Pramana – J. Phys. (2020) 94:138
https://doi.org/10.1007/s12043-020-02006-8

A generalised short pulse equation: Darboux transformation and exact solutions

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MS received 20 March 2020; revised 12 May 2020; accepted 1 July 2020

Abstract. We construct infinitely many conservation laws of the generalised short pulse equation with the help of its Lax pairs. By a reciprocal transformation, the generalised short pulse equation was transformed to the first negative flow of Sawada–Kotera hierarchy. On the basis of Darboux and reciprocal transformations, we obtain some exact solutions of the generalised short pulse equation.

Keywords. Generalised short pulse equation; conservation laws; Darboux transformation; exact solutions.

PACS Nos 02.30.Ik; 04.20.Jb; 05.45.Yv

1. Introduction

The integrable short pulse equation

$$u_{xt} = u + \tfrac{1}{6}(u^3)_{xx} \tag{1}$$

was first discovered in differential geometry to describe pseudospherical surfaces [1,2]. Later, it was found again in physics to model the propagation of ultra-short infrared light pulses in silica optical fibres [3,4]. After that, various aspects of the short pulse equation have been studied, including Lax pair [5], bi-Hamiltonian structure [6], solitons [7,8], generalisations [9,10] and many others.

Recently, by considering the classification of integrable generalised short pulse (GSP) equations with quadratic and cubic nonlinear terms, Hone *et al* [11] found two new integrable GSP equations. One of them is

$$u_{xt} = u + (u^2 - 4u^2 u_x)_x \tag{2}$$

which was shown to have reciprocal link with the first negative flow of Sawada–Kotera hierarchy ($(SK)_{-1}$). Using the direct method, Matsuno [12] obtained parametric solutions of this GSP equation. As is well-known, Darboux transformation [13,14] is one of the most direct and powerful methods for finding the solutions of nonlinear integrable systems. The purpose of the present paper is to apply the reciprocal and Darboux transformations to study the exact solutions of the GSP equation (2) under the zero boundary condition as $|x| \to \infty$.

The paper is organised as follows. In §2, we establish infinite conservation laws of the GSP equation. In §3, resorting to the reciprocal transformation, we relate the GSP equation to $(SK)_{-1}$ equation. In §4, by virtue of the Darboux transformation of $(SK)_{-1}$ equation, reciprocal transformation and the asymptotic behaviours of wave functions, we arrive at exact solutions of the GSP equation. Section 5 gives conclusions.

2. Infinite conservation laws

In this section, we shall construct infinite conservation laws of the GSP equation. To this end, we present the 3×3 matrix Lax pairs of the GSP equation (2) [11]

$$\psi_x = \begin{pmatrix} 0 & 1-2u_x & 0 \\ 0 & 0 & 1+4u_x \\ \lambda(1-2u_x) & 0 & 0 \end{pmatrix} \psi,$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}, \tag{3}$$

Published online: 21 September 2020

$$\psi_t=\begin{pmatrix} 0 & -4u^2(1-2u_x) & 1/3\lambda \\ 1/3 & 2u & -4u^2(1+4u_x) \\ -4\lambda u^2(1-2u_x) & 1/3 & -2u \end{pmatrix}\psi. \tag{4}$$

Defining

$$\mu = \frac{\psi_2}{\psi_1}, \quad \nu = \frac{\psi_3}{\psi_1},$$

it follows from eqs (3) and (4) that μ and ν satisfy equations

$$\mu_x - (1+4u_x)\nu + (1-2u_x)\mu^2 = 0,$$
$$\nu_x + (1-2u_x)\mu\nu - \lambda(1-2u_x) = 0. \tag{5}$$

