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A new combined soliton solution of the modified Korteweg–de Vries equation

JIANPING WU

School of Science, Zhengzhou University of Aeronautics, Zhengzhou 450046, Henan, People's Republic of China

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Abstract. In this paper, the Riemann–Hilbert problem of the modified Korteweg–de Vries (mKdV) equation is studied, from which a new combined soliton solution is obtained. In addition, to illustrate the dynamics of the new combined soliton solution, an algebra technique is developed to demonstrate the soliton interactions using *Mathematica* symbolic computations. The proposed method is effective in deriving and investigating new soliton solutions of the mKdV equation. The results also expand the understanding of the soliton structure of the mKdV equation.

Keywords. Modified Korteweg–de Vries equation; combined soliton solution; soliton dynamics.

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1. Introduction

It is known that the modified Korteweg–de Vries (mKdV) equation

$$u_t + 6u^2 u_x + u_{xxx} = 0 \tag{1.1}$$

is an important soliton equation in plasma physics [1], lattice dynamics [2] and in traffic flow dynamics [3]. For the mKdV equation, $u = u(x, t)$ is a real-valued function of x, t. In ref. [4], the mKdV equation was used to construct infinitely many conservation laws of the Korteweg–de Vries (KdV) equation which is an important integrable system in soliton theory. The mKdV equation (1.1) is completely integrable and can be solved by the inverse scattering transform (IST) [5–7]. Moreover, various effective approaches such as the Hirota method [8,9], the Wronskian technique [10,11], the Darboux transformation [12], and others [13–20], have been used to study the mKdV equation (1.1). For exact solutions, it is well-known that there exist two elementary soliton solutions for the mKdV equation. One is the breather soliton solution

$$u = 2\left(\arctan \frac{k_2 \sin[k_1 x + k_1(k_1^2 - 3k_2^2)t]}{k_1 \cosh[k_2 x + k_2(3k_1^2 - k_2^2)t]}\right)_x,$$

and the other is the bell soliton solution

$$u = k\,\mathrm{sech}(kx - k^3 t).$$

The breather soliton is an oscillating wave moving along a straight line. The bell soliton is a travelling wave keeping its form along the right-travelling process. However, to our knowledge, there are few reports on the combined forms of solutions of multibreather solitons and multibell solitons for the mKdV equation (1.1).

From the viewpoint of the IST [7], it is known that the eigenvalues situated symmetrically across the imaginary axis yield breather solitons and the purely imaginary eigenvalues give bell solitons for the mKdV equation. Naturally, a question arises: Can new combined soliton solutions be obtained for (1.1) by considering the eigenvalues involving both these two kinds? To answer this question, we aim to study the new zero structure of (1.1) using the IST via Riemann–Hilbert (RH) problems [21–29]. Then, corresponding to the new zero structure, we shall derive a new combined soliton solution for the mKdV equation (1.1). In addition, to illustrate the dynamics of the obtained soliton solution, an algebra technique will be developed to demonstrate the interaction behaviours using *Mathematica* symbolic computations.

This paper is organised as follows. In §2, the RH problem will be formulated by investigating the direct scattering transform of the mKdV equation (1.1). Then the RH problems corresponding to the reflectionless cases will be exactly solved under a new zero structure. In §3, a new combined soliton solution will be

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derived for the mKdV equation. In addition, an algebra technique will be developed to illustrate the multisoliton interaction dynamics. Conclusions are given in §4.

2. New zero structure

In this section, we shall investigate the direct scattering transform of the mKdV equation (1.1) by formulating RH problems. The RH problems and the new zero structures are the basis for the derivation of new soliton solutions. It is known that the mKdV equation (1.1) admits a Lax pair formulation

$$\Psi_x = \begin{pmatrix} i\lambda & u \\ -u & -i\lambda \end{pmatrix} \Psi, \tag{2.1a}$$

$$\Psi_t = \begin{pmatrix} 4i\lambda^3 + 2i\lambda u^2 & 4\lambda^2 u + 2i\lambda u_x - 2u^3 - u_{xx} \\ -4\lambda^2 u + 2i\lambda u_x + 2u^3 + u_{xx} & -4i\lambda^3 - 2i\lambda u^2 \end{pmatrix} \Psi, \tag{2.1b}$$

where Ψ is a column vector function of the spectral parameter λ. Since (2.1a) is just a special reduction of the Ablowitz–Kaup–Newell–Segur (AKNS) spectral problem, the generic RH problem established for the AKNS system [21] is still valid for (1.1). However, we should notice a key point here. That is, there will be more symmetry relations in the scattering data when solving the corresponding RH problems because u is required to be real in the mKdV equation.

