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Thermal entanglement and teleportation via thermally atomic entangled state in cavity QED

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Abstract. In this article, we study thermal entanglement between two-coupled two-level atoms interacting with a single-mode cavity field. The cavity mode is assumed to be initially in the vacuum state and the detuning parameter is large. The thermal entanglement in this two-qubit system is quantified in terms of the concurrence, as a proper measure for two-qubit mixed states. We investigate how the thermal entanglement depends on temperature, dipole coupling between the atoms and detuning parameter. In addition, we shall use the generated thermally atomic entangled state as a quantum channel for teleportation and we mainly concentrate on the fidelity of teleportation. Finally, the relationship between the quantum coherence and entanglement will be discussed.

Keywords. Quantum coherence; concurrence; fidelity; Jaynes–Cummings model; teleportation; thermal entanglement.

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1. Introduction

Quantum entanglement as an important resource is one of the most striking features of quantum mechanics. It plays a central role in quantum information theory as well as in quantum teleportation [1], communication and quantum computing [2,3]. Due to its fundamental role, the generation of entangled state between separated subsystems is an important subject. In recent years, several methods have been proposed for creating entangled states. One of them is the Jaynes–Cummings model (JCM). The JCM explains the interaction between the quantised electromagnetic field and the atom [4]. The JCM is a simple but applicable tool. Over the last two decades, many efforts have been devoted to the applications of JCM in quantum information [5–7] and quantum teleportation [8]. The entangled states induced from the JCM have been used as quantum channels [9]. Zang *et al* [10] used the interaction between two-level atoms and a single-mode cavity field with large detuning for transforming bipartite non-maximally entangled states into a W-state. Dynamics of entanglement between the atom interacting with one-mode electromagnetic cavity field have been studied [11]. Due to the significance of the JCM in quantum optics, it has been extended to multilevel atom and deformed JCM [12,13]. A surprising phenomenon was termed as entanglement sudden death (ESD) [11,14]. From both theoretical and experimental points of view [15], ESD has attracted much attention. Interaction of the system with the environment is an inevitable problem of real physical systems. For instance, one can consider the entanglement between two subsystems when they interact with a thermal environment. In this case, the entanglement is called thermal entanglement. The thermal entanglement has been extensively studied for various systems such as isotropic and anisotropic Heisenberg and Ising models in the magnetic field [16–21]. Quantum discord is a measure of the difference between total correlation and the classical correlation which measures all non-classical correlations. Recently, some papers have been devoted to study the thermal quantum discord in the spin chain systems [22,23]. In the context of quantum optics, the thermal entanglement between atoms and fields has been studied. For example, thermal atom–atom entanglement in a cavity filled with a Kerr medium has been investigated in [24,25].

In this work, we consider two identical two-level atoms interacting with a single-mode cavity field [5, 26,27]. Here it is supposed that the cavity mode is

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initially in the vacuum state $|0\rangle$, and the detuning between the radiation field and the atomic transition is large. By calculating the eigenvalues and eigenvectors of the Hamiltonian, we construct the thermal density matrix. The atom–atom thermal entanglement at an equilibrium temperature T can be attained by concurrence measure, introduced by Wootters [28,29]. The effects of temperature, atom–atom coupling constant and detuning parameter on entanglement between two atoms are investigated. Moreover, we consider our model as a quantum channel for teleportation and discuss the behaviour of average fidelity with respect to temperature, coupling constant and detuning parameter. Due to the importance of the relationship between the quantum coherence and entanglement, we study the coherence for our model.

The outline of this paper is as follows: In §2, we introduce the model. The thermal density operator and the thermal entanglement are studied in §3. In §4, we recall our model as the quantum teleportation channel and explore the dependence of the average fidelity on temperature, coupling constant and detuning parameter. Quantum coherence for thermal entangled state is studied in §5. Finally, conclusions are given in the last section.