Substituting the following ansatzes

$$\mu = \sum_{j=-1}^{\infty} \mu_j \zeta^j, \quad \nu = \sum_{j=-2}^{\infty} \nu_j \zeta^j, \quad \lambda = \zeta^{-3}, \tag{6}$$

into (5), we have

$$\mu_{-1} = \left(\frac{1+4u_x}{1-2u_x}\right)^{1/3},$$
$$\nu_{-2} = \left(\frac{1+4u_x}{1-2u_x}\right)^{-1/3},$$
$$\mu_0 = 0, \quad \nu_{-1} = \frac{\mu_{-1,x}}{1+4u_x},$$
$$\mu_{j+1} = \frac{1}{2}(1-2u_x)^{-2/3}(1+4u_x)^{-1/3} \times \left[-\mu_{j,x} + (1+4u_x)\nu_j - (1-2u_x)\sum_{i=0}^{j}\mu_i\mu_{j-i}\right],$$
$$\nu_j = (1-2u_x)^{-2/3}(1+4u_x)^{-1/3} \times \left[-\nu_{j-1,x} - (1-2u_x)\sum_{i=0}^{j+1}\mu_i\nu_{j-1-i}\right],$$
$$j \geq -1. \tag{7}$$

If

$$\rho = (\ln\psi_1)_x = (1-2u_x)\mu,$$
$$\theta = (\ln\psi_1)_t = -4u^2(1-2u_x)\mu + \frac{1}{3\lambda}\nu, \tag{8}$$

it is easy to get the conservation law formula

$$\rho_t = \theta_x. \tag{9}$$

In view of (6) and (8), one infers that

$$\rho = \sum_{j=-1}^{\infty} \rho_j \zeta^j, \quad \theta = \sum_{j=-1}^{\infty} \theta_j \zeta^j,$$

in which the conserved densities ρ_j and fluxes θ_j are given by

$$\rho_{-1} = (1-2u_x)^{2/3}(1+4u_x)^{1/3},$$
$$\theta_{-1} = -4u^2\rho_{-1},$$
$$\rho_0 = \theta_0 = 0,$$
$$\rho_j = (1-2u_x)\mu_j,$$
$$\theta_j = -4u^2(1-2u_x)\mu_j + \frac{1}{3}\nu_{j-3}, \ j \geq 1.$$

3. A reciprocal transformation

In this section, we shall introduce a reciprocal transformation which relates the GSP equation (2) to the $(SK)_{-1}$ equation.

The first conservation law of GSP equation in (9)

$$p_t = -4(u^2 p)_x, \quad p = \rho_{-1},$$

allows us to introduce a reciprocal transformation

$$\mathrm{d}y = p\,\mathrm{d}x - 4u^2 p\,\mathrm{d}t, \quad \mathrm{d}\tau = \mathrm{d}t, \tag{10}$$

from which we have

$$\partial_x = p\partial_y, \quad \partial_t = \partial_\tau - 4u^2 p\partial_y. \tag{11}$$

Applying the reciprocal transformation (10) to eqs (3) and (4), we get

$$\psi_y = \begin{pmatrix} 0 & W & 0 \\ 0 & 0 & W^{-2} \\ \lambda W & 0 & 0 \end{pmatrix}\psi, \tag{12}$$

$$\psi_\tau = \begin{pmatrix} 0 & 0 & 1/3\lambda \\ 1/3 & 2u & 0 \\ 0 & 1/3 & -2u \end{pmatrix}\psi \tag{13}$$

with

$$W = p^{-1} - 2u_y, \quad W^{-2} = p^{-1} + 4u_y.$$

On the other hand, under the reciprocal transformation (10), the GSP equation was transformed to the so-called associated GSP (AGSP) equation

$$W_\tau + 2uW = 0 \tag{14}$$

which coincides with the compatibility condition of (12) and (13): $\psi_{y\tau} = \psi_{\tau y}$. We recast the matrix Lax pairs (12) and (13) into the scalar form

$$\phi_{yyy} + U\phi_y = \lambda\phi, \tag{15}$$

$$\phi_\tau = \frac{1}{3\lambda}(W\phi_{yy} - W_y\phi_y), \tag{16}$$

where $\phi = \psi_1$, $U = -W_{yy}/W$. A direct calculation shows that the compatibility condition of (15) and (16) gives rise to the $(SK)_{-1}$ equation

$$U_\tau + W_y = 0, \quad W_{yy} + UW = 0. \tag{17}$$

4. Darboux transformation and exact solutions

In this section, we shall derive exact solutions of the GSP equation with the help of the Darboux transformation of $(SK)_{-1}$ equation and the reciprocal transformation.

