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On the solitary wave solutions of modified Benjamin–Bona–Mahony equation for unidirectional propagation of long waves

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Abstract. In this article, generalised Kudryashov technique has been implemented to construct new soliton solutions of modified Benjamin–Bona–Mahony (mBBM) equation. Also, the optimal homotopy asymptotic method (OHAM) has been employed to estimate the numerical solution. The solutions thus acquired by the above methods are illustrated graphically. The techniques considered here are efficacious, plausible and can be employed in mathematical physics to compute new exact and numerical solutions of NPDEs.

Keywords. Nonlinear evolution equation; modified Benjamin–Bona–Mahony equation; generalised Kudryashov method; optimal homotopy asymptotic method; soliton solution.

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1. Introduction

With the advancement of science and technology, nonlinear evolutionary equation (NLEE) modelling has come a long way in the field of science related to mathematical physics, engineering, fluid dynamics, quantum mechanics, geological studies etc. Various aspects of nonlinear evolutionary equations are studied by scientists and researchers [1–6]. Due to the remarkable study of nonlinear evolutionary PDEs in recent years, it was found that many equations possess special solutions that retain a pulse-like shape and has constant velocity called solitons. In the study of nonlinear physical phenomena, the analysis of travelling wave solutions for nonlinear PDEs also imparts a significant role.

One such NLEE is the (1+1)-dimensional generalised modified Benjamin–Bona–Mahony (mBBM) equation [7], considered as follows:

$$u_t + u_x + au^n u_x + u_{xxt} = 0. \tag{1.1}$$

When $n = 1$, the equation is termed as the BBM equation, given by

$$u_t + u_x + auu_x + u_{xxt} = 0. \tag{1.2}$$

Here a is an arbitrary constant, $u(x, t)$, the dependent variable, depends on the spatial variable (x) and temporal variable (t) respectively. The BBM equation was first studied by Benjamin *et al* [8] in 1972 as an enhancement to the KdV equation, to eliminate the problem with small amplitudes which showed that the waves propagated in the negative x direction, denying the proposition that waves only progress in positive x direction. Thus, the BBM equation imprecisely describes unidirectional propagation of long waves in certain nonlinear dispersive media, as discussed in [9].

When $n = 2$, eq. (1.1) is termed as the mBBM equation, which is an advancement of BBM equation to model long waves of small amplitude in $(1 + 1)$ dimensions, given by [10]

$$u_t + u_x + au^2 u_x + u_{xxt} = 0. \tag{1.3}$$

Equation (1.3) was initially acquired to depict the estimation of surface long waves in nonlinear dispersive media. The mBBM equation can be utilised to construct surface waves of long wavelengths in liquids, hydromagnetic waves in cold plasma and acoustic gravity waves in compressible fluids, to characterise acoustic waves in anharmonic crystals etc. [8,11].

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Various iterative and analytical methods such as improved extended (G'/G)-expansion method [10], auxiliary equation method [12], F-expansion method [13], solitary wave ansatz [7] and exp-function method [14] have been implemented to acquire the solution of mBBM equation. In this paper, the aim is to implement Kudryashov method and OHAM to exhibit the ability of the proposed techniques to solve NLEEs, and hence it can be applied to solve other NLEEs.

The manuscript is structured as follows. Introduction to mBBM equation is outlined in §1. In §2, the description of the generalised Kudryashov method and basic idea of OHAM technique is discussed. Next, §3 and 4 respectively applied the Kudryashov method and OHAM for solving the mBBM equation. The convergence of OHAM is discussed in §5. Section 6 exhibits the results and §7 gives conclusion.

2. Basic idea of the proposed techniques

2.1 *Description of the generalised Kudryashov method*

The fundamental steps of the generalised Kudryashov method are as follows:

Step 1: Consider the following general nonlinear partial differential equation of the form

$$F(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, u_{xt}, u_{xxx}, u_{xxt}, \ldots) = 0, \tag{2.1}$$

where F is a polynomial in $u(x, y, z, t, \ldots)$ and its partial derivatives, comprising its nonlinear terms and the highest-order derivative.

Step 2: Employ the travelling wave transformation [15, 16] in eq. (2.1),

$$u(x, y, z, t, \ldots) = u(\xi, X),$$
$$\xi = kx + ny + mz + \omega t + \cdots, \tag{2.2}$$

where k, n, m, and ω are arbitrary constants.

