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An efficient algorithm for inferring functional connectivity between the drive and the noisy response chaotic oscillators with significant frequency mismatch

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Abstract. In this study, we present a new method for assessing the functional connectivity between the drive and the noisy response chaotic oscillators with significant frequency mismatch. This method is based on the footprint of the drive oscillator on the response oscillator. Specifically, we calculate the magnitude squared coherence between the intrinsic frequency of the drive oscillator and the oscillation frequency of the dynamics of the spectral energy of the response oscillator. The spectral energy of the response oscillator is estimated using a modified S-transform algorithm with a short sliding window. We apply the proposed approach to master–slave Rössler systems and coupled Van der Pol oscillators with a frequency ratio of $\sim$ 1:4 and show that when the slave signal is contaminated with white Gaussian noise at different signal-to-noise ratios (SNR), the new algorithm is well suited to assess the presence of coupling with *a priori* known direction in noisy, unidirectionally coupled chaotic oscillators, especially in the case of weak and moderate coupling.

Keywords. Chaotic oscillators; functional connectivity; causality; unidirectional coupling.

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1. Introduction

In recent years, various time series analysis methods have been proposed to identify interactions between complex systems, and particularly for the study of causality, which has attracted significant attention from researchers. In terms of dynamical systems science, the question arises as to which dynamic variables influence other variables [1], and an estimation of the interdependence between the observed variables can provide valuable knowledge of the processes that generate time series. In the past few decades, a variety of approaches have been proposed in an attempt to detect causal relations from purely observational data, including methods from dynamical systems, information theory and time series analysis, to name only a few. A comprehensive review can be found in [1]. A significant proportion of the works on causal effects in coupled chaotic oscillators consider the causation problem from the perspective of the dynamical attractors underlying nonlinear dynamical systems and the concept of generalised synchronisation [8–11]. These model-free methods are based on Takens' embedding theorem, and exploit the geometry of attractors of coupled dynamical systems. More specifically, they rely on the existence of a continuous mapping from the reconstructed attractor of the response ('slave') system to the reconstructed attractor of the driving ('master') system. In one implementation, the convergent cross mapping (CCM) method [2–11] is used to test for causation between a driver system X and a response system Y by measuring the extent to which the historical states of the reconstructed state space M_Y can reliably estimate the states of the reconstructed state space M_X, and vice versa by measuring the extent to which the historical states of the reconstructed state space M_X can reliably estimate the states of the reconstructed state space M_Y. The direction of coupling is then inferred based on the asymmetries emerging from the calculations of the two possible causal directions. Most of the existing tools for detecting causality can make determinations of directionality in idealised conditions of sufficiently long, noise-free time series with closely related natural frequencies, but these determinations are relatively fragile in the presence of noise

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[12]. However, real-world datasets possess instrumental and environmental noise and may lack sufficient data. Moreover, the problem of detecting directionality with a significant frequency mismatch between oscillators is much more challenging than in the case of two systems with more closely related natural frequencies [2,11].

In this study, a new effective algorithm is presented for assessing the presence of functional connectivity in the specific case where two non-identical chaotic oscillators with a frequency ratio of ∼1:4 are unidirectionally coupled with *a priori* known causal direction from a low- to a high-frequency variable and where the slave signal is contaminated with white Gaussian noise. A reasonably high sampling rate is used to achieve the highest possible performance of the modified S-transform (MST).