Assume that ϕ and $f_j (1 \le j \le N)$ satisfy (15) and (16) with initial values $U = 0$, $W = 1$ and spectral parameters λ and λ_j, respectively. According to [15,16], the Darboux transformation for the $(SK)_{-1}$ equation (17) reads as

$$U[N] = 6[\ln P(f_1, f_2, \ldots, f_N)]_{yy},$$
$$W[N] = 1 - 6[\ln P(f_1, f_2, \ldots, f_N)]_{y\tau},$$
$$\phi[N] = \frac{P(f_1, f_2, \ldots, f_N, \phi)}{P(f_1, f_2, \ldots, f_N)}. \tag{18}$$

Here $P(f_1, f_2, \ldots, f_n)$, if n is even, denotes the Pfaffian of the $n \times n$ skew-symmetric matrix A with (i, j)th entry

$$a_{ij} = \int^y (f_{i,y} f_j - f_i f_{j,y})\, \mathrm{d}y,$$

in which the integral constants are identified to be zero. That is

$$P(f_1, f_2, \ldots, f_n) = \sum_\sigma \operatorname{sgn}(\sigma) a_{\sigma(1)\sigma(2)} \cdots a_{\sigma(n-1)\sigma(n)},$$

where the sum is over permutations of $\{1, \ldots, n\}$ that satisfy

$$\sigma(1) < \sigma(2), \sigma(3) < \sigma(4), \ldots,$$
$$\sigma(n-1) < \sigma(n); \quad \sigma(1) < \sigma(3) < \cdots < \sigma(n-1).$$

If n is odd, $P(f_1, f_2, \ldots, f_n)$ is defined as $P(f_1, f_2, \ldots, f_n, 1)$.

Let α_j, β_j and $-(\alpha_j + \beta_j)$ be solutions of $r^3 = \lambda_j$,

$$f_j = \mathrm{e}^{\frac{\gamma_j + \delta_j}{2}} (\mathrm{e}^{\xi_j} + \kappa_j \mathrm{e}^{-\xi_j}),$$
$$\gamma_j = \alpha_j y + \frac{1}{3\alpha_j}\tau + \gamma_{j0},$$
$$\delta_j = \beta_j y + \frac{1}{3\beta_j}\tau + \delta_{j0},$$
$$\xi_j = \frac{p_j}{2}\left(y + \frac{1}{p_j^2}\tau\right) + \xi_{j0},$$
$$p_j = \alpha_j - \beta_j,$$

κ_j be arbitrary constants. Here we can limit $\mathrm{Re}(\lambda_j) = 0$ in order to obtain the real solutions of the GSP equation. Based on the above preparation, the following theorem holds.

Theorem 1. *The GSP equation (2) admits exact solutions represented by*

$$u = -\frac{1}{2}(\ln W[N])_\tau, \tag{19}$$
$$x = -(\ln W[N])_\tau + \phi_2[N] + \text{const.}, \tag{20}$$

where

$$\phi_2[N] = \frac{P(f_1, f_2, \ldots, f_N, y)}{P(f_1, f_2, \ldots, f_N)}.$$

Proof. Equation (19) can be easily obtained from the AGSP equation (14). Taking $U = 0$, $\lambda = 0$ in eq. (15), one obtains

$$\phi_{yyy} = 0,$$

which has three fundamental solutions, $1, y, y^2$. Using Darboux transformation (18), we get three fundamental solutions of (15) with $\lambda = 0$

$$\phi_1[N] = \frac{P(f_1, f_2, \ldots, f_N, 1)}{P(f_1, f_2, \ldots, f_N)},$$
$$\phi_2[N] = \frac{P(f_1, f_2, \ldots, f_N, y)}{P(f_1, f_2, \ldots, f_N)},$$
$$\phi_3[N] = \frac{P(f_1, f_2, \ldots, f_N, y^2)}{P(f_1, f_2, \ldots, f_N)},$$

whose linear combination yields its general solution ϕ. On the other hand, rewriting (3) into scalar form and taking $\lambda = 0$, we have

$$\phi_{xxx} - 3\frac{p_x}{p}\phi_{xx} + \frac{2}{1 - 2u_x}\left(u_{xxx} - 3u_{xx}\frac{p_x}{p}\right)\phi_x = 0. \tag{21}$$