2. Description of the model's Hamiltonian

The interaction of two-coupled two-level atoms with a single-mode cavity field can be described by the following Hamiltonian

$$H = \hbar\omega a^{\dagger}a + \frac{\hbar\omega_0}{2}(\sigma_z^1 + \sigma_z^2) + \hbar g \sum_{j=1,2}(a^{\dagger}\sigma_-^j + a\sigma_+^j), \tag{1}$$

where ω and ω_0 are the cavity and the atomic transition frequencies, respectively and g is the atom–field coupling constant. The operators a and $a^{\dagger}$ are annihilation and creation operators for the quantised field. Also,

$$\sigma_z^j = |e_j\rangle\langle e_j| - |g_j\rangle\langle g_j| \tag{2}$$

is the z-component of Pauli operator, $\sigma_+^j = |e_j\rangle\langle g_j|$ and $\sigma_-^j = |g_j\rangle\langle e_j|$ are raising and lowering operators of jth two-level atom with $|e_j\rangle$ and $|g_j\rangle$ ($j = 1, 2$) being the excited and ground states of the atom. For large detuning parameter, $\delta = \omega_0 - \omega \gg g\sqrt{\bar{n}}$, with $\bar{n}$ being the mean photon number of the cavity field [5], there is no energy exchange between the atomic systems. Then the effective Hamiltonian takes the form

$$H_{\text{eff}} = \lambda\left[\sum_{j=1,2}(|e_j\rangle\langle e_j|aa^{\dagger} - |g_j\rangle\langle g_j|a^{\dagger}a) + (\sigma_+^1\sigma_-^2 + \sigma_-^1\sigma_+^2)\right], \tag{3}$$

where $\lambda = \hbar g^2/\delta$. The first and second terms of this Hamiltonian describe the photon number dependent Stark shifts, and the third term describes the dipole coupling between the atoms in the cavity. Let us assume that the cavity mode is initially in the vacuum state. As $aa^{\dagger}|0\rangle = |0\rangle$ and $a^{\dagger}a|0\rangle = 0$, the effective Hamiltonian reads as

$$H_{\text{eff}} = \lambda(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + \sigma_+^1\sigma_-^2 + \sigma_-^1\sigma_+^2). \tag{4}$$

So cavity mode will remain in the vacuum state throughout the procedure. The Hilbert space of coupled two-level atoms is spanned by the basis

$$\{|g_1, g_2\rangle, |g_1, e_2\rangle, |e_1, g_2\rangle, |e_1, e_2\rangle\}. \tag{5}$$

Therefore, in the basis of the product states (5) the non-zero matrix elements of the effective Hamiltonian are

$$\begin{aligned} &h_{11} = 2\lambda, \\ &h_{22} = h_{23} = h_{32} = h_{33} = \lambda. \end{aligned} \tag{6}$$

To determine the thermal state of the present system, we have to find the eigenvalues and eigenvectors of the Hamiltonian. The resulting eigenvalues and the corresponding eigenvectors of the system are

$$E_{1,2} = 2\lambda,\ E_{3,4} = 0, \tag{7}$$

and

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}}(|e_1g_2\rangle + |g_1e_2\rangle), \\ |\psi_2\rangle &= |e_1e_2\rangle, \\ |\psi_3\rangle &= |g_1g_2\rangle, \\ |\psi_4\rangle &= \frac{1}{\sqrt{2}}(|e_1g_2\rangle - |g_1e_2\rangle). \end{aligned} \tag{8}$$

In the next section, eqs (7) and (8) are used to form the thermal density operator.

3. Thermal atom–atom density operator and entanglement

The state of the system at thermal equilibrium (temperature T) is given by

$$\rho(T) = \frac{1}{Z}\mathrm{e}^{-H/K_{\mathrm{B}}T}, \tag{9}$$

in which the partition function is $Z = \mathrm{Tr}(\mathrm{e}^{-H/K_{\mathrm{B}}T})$ and K_{B} is the Boltzmann's constant. The matrix elements of the thermal density operator in the standard basis,