2. Description of the algorithm

To reliably infer the causal relation between two unidirectionally coupled, non-identical, noisy chaotic oscillators with significant frequency mismatch, it is necessary that the applied algorithm should be allowed to discover the effect of system X on system Y. Numerical simulations with two unidirectionally coupled Rössler oscillators with a frequency ratio of ∼1:4 or ∼1:5 show that the master oscillator has a negligible effect on the phase dynamics of the slave oscillator. The effect on the frequency of the slave oscillator can be detected only by changing the strength of the coupling, and the larger the coupling strength, the stronger is the 'dragging' of the trajectory of the driven system towards the driver. However, for a fixed coupling strength, this effect is not significant. This means that methods based on the phase dynamics of the two subsystems, e.g., the evolution map approach (EMA) [13,14], are not reliable for this type of data. Approaches that rely on evaluating the distances of conditioned neighbours in reconstructed state spaces, e.g., CCM, correctly detect causal links for noise-free data [11] but fail to identify the correct causal influence for noisy oscillators, especially in the case of weak coupling. This can be explained by the fact that the master oscillator also has a weak effect on the amplitude dynamics of the slave oscillator, i.e., the phase-amplitude modulation index is low, and this effect is already suppressed at low noise, especially in the case of weak coupling. Nonlinear filtering of the slave oscillator does not help, due to unavoidable distortions in the local chaotic dynamics during the de-noising process. To determine the relationship between the master and the noisy slave oscillators, we propose to evaluate the dynamics of the spectral energy of the response oscillator using the time–frequency (TF) technique. TF signal analysis provides numerous tools for signal representation and estimation of the spectral energy around the instantaneous frequency (IF), and has significant practical importance in many fields, including chaotic time series analysis [15]. In this paper, we use a computationally efficient MST method with a moving window of length W approximately equal to two thirds of the period of the master oscillator and covering several periods of the slave oscillator. The MST [16] is a TF-based technique, a signal-dependent version of the standard ST with improved TF resolution. Given a time series $x(t)$, the MST is defined as follows [16]:

$$\mathrm{MST}\,(t, f, m, k) = \cdots \int_{-\infty}^{+\infty} x(\tau) g(t-\tau, f, m, k) \mathrm{e}^{-j2\pi f\tau} \mathrm{d}\tau, \quad (1)$$

where $g\,(t-\tau, f, m, k)$, the window function of the MST, is given as follows:

$$g(t-\tau, f, m, k) = \frac{|f|}{(mf+k)\sqrt{2\pi}} \mathrm{e}^{\frac{-f^2(t-\tau)^2}{2(mf+k)^2}}, \quad (2)$$

where m is the slope and k is the intercept for a linear change in frequency. For more details, see [16]. The energy concentration along the frequency axis is obtained as the mean value $\bar{S}$ of the absolute MST over the window length W, and the peak values of $\bar{S}$ summed with the two neighbouring values are used to form a time series $\hat{S}(t)$ by shifting the sliding window by one sample. To reliably assess the interdependence between the processes, the time series $\hat{S}(t)$ is smoothed by two applications of the Savitzky–Golay filter with polynomial order four. This allows the high-frequency component of $\hat{S}(t)$ to be suppressed, and helps to highlight the frequencies related to the main frequency of the master oscillator. Figure 1a shows an example of a time series $\hat{S}(t)$ compared with the signal of the master oscillator when two Rössler systems are unidirectionally weakly coupled. For clarity, both curves are scaled to the same magnitude. Finally, the magnitude-squared coherence (MSC) between the signal of the drive oscillator and the time series $\hat{S}(t)$ is calculated, and conclusions on the functional connectivity are drawn based on the MSC at the intrinsic frequency of the drive oscillator. Figure 1b shows an example of the MSC when two Rössler systems with frequency ratio 1:3.697 are unidirectionally coupled with a coupling strength of 0.5. The peak in the MSC is close to the unit with a normalised frequency of 0.0614, i.e., the intrinsic frequency of the drive oscillator, implying functional connectivity between the oscillators. Since the MSC is non-directional, the direction of the relationship between the signals must be known in advance.