Note that $x - 2u$ is a solution of (21), and so it can be represented as

$$x - 2u = c_1\phi_1[N] + c_2\phi_2[N] + c_3\phi_3[N],$$

where c_1, c_2 and c_3 are independent of x. Combining asymptotic behaviours: $x - 2u \sim x$, $\phi_1[N] \sim c (c = 0 \text{ or } 1)$, $\phi_2[N] \sim y$, $\phi_3[N] \sim y^2$ as $x, y \to \pm\infty$ with reciprocal transformation (10), one concludes that $c_2 = 1$, $c_3 = 0$. Consequently, we have

$$x - 2u = \phi_2[N] + \text{const.},$$

which together with (19) proves (20). □

Example 1. For $N = 1$, started with $U = 0$, $W = 1$, and spectral parameter $\lambda = 0$, the Darboux transformation (18) permits us to arrive at

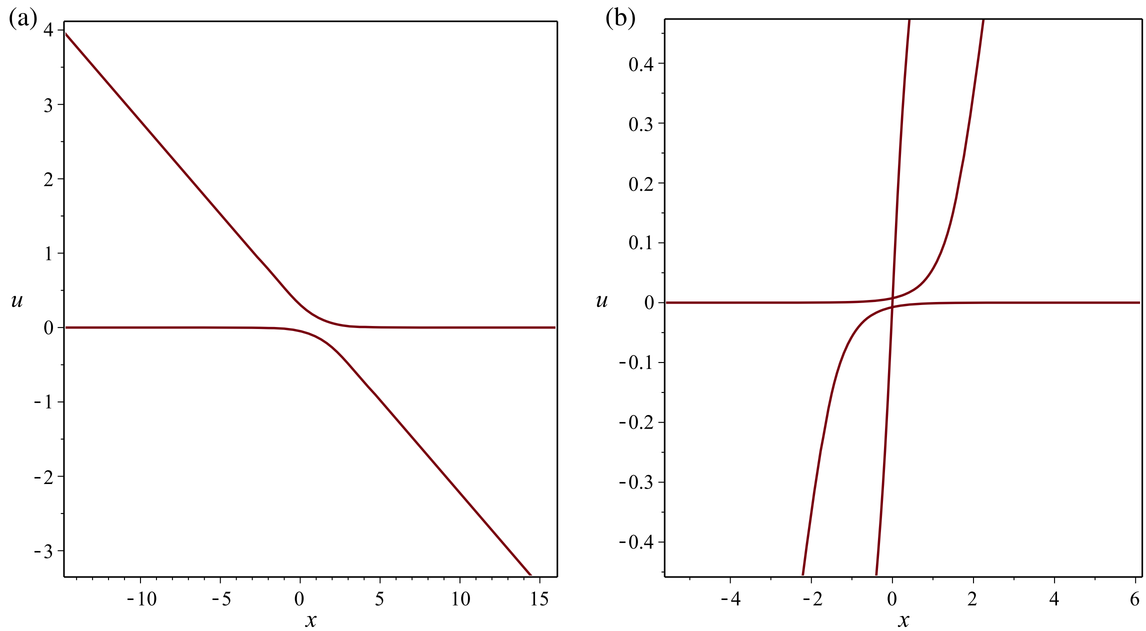

Figure 1. (**a**) The solution u when $\kappa_1 = -3$, $p = 1$, $\xi_{10} = 0$, $t = 0$ and (**b**) the solution u when $\kappa_1 = 1$, $p = 1$, $\xi_{10} = 0$, $t = 0$.

