdV equation (2) remain the same but fluxes vary.
(iv) The conserved densities of the DKdV equation (1) and the D5oKdV equation (2) remain the same but fluxes vary.
(v) The dimension of the symmetry algebra admitted by KdV equation [25] and the dimension of the symmetry algebra admitted by the DKdV equation (1) are different.
(vi) The fifth-order Lax equation [26–28] (when $\alpha = 0, \beta = 1$) and its deformed version admit symmetry algebras with the same dimension.
(vii) The introduction of deformation function may or may not reduce the dimension of the symmetry algebra. In future, we wish to study other integrable properties of the D5oKdV equation (2).

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