Thus eq. (2.1) reduces to an ODE of the form

$$F(u, \omega u_\xi, k u_\xi, n u_\xi, m u_\xi, k^2 u_{\xi\xi}, n^2 u_{\xi\xi}, m^2 u_{\xi\xi}, k\omega u_{\xi\xi}, k^3 u_{\xi\xi\xi}, k^2 \omega u_{\xi\xi\xi}, \ldots) = 0. \tag{2.3}$$

Step 3: Infer that the solution of the reduced equation has the following form:

$$u(\xi, X) = \sum_{i=0}^{N} a_i X^i(\xi), \tag{2.4}$$

where $a_i, i = 0, 1, 2, \ldots, N$ $(a_i \neq 0)$ are constants to be determined. Here highest-order derivative and the nonlinear term in eq. (2.3) are equated to obtain the value of N and the function X is of the form,

$$X(\xi) = \frac{1}{1 + e^{\xi}},$$

being the solution of the equation

$$X_\xi(\xi) = X^2(\xi) - X(\xi). \tag{2.5}$$

Step 4: Substituting eq. (2.4) into eq. (2.3) and then equating coefficients of $X^i (i = 0, 1, 2, \ldots)$ to 0, we attain a system of equations. The solution of eq. (2.1) can be acquired by solving the obtained system of equations.

2.2 *Optimal homotopy asymptotic method (OHAM)*

The NPDEs can be solved by various perturbation methods. Although the techniques are simple, the problem lies in fixing small parameters as there is no proper way of its selection. This problem was solved by the OHAM due to the ease of its convergence control parameters. As opposed to other perturbation methods, OHAM conceptualises a union between computation algorithm and HPM aimed to optimally control the convergence of the solution by introducing an additional function $H(p)$. Presently, this method has been employed to various problems in the field of engineering and science [17–19].

The basic concept of OHAM can be obtained in refs [20–22]. To illustrate the basic idea of OHAM, we consider the following nonlinear differential equation:

$$A(u(x, t)) + g(x, t) = 0, \quad x \in \Omega \tag{2.6}$$

with the boundary conditions

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, \quad x \in \Gamma, \tag{2.7}$$

where A is a differential operator, B is a boundary operator, $u(x, t)$ is an unknown function, Γ is the boundary of the domain Ω and $g(x, t)$ is a known analytic function.

The operator A can be decomposed as

$$A = L + N$$

where L is a linear operator and N is a nonlinear operator.

We construct a homotopy $\varphi(x, t; p) : \Omega \times [0, 1] \to \Re$ which satisfies

$$H(\varphi(x, t; p), p) \equiv (1 - p)[L(\varphi(x, t; p)) + g(x, t)] - H(p)\,[A(\varphi(x, t; p)) + g(x, t)] = 0, \tag{2.8}$$

where $p \in [0, 1]$ is an embedding parameter, $H(p)$ is a non-zero auxiliary function for $p \neq 0$ and $H(0) = 0$. When $p = 0$ and 1, we have $\varphi(x, t; 0) = u_0(x, t)$ and $\varphi(x, t; 1) = u(x, t)$ respectively. Thus, as p varies from 0 to 1, the solution $\varphi(x, t; p)$ approaches from $u_0(x, t)$ to $u(x, t)$.

Here $u_0(x, t)$ is obtained from eq. (2.8) and eq. (2.7) with $p = 0$ yields

$$L(\varphi(x, t; 0)) + g(x, t) = 0,$$
$$B\left(u_0, \frac{\partial u_0}{\partial t}\right) = 0. \tag{2.9}$$

The auxiliary function $H(p)$ is chosen in the form

$$H(p) = C_1 p + C_2 p^2 + C_3 p^3 + \cdots, \tag{2.10}$$

where $C_1, C_2, C_3, \ldots$ are convergence control parameters to be determined. To get an approximate solution, $\tilde{\varphi}(x, t; C_1, C_2, C_3, \ldots)$ is expanded in a series about p as

$$\tilde{\varphi}(x, t; p, C_1, C_2, C_3, \ldots) = u_0(x, t) + \sum_{i=1}^{\infty} u_i(x, t, C_1, C_2, C_3, \ldots)\, p^i. \tag{2.11}$$

Substituting eq. (2.11) in eq. (2.8) and equating the coefficients of like powers of p, we shall have the following equations:

$$L(u_1(x, t) + g(x, t)) = C_1 N_0(u_0(x, t)),$$
$$B\left(u_1, \frac{\partial u_1}{\partial t}\right) = 0. \tag{2.12}$$

$$L(u_2(x, t)) - L(u_1(x, t)) = C_2 N_0(u_0(x, t)) + C_1 (L(u_1(x, t)) + N_1(u_0(x, t), u_1(x, t))),$$
$$B\left(u_2, \frac{\partial u_2}{\partial t}\right) = 0 \tag{2.13}$$

and hence the general governing equations for $u_j(x, t)$ is given by

$$L\big(u_j(x, t)\big) = L\big(u_{j-1}(x, t)\big) + C_j N_0(u_0(x, t)) + \sum_{i=1}^{j-1} C_i \left[L\left(u_{j-i}(x, t)\right) + N_{j-i}\left(u_0(x, t), \ldots, u_{j-1}(x, t)\right)\right]; \quad j = 2, 3, \ldots \tag{2.14}$$

where $N_j\left(u_0(x, t), \ldots, u_j(x, t)\right)$ is the coefficient of p^j in the expansion of $N(\varphi(x, t; p))$ about the embedding parameter p and

$$N(\varphi(x, t; p, C_1, C_2, C_3, \ldots)) = N_0(u_0(x, t)) + \sum_{j=1}^{\infty} N_j\left(u_0, u_1, \ldots, u_j\right) p^j. \tag{2.15}$$