$$U[1]=\frac{6\kappa_1 p_1^2}{(e^{\xi_1}+\kappa_1 e^{-\xi_1})^2},\quad W[1]=\frac{e^{2\xi_1}+\kappa_1^2 e^{-2\xi_1}-4\kappa_1}{(e^{\xi_1}+\kappa_1 e^{-\xi_1})^2},$$
$$\phi_1[1]=1,\quad \phi_2[1]=y-\frac{3(e^{\xi_1}-\kappa_1 e^{-\xi_1})}{p_1(e^{\xi_1}+\kappa_1 e^{-\xi_1})}-\frac{p_1^2}{3\lambda_1},$$
$$\phi_3[1]=y^2-\left[2\frac{p_1^2}{3\lambda_1}+\frac{6(e^{\xi_1}-\kappa_1 e^{-\xi_1})}{p_1(e^{\xi_1}+\kappa_1 e^{-\xi_1})}\right]y +\frac{2p_1(e^{\xi_1}-\kappa_1 e^{-\xi_1})}{\lambda_1(e^{\xi_1}+\kappa_1 e^{-\xi_1})}+\frac{6}{p_1^2},$$

which, according to Theorem 1, yields an exact solution of GSP equation

$$\begin{aligned} u &= \frac{-3\kappa_1(e^{\xi_1}-\kappa_1 e^{-\xi_1})}{p_1(e^{\xi_1}+\kappa_1 e^{-\xi_1})(e^{2\xi_1}+\kappa_1^2 e^{-2\xi_1}-4\kappa_1)},\\ x &= y-\frac{3(e^{\xi_1}-\kappa_1 e^{-\xi_1})^3}{p_1(e^{\xi_1}+\kappa_1 e^{-\xi_1})(e^{2\xi_1}+\kappa_1^2 e^{-2\xi_1}-4\kappa_1)}\\ &\quad +\text{const.} \end{aligned} \tag{22}$$

Remark.
1. If $\kappa_1 < 0$, u has one pole, denoted as ξ^0, which means u admits two branches. As $\xi_1 \to \xi^0$,

$$u=\frac{\kappa_1(x-y+\text{const.})}{(e^{\xi_1}-\kappa_1 e^{-\xi_1})^2}\to -\frac{x}{4}-\frac{t}{4p_1^2}+\text{const.}$$

Actually,

$$u=-\frac{x}{4}-\frac{t}{4p_1^2}+\text{const.}$$

is a solution of GSP equation (2).
2. If $\kappa_1 > 0$, u has two poles ξ^1 and ξ^2, then there are three branches for u. As $\xi_1 \to \xi^1$ (or ξ^2),

$$u\to\frac{x}{2}+\frac{t}{2p_1^2}+\text{const.}$$

Indeed this is also a solution of GSP equation (2) (see [12] for more details about solutions of the GSP equation). The profiles of u are given in figure 1.

Example 2. For $N = 2$, a direct calculation gives

$$U[2]=\frac{g_0}{g^2},\quad W[2]=\frac{g_1}{g^2},$$
$$\phi_1[2]=0,$$
$$\phi_2[2]=y-\frac{6}{g}\left[\left(\frac{1}{p_1}+\frac{1}{p_2}\right)e^{2(\xi_1+\xi_2)} +\frac{k_2}{p_1}e^{2\xi_1}+\frac{k_1}{p_2}e^{2\xi_2}\right]-2\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right),$$
$$\begin{aligned}\phi_3[2]=2y\phi_2[2]-y^2+\frac{4}{g}\Bigg[&\left(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right)^2 e^{2\xi_1+2\xi_2}\\ &+k_2\left(\frac{1}{\alpha_1}+\frac{1}{\beta_2}\right)^2 e^{2\xi_1}\\ &+k_1\left(\frac{1}{\alpha_2}+\frac{1}{\beta_1}\right)^2 e^{2\xi_2}\\ &+k_1k_2\left(\frac{1}{\beta_1}+\frac{1}{\beta_2}\right)^2 h\Bigg],\end{aligned} \tag{23}$$