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}, \tag{10}$$

are

$$\rho_{11}(T) = \frac{1}{Z}\mathrm{e}^{-E_1/K_{\mathrm{B}}T},$$
$$\rho_{22}(T) = \rho_{33}(T) = \frac{1}{2Z}(1+\mathrm{e}^{-E_1/K_{\mathrm{B}}T}),$$
$$\rho_{23}(T) = \rho_{32}(T) = \frac{1}{2Z}(-1+\mathrm{e}^{-E_1/K_{\mathrm{B}}T}),$$
$$\rho_{44}(T) = \frac{1}{Z}, \tag{11}$$

where $Z = 2(1+\mathrm{e}^{-E_1/K_{\mathrm{B}}T})$. Here we shall quantify the thermal entanglement between atomic states. One of the suitable measures for characterising the degree of entanglement for two-qubit mixed state is concurrence which was introduced by Wootters [28] as follows:

$$C(\rho) = \max\left\{0, \sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\right\}, \tag{12}$$

where λ_i $(i = 1, 2, 3, 4)$ are the non-negative eigenvalues of $\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)$, in decreasing order, σ_y represents the second Pauli matrix and ρ^* is the complex conjugate of ρ. For the thermal atomic density matrix, eq. (11), the concurrence is obtained as [30]

$$C(T) = 2\max\{0, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}\}. \tag{13}$$

Note that $0 \le C(T) \le 1$. $C(T) = 1$ corresponds to the maximally entangled state, while $C(T) = 0$ indicates that the two atoms are separable. To investigate the effect of atom–field coupling constant and detuning parameter on thermal atom–atom entanglement, we plot concurrence as a function of scaled temperature, $\tau = K_{\mathrm{B}}T$, for given g and δ in figure 1.

According to figures 1a and 1b, when temperature $T = 0$, the thermal state of the atom–atom system is entangled and $C(T = 0) = 0.5$. This issue is expected, as the state of combined system collapses into its ground state at absolute zero temperature. By increasing the temperature, some excited states become mixed with the ground state, and thus the amount of entanglement varies. At very high temperatures, the thermal state of the system will be a fully mixed state and consequently, the entanglement disappears. So the thermal concurrence abruptly drops to zero, and ESD occurs at high temperatures. From figure 1a, it can be seen that by increasing the atom–atom coupling constant the thermal concurrence disappears at higher temperature. This is very important for generating thermally resistant atomic entangled state. Moreover, figure 1b shows that by increasing the detuning parameter the thermal entanglement suffers from sudden death at lower temperature.

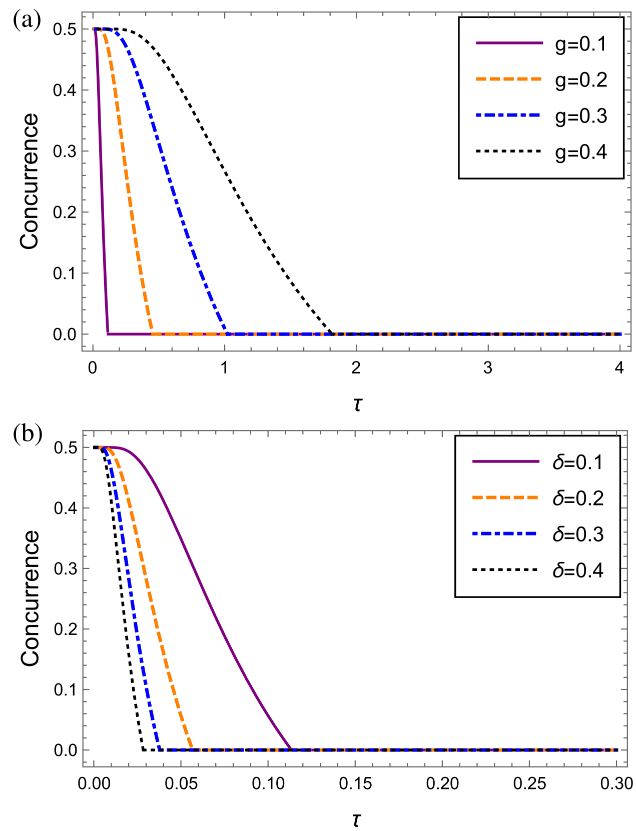

Figure 1. Concurrence $C(T)$ vs. scaled temperature τ for (**a**) $\delta = 0.1$ and (**b**) $g = 0.1$.