According to ref. [21], we know that the RH problem for the mKdV equation (1.1) is

$$P^-(\lambda)P^+(\lambda) = \begin{pmatrix} 1 & s_{12}e^{2i\lambda x} \\ r_{21}e^{-2i\lambda x} & 1 \end{pmatrix}, \quad \lambda \in \mathbb{R}, \tag{2.2}$$

$$P_1(\lambda) \to \mathbb{I}, \qquad \lambda \in \mathbb{C}^+ \to \infty, \tag{2.3}$$

$$P_2(\lambda) \to \mathbb{I}, \qquad \lambda \in \mathbb{C}^- \to \infty, \tag{2.4}$$

where eqs (2.3) and (2.4) are called the canonical normalisation conditions. Here $P_1(\lambda)$ and $P_2(\lambda)$ are two matrix functions which are analytic in the upper half λ-plane $\mathbb{C}^+$ and the lower half λ-plane $\mathbb{C}^-$, respectively. $P^+(\lambda)$ is the limit of $P_1(\lambda)$ taken from the LHS of the real λ-axis, while $P^-(\lambda)$ is the limit of $P_2(\lambda)$ taken from the RHS of the real λ-axis. In addition, s_{12} and r_{21} are two reflection coefficients defined on the real λ-axis as we shall see below. In what follows, for completeness, we list the main steps for establishing the RH problem (2.2)–(2.4). We refer to ref. [21] for details of the RH formulation.

Step 1. Extending Ψ in (2.1) to a matrix form and then introducing a new matrix spectral function $J = J(x, t; \lambda)$ defined by

$$\Psi = Je^{i\lambda\sigma_3 x + 4i\lambda^3\sigma_3 t}, \tag{2.5}$$

where $\sigma_3 = \mathrm{diag}(1, -1)$, we can rewrite the original Lax pair (2.1) in the following form:

$$J_x = i\lambda[\sigma_3, J] + QJ, \tag{2.6a}$$

$$J_t = 4i\lambda^3[\sigma_3, J] + \tilde{Q}J, \tag{2.6b}$$

where the square bracket is the matrix commutator, and

$$Q = \begin{pmatrix} 0 & u \\ -u & 0 \end{pmatrix},$$

$$\tilde{Q} = 4\lambda^2 Q - 2i\lambda(Q^2 + Q_x)\sigma_3 + 2Q^3 - Q_{xx}.$$

Step 2. Consider the direct scattering transform by introducing two Jost solutions $J_\pm = J_\pm(x, \lambda)$ of (2.6a)

$$J_\pm = ([J_\pm]_1, [J_\pm]_2), \tag{2.7}$$

with the asymptotic conditions

$$J_\pm \to \mathbb{I}, \quad x \to \pm\infty. \tag{2.8a}$$

One can verify that $[J_+]_1, [J_-]_2$ allow analytic extensions to $\mathbb{C}^+$. On the other hand, $[J_-]_1, [J_+]_2$ are analytically extendible to $\mathbb{C}^-$. Moreover, we have

$$J_-e^{i\lambda\sigma_3 x} = J_+e^{i\lambda\sigma_3 x} \cdot S(\lambda), \quad \lambda \in \mathbb{R}, \tag{2.9}$$

where $S(\lambda)$ is the scattering matrix $S(\lambda) = (s_{kj})_{2\times 2}$. In addition, spectral analysis shows that s_{22} and s_{11} allow analytic extensions to $\mathbb{C}^+$ and $\mathbb{C}^-$, respectively. In general, s_{12}, s_{21} cannot be extended off the real λ-axis.

Step 3. Consider the matrix inverses of $J_\pm$ as

$$J_\pm^{-1} = \begin{pmatrix} [J_\pm^{-1}]^1 \\ [J_\pm^{-1}]^2 \end{pmatrix}, \tag{2.10}$$

where each $[J_\pm^{-1}]^l$ $(l = 1, 2)$ denotes the lth row of $J_\pm^{-1}$, respectively. It is easy to verify that $J_\pm^{-1}$ satisfy the linear equation of K

$$K_x = i\lambda[\sigma_3, K] - KQ, \tag{2.11}$$

which is an adjoint equation of (2.6a). Resorting to eq. (2.11), one can see that the two rows $[J_+^{-1}]^1, [J_-^{-1}]^2$ are analytically extendible to $\mathbb{C}^-$, whereas $[J_-^{-1}]^1, [J_+^{-1}]^2$ allow analytic extensions to $\mathbb{C}^+$. Moreover, from (2.9) it is easy to find that

$$\mathrm{e}^{-i\lambda\sigma_3 x} J_-^{-1} = R(\lambda) \cdot \mathrm{e}^{-i\lambda\sigma_3 x} J_+^{-1}, \quad \lambda \in \mathbb{R}, \tag{2.12}$$

where $R(\lambda) \equiv (r_{kj})_{2\times 2} = S^{-1}(\lambda)$. Additionally, spectral analysis shows that r_{22} and r_{11} allow analytic extensions to $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. In general, r_{12} and r_{21} cannot be extended off the real λ-axis.