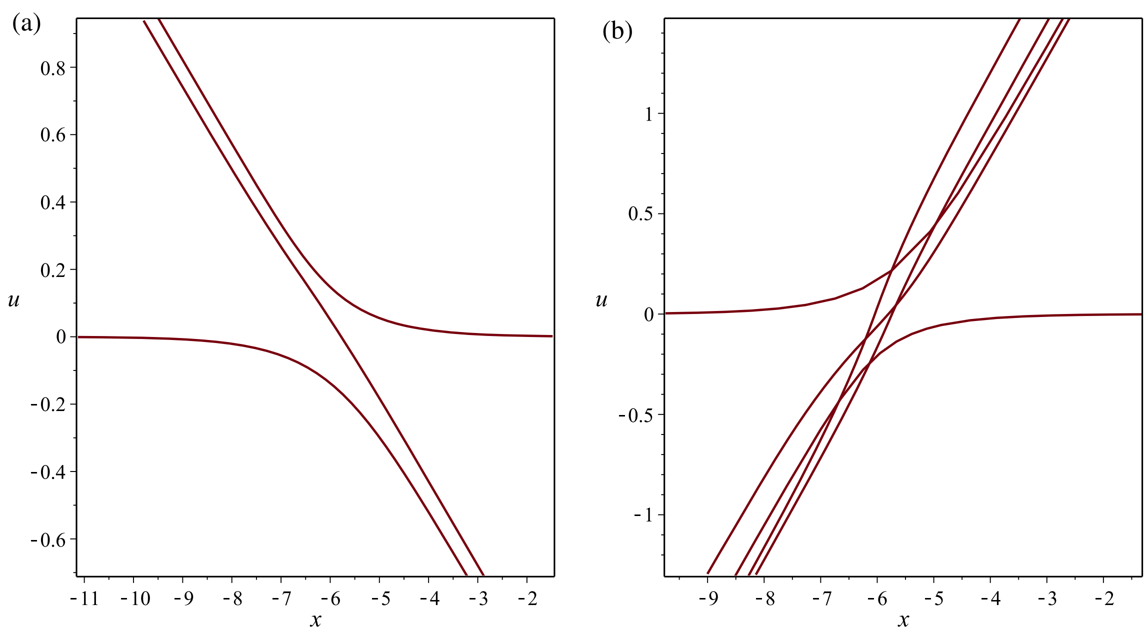

Figure 2. (**a**) The solution u when $k_1 = -1$, $k_2 = -6$, $p_1 = 1$, $p_2 = 2$, $\xi_{10} = 0$, $\xi_{20} = 0$, $t = 0$ and (**b**) the solution u when $k_1 = 1$, $k_2 = 12$, $p_1 = 1$, $p_2 = 2$, $\xi_{10} = 0$, $\xi_{20} = 0$, $t = 0$.

where

$$g = e^{2(\xi_1+\xi_2)} + k_1 e^{2\xi_2} + k_2 e^{2\xi_1} + k_1 k_2 h,$$

$$h = \frac{(p_1-p_2)^2(p_1^2+p_2^2-p_1p_2)}{(p_1+p_2)^2(p_1^2+p_2^2+p_1p_2)},$$

$$\begin{aligned} g_0 &= 6k_1 p_1^2 e^{2\xi_1+4\xi_2} + 6k_2 p_2^2 e^{4\xi_1+2\xi_2} \\ &\quad + 6k_1k_2[h(p_1+p_2)^2 + (p_1-p_2)^2]e^{2\xi_1+2\xi_2} \\ &\quad + 6k_1^2 k_2 p_2^2 h e^{2\xi_2} + 6k_1 k_2^2 p_1^2 h e^{2\xi_1}, \end{aligned}$$

$$\begin{aligned} g_1 &= e^{4\xi_1+4\xi_2} - 4k_2 e^{4\xi_1+2\xi_2} \\ &\quad - 4k_1 e^{2\xi_1+4\xi_2} + k_2^2 e^{4\xi_1} + k_1^2 e^{4\xi_2} \\ &\quad - 4k_1 k_2^2 h e^{2\xi_1} - 4k_1^2 k_2 h e^{2\xi_2} \\ &\quad + k_1 k_2 \left[6\left(\frac{p_1}{p_2} + \frac{p_2}{p_1}\right) \right. \\ &\quad \left. \times (1-h) - 10(1+h) \right] e^{2\xi_1+2\xi_2} + k_1^2 k_2^2 h^2, \end{aligned}$$

$$k_1 = \kappa_1 \frac{(\beta_1-\alpha_2)(\alpha_1+\alpha_2)}{(\beta_1+\alpha_2)(\alpha_1-\alpha_2)},$$

$$k_2 = \kappa_2 \frac{(\alpha_1-\beta_2)(\alpha_1+\alpha_2)}{(\alpha_1+\beta_2)(\alpha_1-\alpha_2)}. \tag{24}$$