4. The fidelity of teleportation

Quantum teleportation is a way to transfer quantum states from one location to another based on entanglement. Let us assume that Alice wants to teleport the following arbitrary pure entangled state to Bob:

$$|\psi_{in}\rangle = \cos\frac{\theta}{2}|10\rangle + \mathrm{e}^{i\varphi}\sin\frac{\theta}{2}|01\rangle, \tag{14}$$

with $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$. The concurrence of the input state is given by

$$C_{\mathrm{in}} = 2\left|\cos\frac{\theta}{2}\sin\frac{\theta}{2}\right|.$$

To this aim Alice and Bob initially share an entangled state given by eq. (11) as a quantum channel. The output state is then given by

$$\rho_{\text{out}} = \sum_{i,j} w_{ij}(\sigma_i \otimes \sigma_j)\rho_{\text{in}}(\sigma_i \otimes \sigma_j), \quad i = 0, 1, 2, 3, \tag{15}$$

where σ_0 is the identity matrix and σ_i $(i = 1, 2, 3)$ are three components of the Pauli matrix. Here

$$w_{ij} = \text{Tr}(E_i\rho(T))\text{Tr}(E_j\rho(T)), \quad \sum_{i,j} w_{ij} = 1 \tag{16}$$

and

$$E^0 = |\psi^-\rangle\langle\psi^-|, \quad E^1 = |\varphi^-\rangle\langle\varphi^-|,$$
$$E^2 = |\varphi^+\rangle\langle\varphi^+|, \quad E^3 = |\psi^+\rangle\langle\psi^+|, \tag{17}$$

in which $|\psi^\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle)$ and $|\varphi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$ are Bell states. Starting from an X-structure density matrix equation (11), the output density matrix can be calculated as

$$\rho^{11}_{\text{out}} = \rho^{44}_{\text{out}} = (\rho_{11} + \rho_{44})(\rho_{22} + \rho_{33}),$$
$$\rho^{22}_{\text{out}} = (\rho_{22} + \rho_{33})^2\cos^2\frac{\theta}{2} + (\rho_{11} + \rho_{44})^2\sin^2\frac{\theta}{2},$$
$$\rho^{33}_{\text{out}} = (\rho_{22} + \rho_{33})^2\sin^2\frac{\theta}{2} + (\rho_{11} + \rho_{44})^2\cos^2\frac{\theta}{2},$$
$$\rho^{14}_{\text{out}} = \rho^{41}_{\text{out}} = (\rho_{14} + \rho_{41})(\rho_{23} + \rho_{32})\sin\theta\cos\varphi,$$
$$\rho^{23}_{\text{out}} = \frac{1}{2}((\rho_{23} + \rho_{32})^2\mathrm{e}^{-i\varphi} + (\rho_{14} + \rho_{41})^2\mathrm{e}^{i\varphi})\sin\theta. \tag{18}$$

To quantify the overlap of initial state with the state receiving to Bob, it is useful to measure the fidelity between ρ_{in} and ρ_{out}, which is defined by

$$F(\rho_{\text{in}}, \rho_{\text{out}}) = \text{Tr}(\rho_{\text{in}}\rho_{\text{out}}), \tag{19}$$

and the average fidelity over all possible pure states in Bloch sphere is given by [31,32]

$$F_{\text{ave}}(\rho_{\text{in}}, \rho_{\text{out}}) = \int_0^\pi \int_0^{2\pi} F(\rho_{\text{in}}, \rho_{\text{out}}) \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\varphi. \tag{20}$$

After some calculation for the X-structure density matrix (18), the average fidelity is obtained as

$$F_{\text{ave}}(\rho_{\text{in}}, \rho_{\text{out}}) = \frac{1 + \mathrm{e}^{2\lambda/\tau} + \mathrm{e}^{4\lambda/\tau}}{3(1 + \mathrm{e}^{2\lambda/\tau})^2}, \tag{21}$$