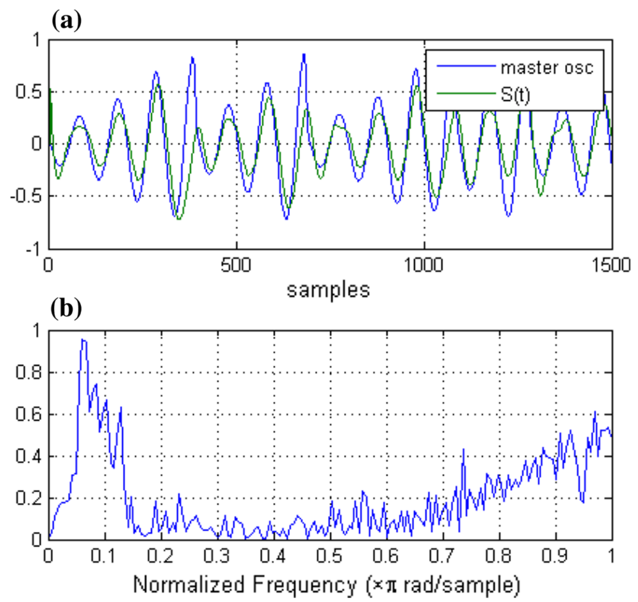

Figure 1. (**a**) Time series of the master oscillator and $\hat{S}(t)$ and (**b**) magnitude squared coherence between time series of the master oscillator and $\hat{S}(t)$.

3. Simulation results

To demonstrate the efficiency of this method, we analyse pairs of unidirectionally coupled deterministic chaotic dynamics and limit-cycle oscillators. The pairs of chaotic dynamics comprised non-identical coupled Rössler dynamics and phase-synchronised Rössler systems with a phase-locking ratio of 1:4. Non-identical Van der Pol oscillators were used as limit-cycle oscillators. All data processing and analyses were performed using Matlab software (MathWorks, Natick, MA).

The example of unidirectionally connected chaotic Rössler systems [11] is considered at a frequency ratio of ∼1:4 and with variable coupling strengths. The oscillators were coupled via a one-way driving relationship between the variable x_1 of the driving system and the variable y_1 of the response system:

$$\begin{aligned}
\dot{x}_1 &= -\omega_1 x_2 - x_3,\\
\dot{x}_2 &= \omega_1 x_1 + a_1 x_2,\\
\dot{x}_3 &= 0.2 + x_3\,(x_1 - 10)\,,\\
\dot{y}_1 &= -\omega_2 y_2 - y_3 + c\,(x_1 - y_1)'\\
\dot{y}_2 &= \omega_2 y_1 + a_2 y_2,\\
\dot{y}_3 &= 0.2 + y_3\,(y_1 - 10)\,,
\end{aligned} \tag{3}$$

where $\omega_1 = 0.985$, $\omega_2 = 3.642$, $a_1 = 0.15$ and $a_2 = 0.72$ when coupling strength $c = 0{-}0.7$, and $a_2 = 0.72{-}1.27$ when $c = 0.8{-}1.2$. The coupling strength c was increased from 0 to 1.2 with a step of 0.1. The data were generated using the fifth-order Runge–Kutta method for integration with a step size of 0.1, and the first 2000 data points were discarded. The total number of data points obtained was 15,000, resulting in approximately 99 samples per average orbit around the attractor of the master oscillator and approximately 23 samples of the slave oscillator. MST was performed using a time window of length 64 and the time series $\hat{S}(t)$ of the slave oscillator was smoothed by applying twice the Savitzky–Golay filter of polynomial order four with a frame size of 101, i.e., approximately equal to the main period of the master oscillator. A Hamming window of length 1024 was used to calculate the MSC between the signal of the drive oscillator and the time series $\hat{S}(t)$. The performance of the algorithm was evaluated using a noise-free slave oscillator, and contaminated with additive zero-mean white Gaussian noise with SNR = 20 dB and 15 dB. The CCM algorithm was chosen for comparison, with 7500 time-delayed vectors of x_1 and y_1, embedding dimension $E = 6$ and time delay $\tau = 8$. The values of embedding dimension and time delay must be understood as optimal. In an analogous way to the proposed algorithm, the slave oscillator was contaminated with additive zero-mean white Gaussian noise with signal-to-noise ratio (SNR) = 15 dB. Consequently, the direction from system X to system Y was denoted as $Y|X$ and the direction from Y to X as $X|Y$. Taking, for example, the measure ρ (correlation coefficient): if X drives Y, the measure $\rho(Y|X)$ is expected to be higher than $\rho(X|Y)$. As can be seen from figure 2, the proposed method accurately determined the functional connectivity between the chaotic Rössler systems when SNR=15 dB already at $c = 0.1$. Meanwhile, the CCM method failed to reliably assess the functional relation between the coupled chaotic oscillators when SNR=15 dB even up to $c = 0.3$.