Step 4. Define two matrix functions $P_1 = P_1(\lambda)$ and $P_2 = P_2(\lambda)$, which are analytic for $\lambda \in \mathbb{C}^+$ and $\lambda \in \mathbb{C}^-$, respectively

$$P_1 = ([J_+]_1, [J_-]_2), \tag{2.13}$$

$$P_2 = \begin{pmatrix} [J_+^{-1}]^1 \\ [J_-^{-1}]^2 \end{pmatrix}. \tag{2.14}$$

Now taking the limit of P_1 from the LHS of the real λ-axis as $P^+(\lambda)$, and the limit of P_2 from the RHS of the real λ-axis as $P^-(\lambda)$, a direct calculation leads to (2.2). In addition, the large-λ asymptotic behaviours of P_1 and P_2 can be obtained, which are (2.3) and (2.4). Therefore, we arrive at the RH problem (2.2)–(2.4).

Now we shall solve the RH problem (2.2)–(2.4). Let us first investigate the zero structure of the RH problem. Note that there are two symmetry conditions for Q in (2.6a)

$$Q^\dagger = -Q, \tag{2.15}$$

$$Q^* = Q. \tag{2.16}$$

Using (2.15) and the definitions of P_1 and P_2, one obtain

$$P_1^\dagger(\lambda^*) = P_2(\lambda), \quad \lambda \in \mathbb{C}^-. \tag{2.17}$$

On the other hand, using (2.16), the following relation also holds:

$$P_1^*(-\lambda^*) = P_1(\lambda), \quad \lambda \in \mathbb{C}^+. \tag{2.18}$$

Therefore, in view of (2.17) and (2.18), we see clearly that if λ is a zero of $\det P_1$, then $\hat{\lambda} = \lambda^*$ is a zero of $\det P_2$. Moreover, we know that if λ is a zero of $\det P_1$, then $-\lambda^*$ is also its zero. Now let us consider a new zero structure of the RH problem which involves both types of eigenvalues: the ones situated symmetrically across the imaginary axis and the purely imaginary ones. That is, $\det P_1$ has $2N + M$ simple zeros λ_j $(1 \le j \le 2N + M)$ in $\mathbb{C}^+$, where $\lambda_{N+l} = -\lambda_l^*$ $(1 \le l \le N)$ and λ_{2N+l} $(1 \le l \le M)$ are purely imaginary ones. Correspondingly, $\det P_2$ has $2N + M$ simple zeros $\hat{\lambda}_j = \lambda_j^*$ $(1 \le j \le 2N + M)$, which are all in $\mathbb{C}^-$. We expect that the new zero structure here might lead to a new combined soliton solution for the mKdV equation (1.1).

To confirm this conjecture, we first solve the RH problem (2.2)–(2.4) by using the continuous scattering data s_{12}, r_{21} and the discrete scattering data $\lambda_j, \hat{\lambda}_j, v_j, \hat{v}_j$ $(1 \le j \le 2N + M)$. Here, v_j and $\hat{v}_j$ are non-zero column and row vectors satisfying $P_1(\lambda_j)v_j = 0$ and $\hat{v}_j P_2(\hat{\lambda}_j) = 0$, respectively. Corresponding to the reflectionless case, i.e., $s_{12} = r_{21} = 0$, the RH problem (2.2)–(2.4) can be solved as

$$P_1(\lambda) = \mathbb{I} - \sum_{k=1}^{2N+M} \sum_{j=1}^{2N+M} \frac{v_k \hat{v}_j (A^{-1})_{kj}}{\lambda - \hat{\lambda}_j}, \tag{2.19a}$$

$$P_2(\lambda) = \mathbb{I} + \sum_{k=1}^{2N+M} \sum_{j=1}^{2N+M} \frac{v_k \hat{v}_j (A^{-1})_{kj}}{\lambda - \lambda_k}, \tag{2.19b}$$

where $A = (a_{kj})$ is a $(2N + M)$th order matrix whose entries are

$$a_{kj} = \frac{\hat{v}_k v_j}{\lambda_j - \hat{\lambda}_k}$$

with $v_j = e^{i\lambda_j\sigma_3 x}\cdot v_{j,0}(t)$ and $\hat{v}_j = v_j^\dagger$. Here, each $v_{j,0}(t)$ is real for $2N+1\le j\le 2N+M$ due to the symmetry relation (2.18).