It is observed that the convergence of series (2.11) depends upon the convergence control parameters $C_1, C_2, C_3, \ldots$. The approximate solution of eq. (2.6) can be written in the following form:

$$\tilde{u}(x, t; C_1, C_2, C_3, \ldots) = u_0(x, t) + \sum_{j=1}^{n-1} u_j(x, t, C_1, C_2, C_3, \ldots). \tag{2.16}$$

Substituting eq. (2.16) in eq. (2.6), we get the following expression for the residual

$$R_n(x, t; C_1, C_2, C_3, \ldots) = L(\tilde{u}(x, t; C_1, C_2, C_3, \ldots)) + N(\tilde{u}(x, t; C_1, C_2, C_3, \ldots)) + g(x, t). \tag{2.17}$$

If $R_n(x, t; C_1, C_2, C_3, \ldots) = 0$, then $\tilde{u}(x, t; C_1, C_2, C_3, \ldots)$ is the exact solution. Generally, such case does not arise for nonlinear problems. The nth-order approximate solution given by eq. (2.16) depends on the convergence control parameters $C_1, C_2, C_3, \ldots$ and these parameters can be optimally determined by various methods such as weighted residual least square method, Galerkin method and collocation method.

3. Application of Kudryashov method for solving mBBM equation

Employing the aforementioned steps to obtain exact solutions for mBBM equation,

$$u_t + \alpha u_x + \beta u^2 u_x - \gamma u_{xxt} = 0. \tag{3.1}$$

After applying transformation $\xi = kx + \omega t$, we obtain the differential of the form

$$\omega u_\xi + \alpha k u_\xi + \beta k u^2 u_\xi - \gamma \omega k^2 u_{\xi\xi\xi} = 0. \tag{3.2}$$

Now, from step IV, we have $N = 1$. Therefore, from eq. (2.4)

$$u(\xi, X) = a_0 + a_1 X(\xi). \tag{3.3}$$

Then from eq. (3.3), substituting the derivatives of $u(\xi, X)$, a system of algebraic equations was obtained after taking the ansatz into account.

Coefficient of X^1:
$a_1 k\alpha - a_0^2 a_1 k\beta - a_1\omega + a_1 k^2 \omega\gamma = 0.$

Coefficients of X^2:
$a_1k\alpha+a_0^2a_1k\beta-2a_0a_1^2k\beta+a_1\omega-7a_1k^2\omega\gamma=0.$
Coefficients of X^3:
$2a_0a_1^2k\beta-a_1^3k\beta+12a_1k^2\gamma\omega=0.$
Coefficients of X^4:
$$a_1^3k\beta-6a_1k^2\gamma\omega=0. \tag{3.4}$$

Solving the above system of equations, the following four cases may be obtained.

Case I:

$$a_0=\frac{\sqrt{3}k\sqrt{\alpha}\sqrt{\gamma}}{\sqrt{-2\beta-k^2\beta\gamma}},\ \ a_1=-\frac{2\sqrt{3}k\sqrt{\alpha}\sqrt{\gamma}}{\sqrt{-\beta(2+k^2\gamma)}}$$

and

$$\omega=\frac{1}{3}\left(-3k\alpha-\frac{9k^3\alpha\beta\gamma}{-2\beta-k^2\beta\gamma}-\frac{9k^5\alpha^2\beta^2\gamma^3}{(-2\beta-k^2\beta\gamma)^2\left(\alpha\gamma-\frac{3\alpha\beta\gamma}{2\beta+k^2\beta\gamma}\right)}-\frac{3k^3\alpha^2\beta\gamma^2}{(-2\beta-k^2\beta\gamma)\left(\alpha\gamma-\frac{3\alpha\beta\gamma}{2\beta+k^2\beta\gamma}\right)}\right).$$

Utilising the coefficients acquired above, we have

$$u(x,t)=\frac{\sqrt{3}k\sqrt{\alpha}\sqrt{\gamma}\tanh\left[\frac{1}{2}k\left(x-\frac{2t\alpha}{2+k^2\gamma}\right)\right]}{\sqrt{-\beta\left(2+k^2\gamma\right)}}. \tag{3.5}$$

Similarly, the following set of solutions may be found.