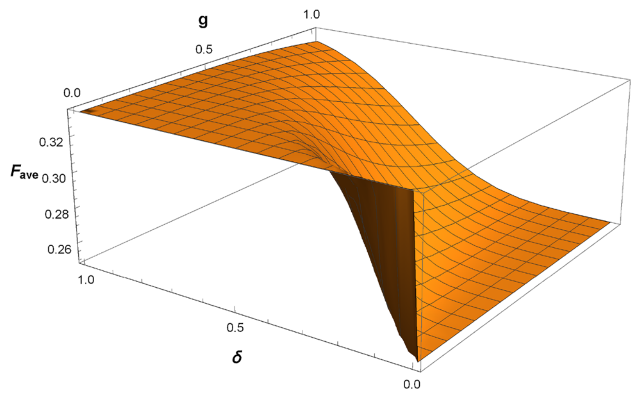

Figure 2. Average fidelity $F_{\text{ave}}(\rho_{\text{in}}, \rho_{\text{out}})$ vs. g and δ for a given scaled temperature $\tau = 0.5$.

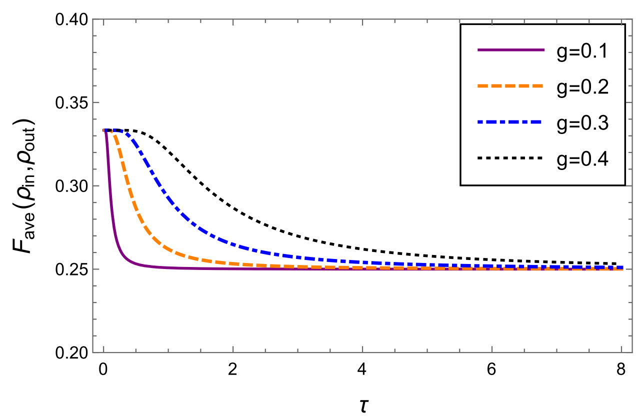

Figure 3. Average fidelity $F_{\text{ave}}(\rho_{\text{in}}, \rho_{\text{out}})$ vs. τ for different values of g when $\delta = 0.1$.

where $0 \leq F_{\text{ave}} \leq 1$. For successful quantum teleportation, the average fidelity must be larger than 2/3, which is the maximum of classical communication. The dependence of average fidelity on coupling constant (g) and detuning parameter (δ) can be easily observed in figure 2.

The average fidelity can reach the maximum value $F_{\text{ave}} \simeq 0.33$ by adjusting g and δ (see figure 2). In figure 3, the behaviour of the average fidelity for given g and $\delta = 0.1$ is plotted vs. scaled temperature τ.

From figure 3, there is no proper value of fidelity i.e., $F_{\text{ave}} < \frac{2}{3}$ and it illustrates that the average fidelity of teleportation using thermal entangled state (11) decreases with temperature. Furthermore, by increasing g, the average fidelity reduces slowly.

5. Quantum coherence for thermal entangled state

A fundamental emanation of quantum superposition principle is quantum coherence. That is important in quantum physics such as quantum coherence optics,