3. The new combined soliton solution

In this section, we shall investigate the IST for (1.1), from which we derive the new combined soliton solution. To this end, we expand $P_1(\lambda)$ in (2.19a) as

$$P_1(\lambda) = \mathbb{I} + \lambda^{-1}P_1^{(1)} + \lambda^{-2}P_1^{(2)} + \cdots, \quad \lambda\to\infty. \tag{3.1}$$

Then substituting it into (2.6a) and equating the $O(1)$ terms, we get

$$Q = -i[\sigma_3, P_1^{(1)}], \tag{3.2}$$

which implies that u can be obtained as

$$u = -2i(P_1^{(1)})_{12}, \tag{3.3}$$

where $(P_1^{(1)})_{12}$ is the $(1,2)$-entry of the matrix function $P_1^{(1)}$. Here, the matrix $P_1^{(1)}$ can be found from (2.19a) as

$$P_1^{(1)} = -\sum_{k=1}^{2N+M}\sum_{j=1}^{2N+M} v_k\hat{v}_j(A^{-1})_{kj}. \tag{3.4}$$

To obtain solutions for the mKdV equation (1.1), we have to consider the temporal evolutions of the scattering data. By noticing (2.6b) and the decaying properties of u, we arrive at

$$v_{j,t} = 4i\lambda_j^3\sigma_3 v_j. \tag{3.5}$$

Then using eq. (3.5), the corresponding vectors v_j and $\hat{v}_j$ can be determined as

$$v_j = \begin{cases} e^{\theta_j\sigma_3}v_{j,0}, & 1\le j\le N,\\ e^{\theta^*_{j-N}\sigma_3}v^*_{j-N,0}, & N+1\le j\le 2N,\\ e^{\theta_j\sigma_3}v_{j,0}, & 2N+1\le j\le 2N+M \end{cases} \tag{3.6}$$

and

$$\hat{v}_j = \begin{cases} v^\dagger_{j,0}e^{\theta^*_j\sigma_3}, & 1\le j\le N,\\ v^T_{j-N,0}e^{\theta_{j-N}\sigma_3}, & N+1\le j\le 2N,\\ v^\dagger_{j,0}e^{\theta^*_j\sigma_3}, & 2N+1\le j\le 2N+M, \end{cases} \tag{3.7}$$

with $\theta_j = i\lambda_j x + 4i\lambda_j^3 t$ ($\lambda_j\in\mathbb{C}^+$), where $\mathrm{Re}(\lambda_j)\neq 0$ $(1\le j\le N)$ and $\mathrm{Re}(\lambda_j)=0$ $(2N+1\le j\le 2N+M)$. Here $v_{j,0}(1\le j\le N)$ are complex constant column vectors and $v_{j,0}(2N+1\le j\le 2N+M)$ are real ones.

Now we are ready to derive new combined soliton solution for (1.1). By assuming that $v_{j,0} = (\alpha_j,1)^T$ $(1\le j\le N)$ are complex and $v_{j,0} = (\alpha_j,1)^T$ $(2N+1\le j\le 2N+M)$ are real, then using eqs (3.6) and (3.7), a new combined soliton solution can be obtained for (1.1) from (3.3)

$$\begin{aligned} u &= 2i\sum_{k=1}^{N}\sum_{j=1}^{N}\alpha_k e^{\theta_k-\theta_j^*}(A^{-1})_{kj}\\ &\quad+2i\sum_{k=1}^{N}\sum_{j=N+1}^{2N}\alpha_k e^{\theta_k-\theta_{j-N}}(A^{-1})_{kj}\\ &\quad+2i\sum_{k=N+1}^{2N}\sum_{j=1}^{N}\alpha^*_{k-N}e^{\theta^*_{k-N}-\theta^*_j}(A^{-1})_{kj}\\ &\quad+2i\sum_{k=N+1}^{2N}\sum_{j=N+1}^{2N}\alpha^*_{k-N}e^{\theta^*_{k-N}-\theta_{j-N}}(A^{-1})_{kj}\\ &\quad+2i\sum_{k=1}^{N}\sum_{j=2N+1}^{2N+M}\alpha_k e^{\theta_k-\theta_j}(A^{-1})_{kj}\\ &\quad+2i\sum_{k=N+1}^{2N}\sum_{j=2N+1}^{2N+M}\alpha^*_{k-N}e^{\theta^*_{k-N}-\theta_j}(A^{-1})_{kj}\\ &\quad+2i\sum_{k=2N+1}^{2N+M}\sum_{j=1}^{N}\alpha_k e^{\theta_k-\theta^*_j}(A^{-1})_{kj}\\ &\quad+2i\sum_{k=2N+1}^{2N+M}\sum_{j=N+1}^{2N}\alpha_k e^{\theta_k-\theta_{j-N}}(A^{-1})_{kj}\\ &\quad+2i\sum_{k=2N+1}^{2N+M}\sum_{j=2N+1}^{2N+M}\alpha_k e^{\theta_k-\theta_j}(A^{-1})_{kj}, \end{aligned} \tag{3.8}$$