Case II:

$$a_0=-\frac{\sqrt{3}k\sqrt{\alpha}\sqrt{\gamma}}{\sqrt{-2\beta-k^2\beta\gamma}},\quad a_1=\frac{2\sqrt{3}k\sqrt{\alpha}\sqrt{\gamma}}{\sqrt{-\beta(2+k^2\gamma)}}$$

and

$$\omega=\frac{1}{3}\left(-3k\alpha-\frac{9k^3\alpha\beta\gamma}{-2\beta-k^2\beta\gamma}-\frac{9k^5\alpha^2\beta^2\gamma^3}{(-2\beta-k^2\beta\gamma)^2\left(\alpha\gamma-\frac{3\alpha\beta\gamma}{2\beta+k^2\beta\gamma}\right)}-\frac{3k^3\alpha^2\beta\gamma^2}{(-2\beta-k^2\beta\gamma)\left(\alpha\gamma-\frac{3\alpha\beta\gamma}{2\beta+k^2\beta\gamma}\right)}\right)$$

$$u(x,t)=-\frac{\sqrt{3}k\sqrt{\alpha}\sqrt{\gamma}\tanh\left[\frac{1}{2}k\left(x-\frac{2t\alpha}{2+k^2\gamma}\right)\right]}{\sqrt{-\beta\left(2+k^2\gamma\right)}}. \tag{3.6}$$

Case III:

$$a_0=\frac{i\sqrt{\alpha}}{\sqrt{\beta}},\quad a_1=-\frac{2i\sqrt{\alpha}}{\sqrt{\beta}},$$

$$k=-\frac{1}{\sqrt{\gamma}}\quad \text{and}\quad \omega=\frac{2\alpha}{3\sqrt{\gamma}}$$

$$u(x,t)=-\frac{i\sqrt{\alpha}\tanh\left[\frac{3x-2t\alpha}{6\sqrt{\gamma}}\right]}{\sqrt{\beta}}. \tag{3.7}$$

Case IV:

$$a_0=\frac{i\sqrt{\alpha}}{\sqrt{\beta}},\quad a_1=-\frac{2i\sqrt{\alpha}}{\sqrt{\beta}},$$

$$k=\frac{1}{\sqrt{\gamma}}\quad \text{and}\quad \omega=-\frac{2\alpha}{3\sqrt{\gamma}}$$

$$u(x,t)=\frac{i\sqrt{\alpha}\tanh\left[\frac{3x-2t\alpha}{6\sqrt{\gamma}}\right]}{\sqrt{\beta}}. \tag{3.8}$$

4. Application of OHAM for solving mBBM equation

The homotopy for eq. (3.1) is constructed as

$$(1-p)\frac{\partial\varphi}{\partial t}=H(p)\left[\frac{\partial\varphi}{\partial t}+\alpha\frac{\partial\varphi}{\partial x}+\beta\varphi^2\frac{\partial\varphi}{\partial x}-\gamma\frac{\partial^3\varphi}{\partial x^2\partial t}\right]. \tag{4.1}$$

Here,

$$\varphi=\sum_{m=0}^{\infty}u_mp^m \tag{4.2}$$

and

$$H(p)=C_1p+C_2p^2+C_3p^3+\cdots. \tag{4.3}$$

Substituting eqs (4.2) and (4.3) in eq. (4.1) and collecting the like powers of p^m, the following system of PDEs is attained.

Coefficient of p^0:

$$\frac{\partial u_0}{\partial t}=0 \tag{4.4}$$

Coefficient of p^1:

$$\frac{\partial u_1}{\partial t}-\frac{\partial u_0}{\partial t}=C_1\left[\frac{\partial u_0}{\partial t}+\alpha\frac{\partial u_0}{\partial x}+\beta u_0^2\frac{\partial u_0}{\partial x}-\gamma\frac{\partial^3u_0}{\partial x^2\partial t}\right]. \tag{4.5}$$

Coefficient of p^2:

$$\frac{\partial u_2}{\partial t}-\frac{\partial u_1}{\partial t}=C_1\left[\frac{\partial u_1}{\partial t}+\alpha\frac{\partial u_1}{\partial x}+2\beta u_0 u_1\frac{\partial u_0}{\partial x}+\beta u_0^2\frac{\partial u_1}{\partial x}-\gamma\frac{\partial^3 u_1}{\partial x^2\partial t}\right]+C_2\left[\frac{\partial u_0}{\partial t}+\alpha\frac{\partial u_0}{\partial x}+\beta u_0^2\frac{\partial u_0}{\partial x}-\gamma\frac{\partial^3 u_0}{\partial x^2\partial t}\right]. \tag{4.6}$$