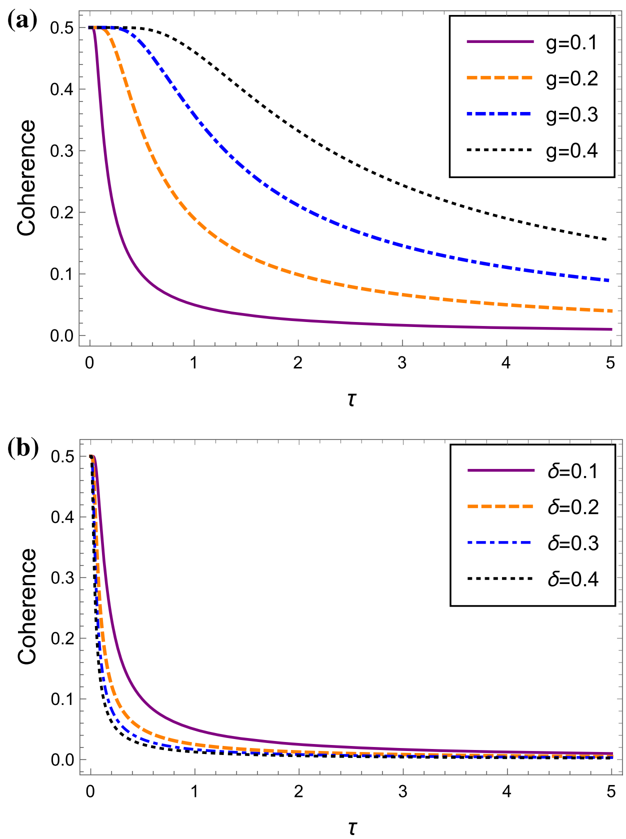

Figure 4. Coherence $c(\rho)$ as a function of τ for (**a**) $\delta = 0.1$ and (**b**) $g = 0.1$.

quantum thermodynamics, etc. Both coherence and entanglement show the quantum nature of physical phenomena. Therefore, it is meaningful to study the relationship between the quantum coherence and entanglement. There are a wide variety of coherence measures. Here we use the following definition [33]:

$$c(\rho) = \sum_{i \neq j} |\rho_{ij}|. \tag{22}$$

In other words, we find the sum of the absolute values of all off-diagonal elements of density matrices. So the density matrices that are diagonal in the specific basis are incoherent [34]. By substituting eq. (11) in definition (22), the coherence is obtained as

$$c(\rho) = \frac{|-1 + \mathrm{e}^{-2\lambda/\tau}|}{2(1 + \mathrm{e}^{-2\lambda/\tau})}. \tag{23}$$

The behaviour of coherence as a function of τ for the given coupling constant and detuning parameter is plotted in figure 4.

From figures 4a and 4b, we conclude that the quantum coherence and thermal entanglement have similar features (see figure 1). When $\tau = 0$, the coherence is maximum ($c(\rho) = 0.5$), then it decreases monotonically by raising temperature. On the other hand, from figure 4a we see that for $\delta = 0.1$, by increasing g, the coherence decreases more slowly. But, for $g = 0.1$, the coherence behaviour is different for various values of δ. In this case, by increasing the detuning parameter, the coherence reduces quickly.

6. Conclusion

In summary, we have examined the thermal entanglement between two-coupled two-level atoms in thermal equilibrium with a heat reservoir. To this aim we used concurrence measure for two-qubit mixed state, introduced by Wootters. The results showed that the thermal entanglement depends on temperature, dipole coupling between the atoms and detuning parameter. Thermal entanglement is maximum at absolute zero temperature ($T = 0$), i.e. $C(T = 0) = 0.5$. Due to the mixing of the excited states with the ground state, the thermal entanglement abruptly vanishes at higher temperature. For $\delta = 0.1$, an increase in the dipole coupling between the atoms leads to ESD at higher temperatures. This is very important for producing thermally robust entangled state. For $g = 0.1$, by increasing the detuning parameter, the thermal entanglement disappears at lower temperature. Furthermore, the thermal density matrix $\rho(T)$ has been used as a quantum channel for teleportation. In order to realise the efficiency of the quantum teleportation, average fidelity has been computed. From figure 3, it can be seen that the average fidelity of teleportation using thermal atomic entangled state can reach the maximum value $F_{\text{ave}} \simeq 0.33$ at $T = 0$ and it gradually decreases and tends to a constant value $F_{\text{ave}} \simeq 0.25$ at high temperature. The slope of decreasing reduces with g. At the end, the relationship between the quantum coherence and entanglement has been studied. We saw that the behaviour of coherence and thermal concurrence are similar in our model and the same results hold for the coherence. Quantum coherence and thermal entanglement exhibit a maximum at absolute zero temperature, $C(T) = c(\rho) = 0.5$ and decrease with temperature.

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