where $A = (a_{kj})_{(2N+M)\times(2N+M)}$ with the matrix entries

$$a_{kj}=\begin{cases}\dfrac{\alpha_k^*\alpha_j e^{\theta_k^*+\theta_j}+e^{-\theta_k^*-\theta_j}}{\lambda_j-\lambda_k^*}, & 1\le k,\ j\le N;\\ \dfrac{\alpha_k^*\alpha_{j-N}^* e^{\theta_k^*+\theta_{j-N}^*}+e^{-\theta_k^*-\theta_{j-N}^*}}{-\lambda_{j-N}^*-\lambda_k^*}, & 1\le k\le N,\ N+1\le j\le 2N;\\ \dfrac{\alpha_k^*\alpha_j e^{\theta_k^*+\theta_j}+e^{-\theta_k^*-\theta_j}}{\lambda_j-\lambda_k^*}, & 1\le k\le N,\ 2N+1\le j\le 2N+M;\\ \dfrac{\alpha_{k-N}\alpha_j e^{\theta_{k-N}+\theta_j}+e^{-\theta_{k-N}-\theta_j}}{\lambda_j+\lambda_{k-N}}, & N+1\le k\le 2N,\ 1\le j\le N;\\ \dfrac{\alpha_{k-N}\alpha_{j-N}^* e^{\theta_{k-N}+\theta_{j-N}^*}+e^{-\theta_{k-N}-\theta_{j-N}^*}}{-\lambda_{j-N}^*+\lambda_{k-N}}, & N+1\le k,\ j\le 2N;\\ \dfrac{\alpha_{k-N}\alpha_j e^{\theta_{k-N}+\theta_j}+e^{-\theta_{k-N}-\theta_j}}{\lambda_j+\lambda_{k-N}}, & N+1\le k\le 2N,\ 2N+1\le j\le 2N+M;\\ \dfrac{\alpha_k\alpha_j e^{\theta_k+\theta_j}+e^{-\theta_k-\theta_j}}{\lambda_j-\lambda_k^*}, & 2N+1\le k\le 2N+M,\ 1\le j\le N;\\ \dfrac{\alpha_k\alpha_{j-N}^* e^{\theta_k+\theta_{j-N}^*}+e^{-\theta_k-\theta_{j-N}^*}}{-\lambda_{j-N}^*-\lambda_k^*}, & 2N+1\le k\le 2N+M,\ N+1\le j\le 2N;\\ \dfrac{\alpha_k\alpha_j e^{\theta_k+\theta_j}+e^{-\theta_k-\theta_j}}{\lambda_j-\lambda_k^*}, & 2N+1\le k,\ j\le 2N+M.\end{cases}$$

Remark 1. It should be pointed out that, after some algebra calculations, one can verify that the new combined soliton solution (3.8) is indeed real.

In what follows, to have a better understanding of (3.8), let us investigate it by using the symbolic computation system *Mathematica*. However, it is not easy to demonstrate (3.8) directly as its representation is rather complicated. To overcome this difficulty, let us develop an algebra technique to rewrite (3.8) in a compact form, i.e., the ratio of two determinants

$$u=-2i\,\frac{\det\ F}{\det\ A}, \tag{3.9}$$

where A is the $(2N+M)$th order matrix defined in (3.8), and $F=\begin{pmatrix}0 & \mathbf{a}\\ \mathbf{b} & M\end{pmatrix}$ with $\mathbf{a}$ being a row vector and $\mathbf{b}$ being a column vector

$$\mathbf{a}=(0,\alpha_1 e^{\theta_1},\ldots,\alpha_N e^{\theta_N},\alpha_1^* e^{\theta_1^*},\ldots,\\ \alpha_N^* e^{\theta_N^*},\alpha_{2N_1+1}e^{\theta_{2N+1}},\ldots,\alpha_{2N+M}e^{\theta_{2N+M}}),$$
$$\mathbf{b}=(0,e^{-\theta_1^*},\ldots,e^{-\theta_N^*},e^{-\theta_1},\ldots,e^{-\theta_N},\\ e^{-\theta_{2N+1}},\ldots,e^{-\theta_{2N+M}})^T.$$

The form of (3.9) is suitable to be written in *Mathematica* commands. Thus, one can plot the combined soliton solution (3.8) easily. In what follows, we shall study the soliton structure of the mKdV equation by investigating some representative cases.