Coefficient of p^3:

$$\frac{\partial u_3}{\partial t}-\frac{\partial u_2}{\partial t}=C_1\left[\frac{\partial u_2}{\partial t}+\alpha\frac{\partial u_2}{\partial x}+\beta u_1^2\frac{\partial u_0}{\partial x}+2\beta u_0 u_2\frac{\partial u_0}{\partial x}+2\beta u_0 u_1\frac{\partial u_1}{\partial x}+\beta u_0^2\frac{\partial u_2}{\partial x}-\gamma\frac{\partial^3 u_1}{\partial x^2\partial t}\right]+C_2\left[\frac{\partial u_1}{\partial t}+\alpha\frac{\partial u_1}{\partial x}+2\beta u_0 u_1\frac{\partial u_0}{\partial x}+\beta u_0^2\frac{\partial u_1}{\partial x}-\gamma\frac{\partial^3 u_1}{\partial x^2\partial t}\right]+C_3\left[\frac{\partial u_0}{\partial t}+\alpha\frac{\partial u_0}{\partial x}+\beta u_0^2\frac{\partial u_0}{\partial x}-\gamma\frac{\partial^3 u_0}{\partial x^2\partial t}\right] \tag{4.7}$$

and so on.

Consider the following initial condition to solve the mBBM equation:

$$u(x,0)=\frac{\sqrt{3}\sqrt{\alpha}\sqrt{\gamma}k\tanh\left(\frac{kx}{2}\right)}{\sqrt{-\beta\left(2+k^2\gamma\right)}}. \tag{4.8}$$

Taking the initial condition and solving eqs (4.4)–(4.7), we get

$$u_0=\frac{\sqrt{3}\sqrt{\alpha}\sqrt{\gamma}k\tanh\left(\frac{kx}{2}\right)}{\sqrt{-\beta\left(2+k^2\gamma\right)}} \tag{4.9}$$

$$u_1=-\frac{\sqrt{3}C_1k^2t\alpha^{\frac{3}{2}}\beta\sqrt{\gamma}\,\mathrm{sech}\left(\frac{kx}{2}\right)^4}{2\left(-\beta\left(2+k^2\gamma\right)\right)^{\frac{3}{2}}}-\frac{\sqrt{3}C_1k^4t\alpha^{\frac{3}{2}}\beta\gamma^{\frac{3}{2}}\,\mathrm{sech}\left(\frac{kx}{2}\right)^4}{\left(-\beta\left(2+k^2\gamma\right)\right)^{\frac{3}{2}}}+\frac{\sqrt{3}C_1k^2t\alpha^{\frac{3}{2}}\beta\sqrt{\gamma}\left(k^2\gamma-1\right)\cosh(kx)\,\mathrm{sech}\left(\frac{kx}{2}\right)^4}{2\left(-\beta\left(2+k^2\gamma\right)\right)^{\frac{3}{2}}} \tag{4.10}$$

and so on.

Hence the approximate solution is obtained by $u=u_0+u_1+u_2+u_3$. Furthermore, collocation method has been employed using mathematical software to acquire optimal values of convergence control parameters.

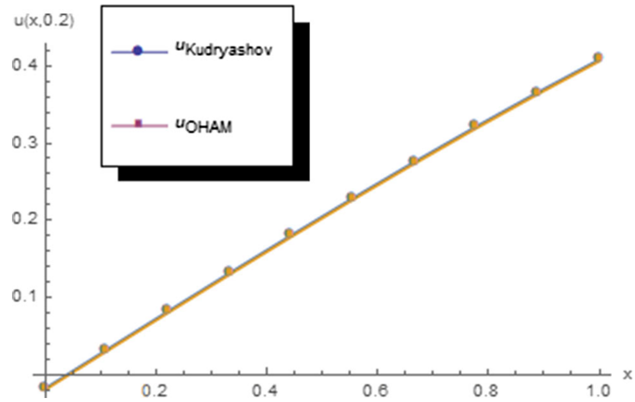

Figure 1. Comparison of the exact and numerical solutions of $u(x,0.2)$ with $\alpha=1$, $\beta=-1$, $\gamma=1$ and $k=1$.

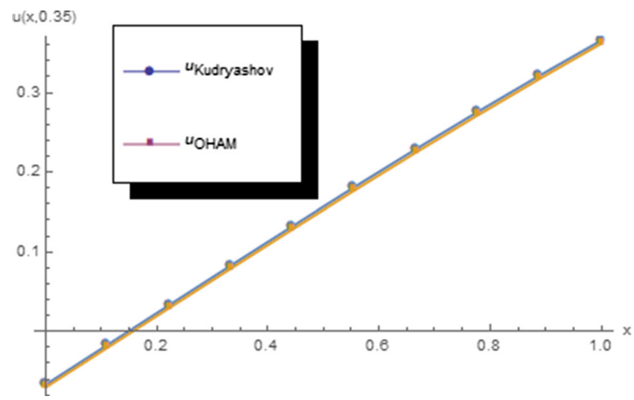

Figure 2. Comparison of the exact and numerical solutions of $u(x,0.35)$ with $\alpha=1$, $\beta=-1$, $\gamma=1$ and $k=1$.