To obtain a phase synchronisation of 1:4 between two unidirectionally coupled chaotic Rössler oscillators, the response oscillator is governed by [17]

$$\begin{aligned}
\dot{y}_1 &= -\omega_2 y_2 - y_3 + c(r_2\cos(\beta\phi_1) - y_1)\\
\dot{y}_2 &= \omega_2 y_1 + a_2 y_2 + c(r_2\sin(\beta\phi_1) - y_2)\,,\\
\dot{y}_3 &= 0.2 + y_3(y_1 - 10)\,,
\end{aligned} \tag{4}$$

where ϕ_1 is the phase of the driver, r_2 is the amplitude of the response system and c is the coupling strength. The value of parameter β is 4, giving the locking ratio as 1:4. The phase and the amplitude of the Rössler attractor are, respectively, defined as

$$\phi_1 = \arctan\,(x_1/x_2)\,, \tag{5}$$

$$r_2 = \sqrt{(y_1)^2 + (y_2)^2}. \tag{6}$$

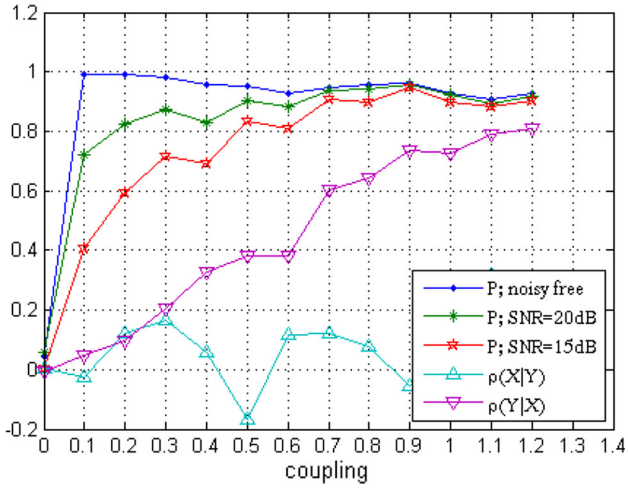

Figure 2. Peak value P of the MSC at the intrinsic frequency of the drive oscillator and the measures $\rho(Y|X)$ and $\rho(X|Y)$ computed for two unidirectionally coupled Rössler systems when $\omega_1 = 0.985$ and $\omega_2 = 3.642$ at different coupling strengths.

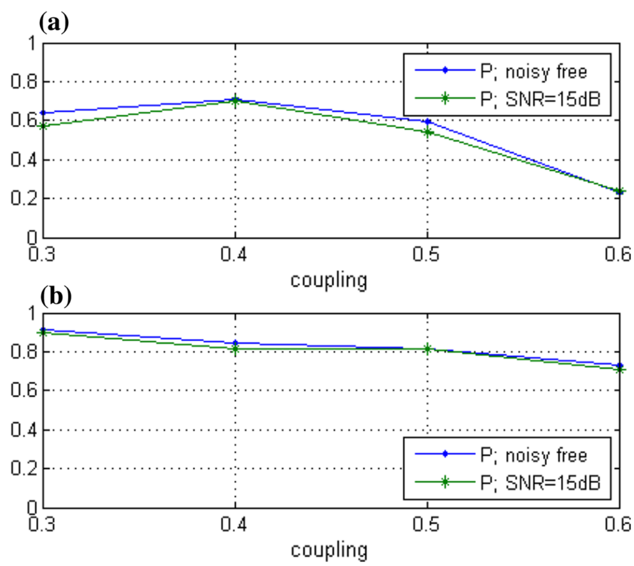

Figure 3. Peak value P of MSC between time series of the master oscillator and $\hat{S}(t)$ for two unidirectionally coupled Rössler systems in the phase synchronisation regime with a locking rate of 1:4 (**a**) at the intrinsic frequency of the drive oscillator and (**b**) at the second harmonic of the drive signal.