Firstly, the corresponding parameters in (3.8) are chosen as

$$N=M=1,\ \alpha_1=i,\ \alpha_3=-1,$$

$$\lambda_1=0.3+0.4i,\ \lambda_3=0.5i. \tag{3.10}$$

In this case, the new combined soliton solution (3.8) represents a one-breather–one-bell soliton solution for the mKdV equation (1.1). This breather–bell soliton interaction is shown in figure 1, from which we see clearly that an elastic collision between a breather and an upward bell soliton occurs. Obviously, both the breather and the upward bell soliton keep their respective forms before and after the interaction.

Furthermore, to compare with the case in (3.10), we select the parameters in (3.8) as

$$N=M=1,\quad \alpha_1=i,\quad \alpha_3=1,$$

$$\lambda_1=0.3+0.4i,\quad \lambda_3=0.5i. \tag{3.11}$$

Note that the only difference between (3.11) and (3.10) is the value of α_3. Under (3.11), the combined soliton solution (3.8) is also a one-breather–one-bell soliton

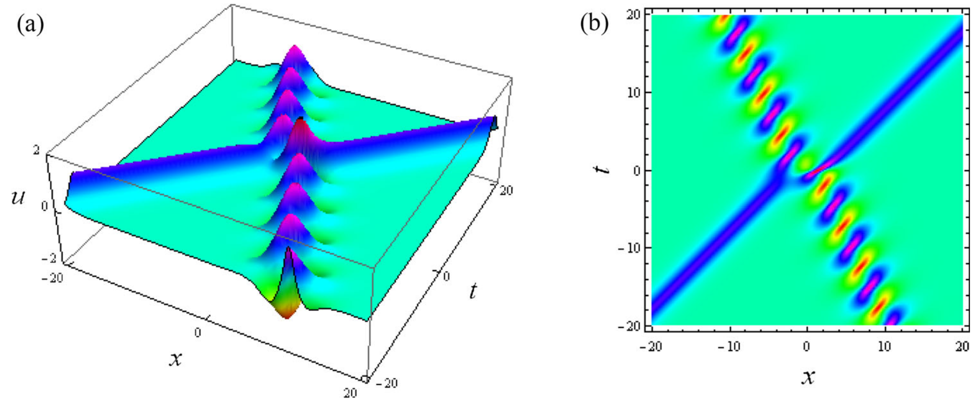

Figure 1. One-breather–one-bell soliton solution via (3.8) with the parameters in (3.10) (one breather and one upward bell soliton). (**a**) 3D plot and (**b**) density plot.

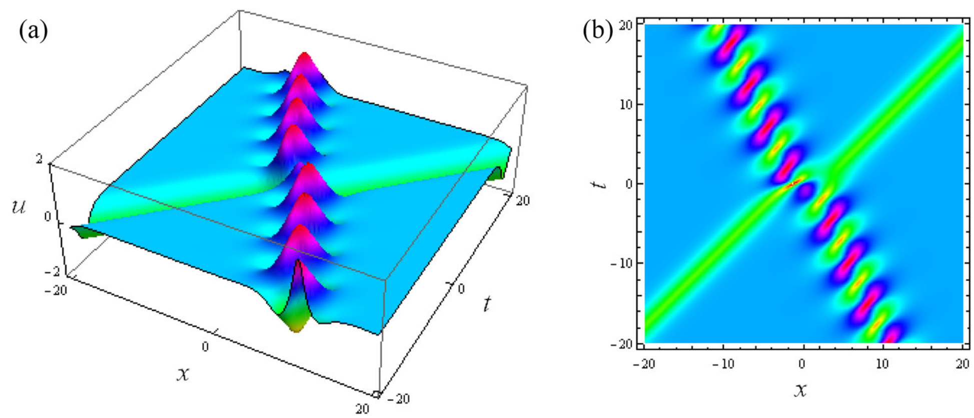

Figure 2. One-breather–one-bell soliton solution via (3.8) with the parameters in (3.11) (one breather and one downward bell soliton). (**a**) 3D plot and (**b**) density plot.