Consequently, we arrive at another exact solution of the GSP equation

$$\begin{aligned} u &= \frac{g_\tau}{g} - \frac{1}{2}\frac{g_{1,\tau}}{g_1}, \\ x &= y - \frac{6}{g}\left[\left(\frac{1}{p_1} + \frac{1}{p_2}\right) e^{2(\xi_1+\xi_2)} \right. \\ &\quad \left. + \frac{k_2}{p_1} e^{2\xi_1} + \frac{k_1}{p_2} e^{2\xi_2} \right] + \frac{2g_\tau}{g} - \frac{g_{1,\tau}}{g_1} + \text{const.} \end{aligned} \tag{25}$$

The profiles of u are given in figure 2.

5. Conclusion

In this paper, we first established infinitely many conservation laws of the GSP equation by considering its Lax pairs. Then the reciprocal transformation, which was constructed from the first conservation law, related the GSP equation to the first negative flow of Sawada–Kotera hierarchy ($(SK)_{-1}$). With the help of Darboux transformation of $(SK)_{-1}$ and the reciprocal transformation, we arrived at some parametric solutions of GSP equation. This paper showed an effective technique for using Darboux transformation to obtain the solutions of those integrable systems whose Darboux transformations cannot be constructed directly.

Acknowledgements

This work was supported by National Natural Science Foundation of China (Project Nos 11871232, 11401230), Cultivation Program for Outstanding Young Scientific talents of the Higher Education Institutions of Fujian Province in 2015, Promotion Program for Young and Middle-aged Teacher in Science and Technology Research of Huaqiao University (Project No. ZQN-PY301), Program for Innovative Research Team in Science and Technology in Fujian Province University, Quanzhou High-Level Talents Support Plan under Grant 2017ZT012 and Subsidized Project for Postgraduates' Innovative Fund in Scientific Research of Huaqiao University.

References

[1] R Beals, M Rabelo and K Tenenblat, *Stud. Appl. Math.* **81**, 125 (1989)
[2] M L Rabelo, *Stud. Appl. Math.* **81**, 221 (1989)
[3] T Schäfer and C E Wayne, *Physica D* **196**, 90 (2004)
[4] Y Chung, C K R T Jones, T Schäfer and C E Wayne, *Nonlinearity* **18**, 1351 (2005)
[5] A Sakovich and S Sakovich, *J. Phys. Soc. Jpn.* **74**, 239 (2005)
[6] J C Brunelli, *Phys. Lett. A* **353**, 475 (2006)
[7] Y Matsuno, *J. Phys. Soc. Jpn.* **76**, 084003 (2007)
[8] E J Parkes, *Phys. Lett. A* **374**, 4321 (2010)
[9] B F Feng, *J. Phys. A* **45**, 085202 (2012)
[10] Y Matsuno, *J. Math. Phys.* **52**, 123702 (2011)
[11] A N W Hone, V Novikov and J P Wang, *Lett. Math. Phys.* **108**, 927 (2018)
[12] Y Matsuno, *J. Phys. A* **53**, 105202 (2020)
[13] V B Matveev and M A Salle, *Darboux transformations and solitons* (Springer, Berlin, 1991)
[14] C H Gu, H S Hu and Z X Zhou, *Darboux transformations in integrable systems: Theory and their applications to geometry* (Springer, Dordrecht, 2005)
[15] C Athorne and J J C Nimmo, *Inverse Problems* **7**, 645 (1991)
[16] S B Leble and N V Ustinov, *Inverse Problems* **10**, 617 (1994)