Figure 3 shows the effect of the driving oscillator on the spectral energy of the response oscillator when SNR = 15 dB in the phase synchronisation regime. The peculiarity of this case is that the MSC between the time series $\hat{S}(t)$ and the second harmonic of the drive signal is greater than between the time series $\hat{S}(t)$ and the first harmonic of the drive signal.

For the unidirectionally coupled Van der Pol oscillators, the equations are [18]

$$\ddot{x}(t) = 0.2\left[1 - x^2(t)\right]\dot{x}(t) - \omega_1^2 x(t)$$
$$\ddot{y}(t) = 0.2\left[1 - y^2(t)\right]\dot{y}(t) - \omega_2^2 y(t)\cdots + c\,|x(t) - y(t)|. \quad (7)$$

In the present paper, this equation is modified by multiplying c by the absolute values of the differences $x(t)$ and $y(t)$. These dynamics were simulated using the ode45 integrator in Matlab. The coupling strength c was chosen in the range 0.3–2.0 with a step of 0.1. The first 2000 data points were discarded and the total number of data points obtained was 15,000. The fundamental period of the master oscillator in the samples was approximately 120, and the fundamental period of the slave oscillator was approximately 32. MST was performed using a time window of length 96. In a similar way to the example discussed above, the performance of the algorithm was evaluated with a noise-free slave oscillator and contaminated with additive zero-mean white Gaussian noise at SNR = 20 dB and 15 dB. Figure 4 shows that the proposed method can determine the functional connectivity between the unidirectionally coupled Van der Pol oscillators when SNR = 15 dB.

4. Conclusions

In this paper we have described an algorithm for assessing the functional connectivity between the drive and the noisy response chaotic oscillators with significant frequency mismatch, at a frequency ratio of ∼1:4. The effectiveness of the proposed method was verified by simulation, and was successfully tested using several well-known systems: two unidirectionally connected chaotic Rössler systems, two unidirectionally coupled chaotic Rössler systems in the phase synchronisation regime with a locking rate of 1:4 and two unidirectionally coupled non-identical Van der Pol oscillators. The simulation showed that the proposed algorithm is an effective method for revealing the presence of functional connectivity with *a priori* known direction between unidirectionally coupled non-identical chaotic oscillators with significant frequency mismatch and where the slave signal is contaminated with white Gaussian noise at an SNR of up to 15 dB.

The simulation using smoothed pseudo-Wigner–Ville distribution (SPWVD) [19] showed that other efficient TF analysis methods can be used instead of the MST method in a particular case. In this work, SPWVD analysis was performed using sliding window of length 512 with 128-overlap, and the time series $\hat{S}(t)$ were composed by concatenating these strings truncated by 64 from both sides to avoid edge effect problems. The SPWVD method is superior to the MST method in the

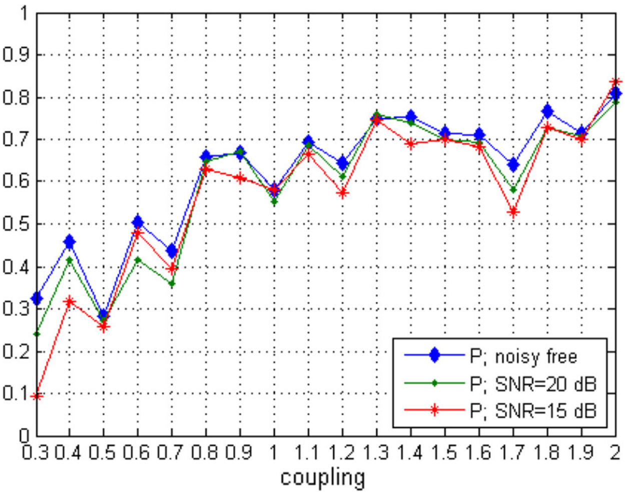

Figure 4. Peak value P of MSC between the time series of the master oscillator and $\hat{S}(t)$ for two unidirectionally coupled Van der Pol oscillators.

case of coupled Van der Pol oscillators, but inferior to MST in the case of coupled Rossler systems with additive Gaussian noise and at low coupling strength.

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