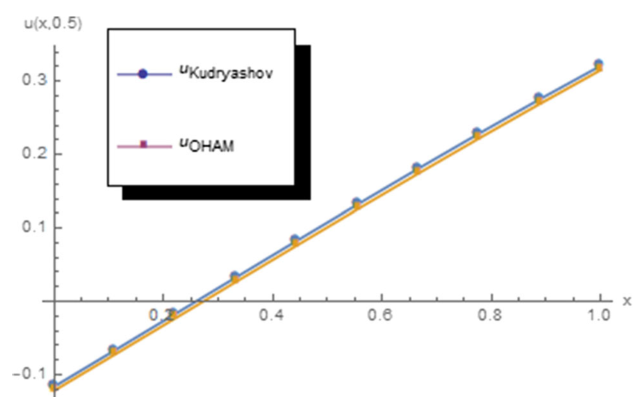

Figure 3. Comparison of the exact and numerical solutions of $u(x,0.5)$ with $\alpha=1$, $\beta=-1$, $\gamma=1$ and $k=1$.

5. Convergence of OHAM [3]

Theorem 5.1. *Let the solution components* $u_0, u_1, u_2, \ldots$ *be defined as given in eqs* (2.12)–(2.14). *The series solution* $\sum_{k=0}^{m-1}u_k(x,t)$ *defined in eq.* (2.16) *converges if there exists* δ, $0<\delta<1$ *such that* $\|u_{k+1}\|\le\delta\|u_k\|$, *for all* $k\ge k_0$ *for some* $k_0\in N$.

Table 1. Comparison of exact and approximate solutions of $u(x,t)$ with $k = 1, \alpha = 1, \beta = -1, \gamma = 1, C_1 = -0.259473, C_2 = -0.230807, C_3 = 0.0258762$.

	$t = 0.05$		$t = 0.10$		$t = 0.15$		$t = 0.20$		$t = 0.25$	
x	$u_{\text{Kudryashov}}$	u_{OHAM}	$u_{\text{Kudryashov}}$	u_{OHAM}	$u_{\text{Kudryashov}}$	u_{OHAM}	$u_{\text{Kudryashov}}$	u_{OHAM}	$u_{\text{Kudryashov}}$	u_{OHAM}
0.1	3.33E−2	3.28E−2	1.66E−2	1.57E−2	0.0	−1.35E−3	−1.66E−2	−1.85E−2	−3.33E−2	−3.56E−2
0.2	8.31E−2	8.26E−2	6.65E−2	6.56E−2	4.99E−2	4.85E−2	3.33E−2	3.14E−2	1.66E−2	1.42E−2
0.3	0.13254	0.13208	0.11614	0.11520	9.96E−2	9.82E−2	8.31E−2	8.12E−2	6.65E−2	6.40E−2
0.4	0.18130	0.18082	0.16514	0.16417	0.14888	0.14741	0.13254	0.13054	0.11614	0.11357
0.5	0.22918	0.22869	0.21333	0.21233	0.19737	0.19583	0.18130	0.17921	0.16514	0.16246
0.6	0.27598	0.27546	0.26052	0.25946	0.24491	0.24331	0.22918	0.22700	0.21333	0.21055
0.7	0.32151	0.32097	0.30649	0.30539	0.29131	0.28964	0.27598	0.27373	0.26052	0.25764
0.8	0.36559	0.36504	0.35107	0.34995	0.33637	0.33467	0.32151	0.31921	0.30649	0.30357
0.9	0.40810	0.40754	0.39411	0.39299	0.37994	0.37826	0.36559	0.36332	0.35107	0.34820
1.0	0.44890	0.44836	0.43550	0.43442	0.42189	0.42027	0.40810	0.40594	0.39411	0.39140

Table 2. Absolute error of $u(x,t)$ with $k = 1, \alpha = 1, \beta = -1, \gamma = 1, C_1 = -0.259473, C_2 = -0.230807, C_3 = 0.0258762$.