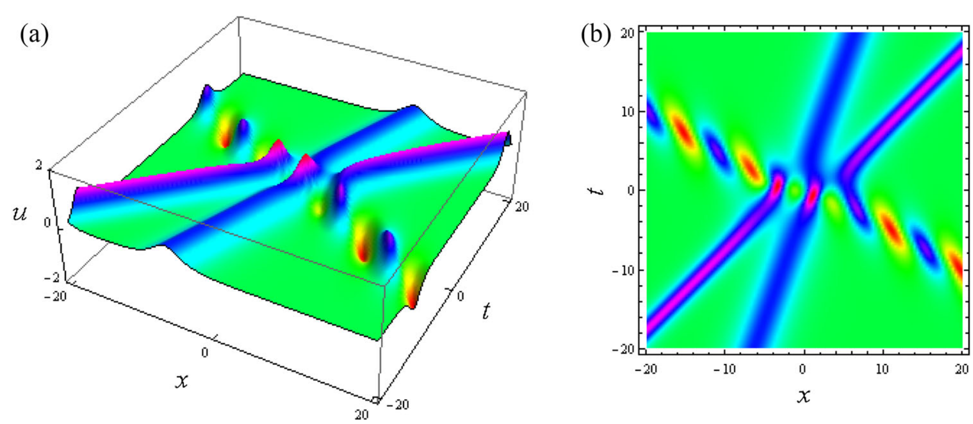

Figure 3. One-breather–two-bell soliton solution via (3.8) with the parameters in (3.12) (one breather and two upward bell solitons). (**a**) 3D plot and (**b**) density plot.

solution for the mKdV equation (1.1). However, in this case, the bell soliton is a downward one. The interaction of the breather and the downward bell soliton is shown in figure 2.

Secondly, if the parameters in (3.8) are

$$N = 1, \quad M = 2, \quad \alpha_1 = i, \quad \alpha_3 = -1, \quad \alpha_4 = 1,$$
$$\lambda_1 = 0.4 + 0.2i, \quad \lambda_3 = 0.5i, \quad \lambda_4 = 0.3i, \tag{3.12}$$

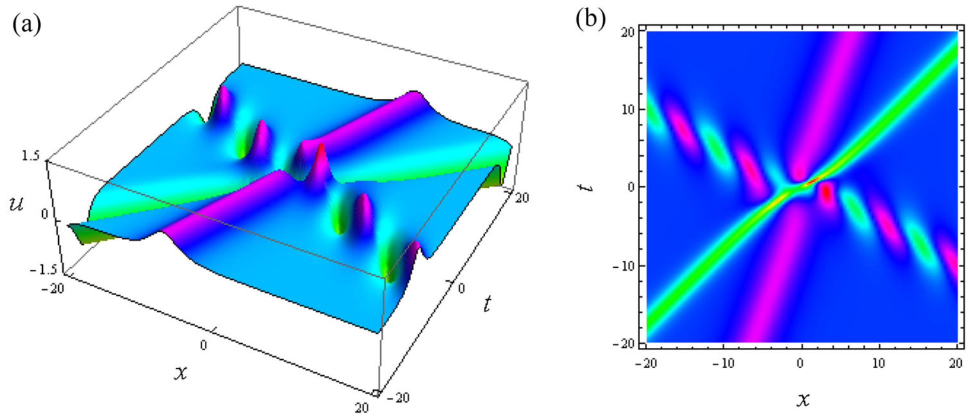

Figure 4. One-breather–two-bell soliton solution via (3.8) with the parameters in (3.13) (one breather, one upward bell soliton and one downward bell soliton). (**a**) 3D plot and (**b**) density plot.

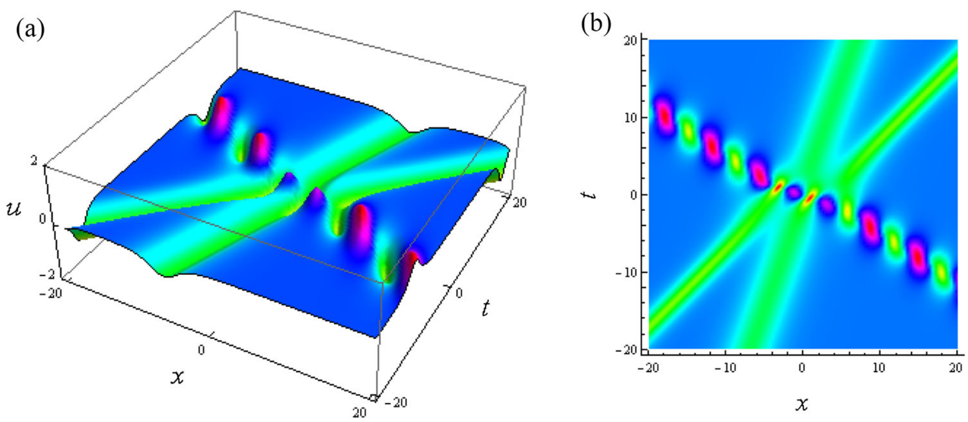

Figure 5. One-breather–two-bell soliton solution via (3.8) with the parameters in (3.14) (one breather and two downward bell solitons). (**a**) 3D plot and (**b**) density plot.