	$\|u_{\text{OHAM}} - u_{\text{Kudryashov}}\|$									
x	$t = 0.05$	$t = 0.10$	$t = 0.15$	$t = 0.20$	$t = 0.25$	$t = 0.30$	$t = 0.35$	$t = 0.40$	$t = 0.45$	$t = 0.50$
0.1	4.42E−4	8.92E−4	1.35E−3	1.84E−3	2.36E−3	2.91E−3	3.51E−3	4.15E−3	4.86E−3	5.63E−3
0.2	4.49E−4	9.07E−4	1.38E−3	1.87E−3	2.40E−3	2.97E−3	3.58E−3	4.24E−3	4.97E−3	5.76E−3
0.3	4.60E−4	9.31E−4	1.42E−3	1.93E−3	2.47E−3	3.06E−3	3.69E−3	4.37E−3	5.12E−3	5.93E−3
0.4	4.77E−4	9.66E−4	1.47E−3	2.00E−3	2.56E−3	3.17E−3	3.82E−3	4.53E−3	5.30E−3	6.14E−3
0.5	4.98E−4	1.00E−3	1.53E−3	2.09E−3	2.67E−3	3.30E−3	3.97E−3	4.70E−3	5.50E−3	6.36E−3
0.6	5.21E−4	1.05E−3	1.60E−3	2.18E−3	2.78E−3	3.43E−3	4.12E−3	4.87E−3	5.67E−3	6.55E−3
0.7	5.42E−4	1.09E−3	1.66E−3	2.25E−3	2.87E−3	3.53E−3	4.23E−3	4.97E−3	5.78E−3	6.65E−3
0.8	5.56E−4	1.12E−3	1.69E−3	2.29E−3	2.91E−3	3.56E−3	4.25E−3	4.98E−3	5.76E−3	6.60E−3
0.9	5.59E−4	1.12E−3	1.68E−3	2.27E−3	2.86E−3	3.48E−3	4.13E−3	4.82E−3	5.54E−3	6.32E−3
1.0	5.43E−4	1.08E−3	1.62E−3	2.16E−3	2.71E−3	3.27E−3	3.85E−3	4.45E−3	5.09E−3	5.75E−3

Table 3. L_2 and L_∞ error norms for $u(x,t)$ at different values of x.

x	L_2	L_∞	x	L_2	L_∞
0.1	1.02E−2	5.63E−3	0.6	1.20E−2	6.55E−3
0.2	1.04E−2	5.76E−3	0.7	1.22E−2	6.65E−3
0.3	1.07E−2	5.93E−3	0.8	1.22E−2	6.60E−3
0.4	1.11E−2	6.14E−3	0.9	1.18E−2	6.32E−3
0.5	1.16E−2	6.36E−3	1.0	1.09E−2	5.75E−3

Proof. Define the sequence $\{S_n\}_{n=0}^{\infty}$ as follows:

$$S_0 = u_0,$$
$$S_1 = u_0 + u_1,$$
$$S_2 = u_0 + u_1 + u_2,$$
$$\dots$$
$$S_n = u_0 + u_1 + u_2 + \cdots + u_n.$$

We have to show $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\Re$. Consider

$$\begin{aligned} \|S_{n+1} - S_n\| &= \|u_{n+1}\| \\ &\le \delta \|u_n\| \\ &\le \delta^2 \|u_{n-1}\| \\ &\dots \end{aligned}$$

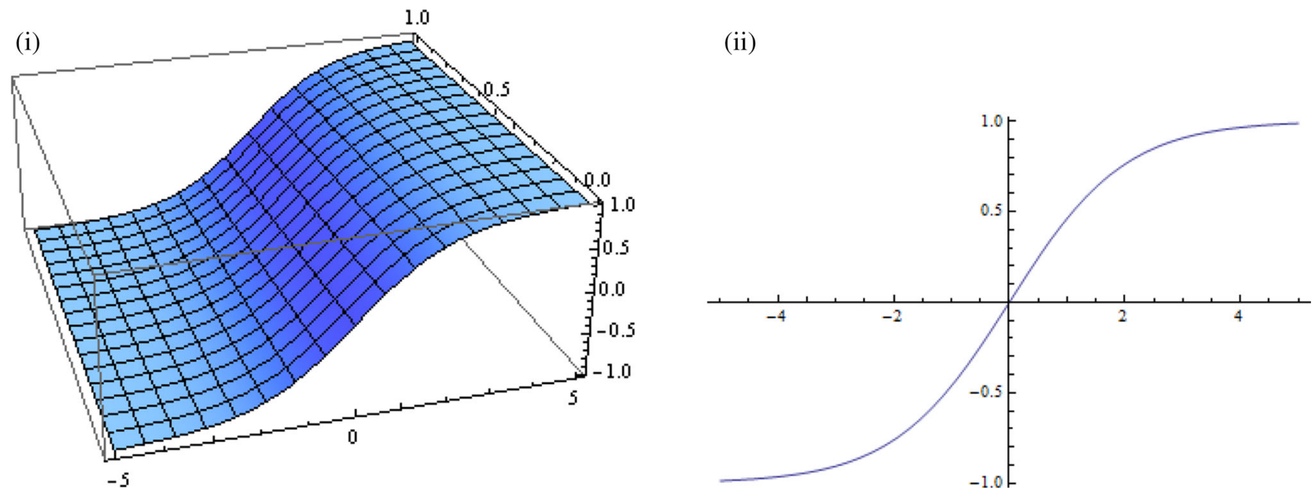

Figure 4. (**i**) Solution of $u(x,t)$ given in eq. (3.5) and (**ii**) $u(x,0)$ with $k=1, \alpha=1, \gamma=1$ and $\beta=-1$.