then a one-breather–two-bell soliton solution will be obtained for the mKdV equation (1.1). It consists of one breather and two upward bell solitons. The interaction characteristic of this solution is demonstrated in figure 3, which shows that the breather and the two upward bell solitons maintain their own forms respectively before and after the collision. Moreover, to compare with the case in (3.12), we further set the parameters in (3.8) as

$$N = 1, \quad M = 2, \quad \alpha_1 = -i, \quad \alpha_3 = 1, \quad \alpha_4 = 1,$$
$$\lambda_1 = 0.4 + 0.2i, \quad \lambda_3 = 0.5i, \quad \lambda_4 = 0.3i. \tag{3.13}$$

Corresponding to (3.13), another one-breather–two-bell soliton solution can be obtained for the mKdV equation (1.1). However, contrary to the case in (3.12), the interaction corresponding to (3.13) is among a breather, an upward bell soliton and a downward bell soliton, as shown in figure 4. Furthermore, if the parameters in (3.8) are chosen as

$$N = 1, \quad M = 2, \quad \alpha_1 = -i, \quad \alpha_3 = 1, \quad \alpha_4 = -1,$$
$$\lambda_1 = 0.4 + 0.2i, \quad \lambda_3 = 0.5i, \quad \lambda_4 = 0.3i. \tag{3.14}$$

Then a solution with one breather and two downward bell solitons will be obtained for (1.1). This kind of one-breather–two-bell soliton collision is illustrated in figure 5.

Thirdly, let us set the parameters in (3.8) to be

$$N = 1, \quad M = 2, \quad \alpha_1 = i, \quad \alpha_3 = -0.1,$$
$$\alpha_4 = 1000, \ \lambda_1 = 0.4 + 0.2i,$$
$$\lambda_3 = 0.5i, \ \lambda_4 = 0.3i. \tag{3.15}$$

Then another type of one-breather–two-bell soliton solution is obtained for the mKdV equation (1.1). There are still a breather and two upward bell solitons, which is the same as the case in (3.12). However, there are three collisions during the interaction process corresponding to (3.15), as can be seen in figure 6.

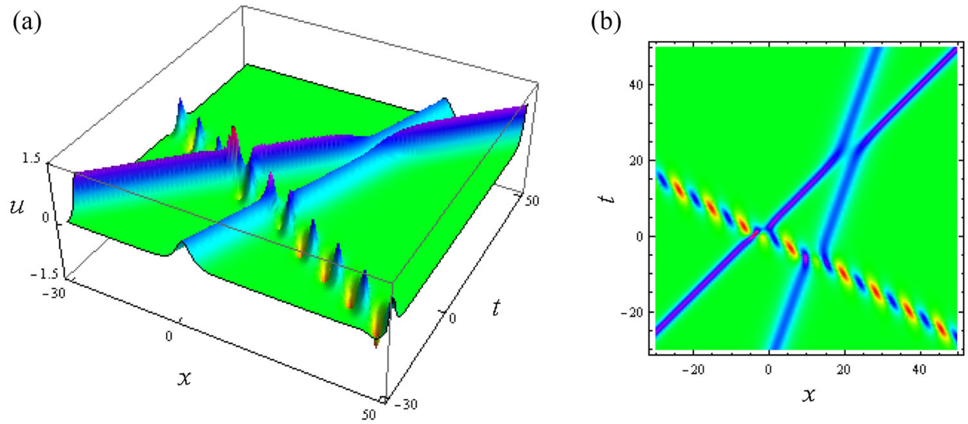

Figure 6. One-breather–two-bell soliton solution via (3.8) with the parameters in (3.15) (one breather and two upward bell solitons with three collisions). (**a**) 3D plot and (**b**) density plot.

4. Conclusions

In this paper, a new combined soliton solution of the mKdV equation (1.1) is obtained by investigating the new zero structure of the corresponding RH problem. The new zero structure involves a combination of eigenvalues situated symmetrically across the imaginary axis and those locating on the imaginary axis, which is more complicated. The new combined soliton solution is (3.8), which is a unified representation of breather–bell soliton solution. This new combined soliton solution is fairly more general as it involves many parameters. To illustrate the dynamics of the new combined soliton solution, an algebra technique is developed to rewrite the solution in a compact form which is convenient to be written in *Mathematica* commands. Upon choosing appropriate parameters in (3.8), a few figures are plotted to demonstrate the multi-breather and multi-bell soliton interactions. To our knowledge, the combined soliton solution (3.8) has not been reported for the mKdV equation (1.1) previously. The results also expand the understanding of the soliton structure of the mKdV equation (1.1). We hope that the proposed method for deriving new multisoliton solutions of the mKdV equation using the new zero structures can be applied to other soliton equations also.

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