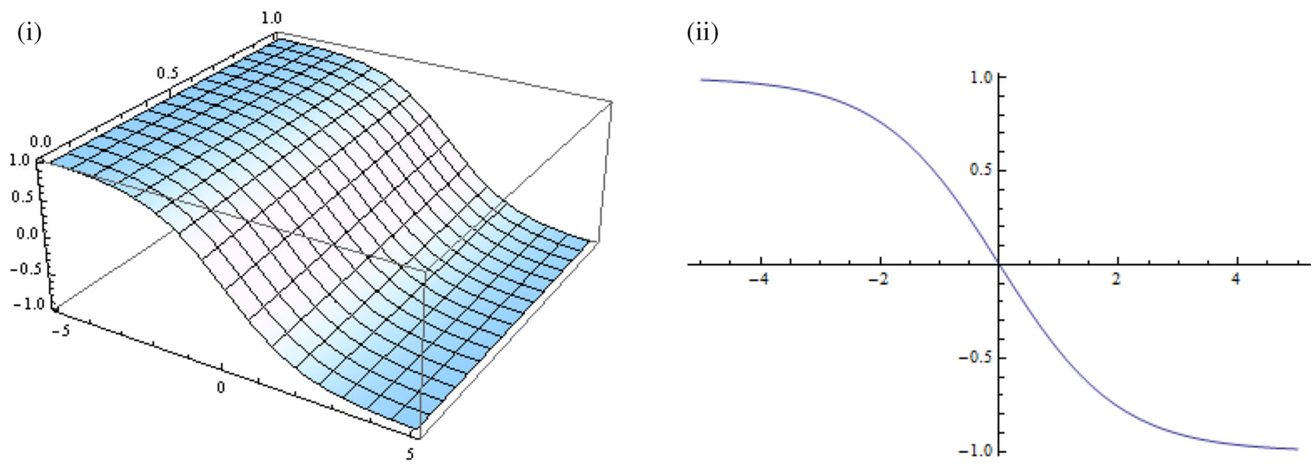

Figure 5. (**i**) Solution of $u(x,t)$ given in eq. (3.6) and (**ii**) $u(x,0)$ with $k=1, \alpha=1, \gamma=1$ and $\beta=-1$.

$$\leq \delta^{n-k_0+1}\left\|u_{k_0}\right\|.$$

Now for every $n, m \in N$, $n \geq m > k_0$

$$\begin{aligned}
&\|S_n - S_m\| \\
&= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + \cdots + (S_{m+1} - S_m)\| \\
&\quad \leq \|S_n - S_{n-1}\| + \|S_{n-1} - S_{n-2}\| \\
&\quad + \cdots + \|S_{m+1} - S_m\| \quad \text{(Triangle inequality)} \\
&\quad \leq \delta^{n-k_0}\left\|u_{k_0}\right\| + \delta^{n-k_0-1}\left\|u_{k_0}\right\| \\
&\quad + \cdots + \delta^{m-k_0+1}\left\|u_{k_0}\right\| \\
&\quad = \left(\frac{1-\delta^{n-m}}{1-\delta}\right)\delta^{m-k_0+1}\left\|u_{k_0}\right\|.
\end{aligned}$$

This implies

$$\lim_{n,m\to\infty}\|S_n - S_m\| = 0 \text{ (since } 0 < \delta < 1).$$

Therefore, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Hilbert space $\Re$ and hence the series solution $\sum_{k=1}^{\infty} u_k(x,t)$ converges. □

6. Numerical results and discussion

The solutions acquired by the proposed techniques for mBBM equation (3.1) have been compared and exhibited in table 1. To assess the validity of the techniques, the absolute errors of eq. (3.1) have been illustrated in table 2 and the corresponding L_2, L_∞ error norms in table 3. Figures 1–3 establish the graphical comparison of OHAM solutions with regard to those attained by Kudryashov technique for eq. (3.1) taking $t = 0.2, 0.35$ and 0.5 respectively. Also, the exact solutions in eqs

(3.5) and (3.6) have been demonstrated in 3D graphs given in figures 4 and 5 respectively. The numerical simulation manifest that the aforementioned techniques are appropriate for acquiring solutions of NPDEs.

7. Conclusion

In this paper, the efficiency to find exact solutions to deal with NLEEs has been demonstrated using the Kudryashov method. Furthermore, we use four term approximate solution of OHAM and compare it with the solution of the Kudryashov method. Tables are presented to show its effectiveness. This paper presents the powerfulness and reliability of OHAM for solving NPDEs compared to other perturbation methods. Finally, an inference can be drawn that these two methods are efficient and reliable in dealing with NPDEs.

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