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Invariant subspaces and exact solutions for some types of scalar and coupled time-space fractional diffusion equations

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Abstract. We explain how the invariant subspace method can be extended to a scalar and coupled system of time-space fractional partial differential equations. The effectiveness and applicability of the method have been illustrated using time-space (i) fractional diffusion-convection equation, (ii) fractional reaction-diffusion equation, (iii) fractional diffusion equation with source term, (iv) two-coupled system of fractional diffusion equation, (v) two-coupled system of fractional stationary transonic plane-parallel gas flow equation and (vi) three-coupled system of fractional Hirota–Satsuma KdV equation. Also, we explicitly showed how to derive more than one exact solution of the equations as mentioned above using the invariant subspace method.

Keywords. Time-space fractional partial differential equations; invariant subspace method; Laplace transformation technique; Mittag–Leffler function.

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1. Introduction

The subject of fractional calculus is one of the most rapidly developing areas of mathematical analysis. The study of fractional differential equations (FDEs) has considerable popularity and importance for the past few decades, mainly due to their widespread applications in various fields of science and engineering such as fluid flow, viscoelasticity, aerodynamics, electromagnetic theory, rheology, signal processing, electrical networks and so on [1–8]. In the last few decades, several analytical and numerical techniques have been developed to construct exact and numerical solutions of nonlinear differential equations. However, the derivation of the exact solution of FDEs is not an easy task, because some properties of fractional derivatives are harder than that of the classical derivative.

For this reason, in recent years, both mathematicians and physicists have paid much attention to study the exact and numerical solutions of nonlinear fractional partial differential equations (FPDEs) using various ad-hoc methods, such as Lie group analysis method [9–15], Adomian decomposition method [16–18], homotopy decomposition method [19], differential transform method [20], function-expansion method [21–23] and so on. However, recent investigations have shown that a new analytic method based on the invariant subspace approach provides an effective tool to derive the exact solution of scalar and coupled system of time-space FPDEs. This method was originally developed by Galaktionov and Svirshchevskii [24] (see also [25–33]) for PDEs and was further extended by Gazizov and Kasatkin [34] (see also [35–46]) for time FPDEs.

The main objective of this article is to demonstrate how the invariant subspace method provides an effective tool to derive exact solution of the following time-space FPDEs: (i) time-space fractional diffusion-convection equation, (ii) time-space fractional reaction-diffusion equation and (iii) time-space fractional diffusion equation with source term.

Here we would like to point out that only a limited number of applications for the coupled system of time-space FPDEs have been investigated through the invariant subspace method. The applicability and effectiveness of the method are illustrated through time-space fractional (i) two-coupled system of diffusion equation, (ii) two-coupled system of stationary transonic plane-parallel gas flow equation [11,24] and (iii) three-coupled system of Hirota–Satsuma KdV equation [12], and their exact solutions are derived.

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The layout of this paper is as follows: In §2, some basic concepts of fractional calculus and a brief description of the invariant subspace method for scalar and m-component coupled system of nonlinear time-space FPDEs in the sense of Riemann–Liouville/Caputo fractional derivative are presented. In §3, the effectiveness of the method is illustrated by solving the above-mentioned scalar and coupled system of time-space FPDEs. Finally, a summary of our results is given in §4.

2. Preliminaries

In this section, we would like to present some basic definitions and results related to the fractional calculus. Also, we present brief details of the invariant subspace method for scalar and coupled system of time-space FPDEs.

DEFINITION 1 [1,2]

The Riemann–Liouville (R–L) fractional derivative of order $\alpha > 0$ of the function $g \in L^1([a,b],\mathbb{R}_+)$ is defined by

$$\frac{{}^{\mathrm{RL}}\mathrm{d}^\alpha g(t)}{\mathrm{d}t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\displaystyle\int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s\right), & n-1<\alpha<n; \\ g^{(n)}(t), & \alpha=n,\ n\in\mathbb{N}, \end{cases}$$

where $L^1([a,b])$ denotes the set of all absolutely integrable functions on $[a,b]$.

Note 1 [1,2]. The R–L fractional derivative of $g(t)=t^\mu$ is as follows:

$$\frac{{}^{\mathrm{RL}}\mathrm{d}^\alpha t^\mu}{\mathrm{d}t^\alpha} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}, \quad \alpha>0,\ \mu>-1,\ t>0. \tag{1}$$

DEFINITION 2 [1,2]

The Caputo fractional derivative of as order $\alpha > 0$ of the function $g \in C^n([a,b])$ is defined as

$$\frac{\mathrm{d}^\alpha g(t)}{\mathrm{d}t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_0^t \frac{g^{(n)}(s)}{(t-s)^{\alpha-n+1}}\mathrm{d}s, & n-1<\alpha<n; \\ g^{(n)}(t), & \alpha=n,\ n\in\mathbb{N}, \end{cases}$$

where $C^n([a,b])$ denotes the set of all continuously n-times differentiable functions on $[a,b]$.

Note 2 [1,2]. The Caputo fractional derivative of $g(t)=t^\mu$ is as follows:

$$\frac{\mathrm{d}^\alpha t^\mu}{\mathrm{d}t^\alpha} = \begin{cases} 0, & \text{if } \mu\in\{0,1,\ldots,n-1\}, \text{ and } n=[\alpha]+1; \\ \dfrac{\Gamma(\mu+1)}{\Gamma(\mu-\alpha+1)}t^{\mu-\alpha}, & \text{if } \mu\in\mathbb{N} \text{ and } \mu\geq n \text{ or } \mu\notin\mathbb{N} \text{ and } \mu>n-1. \end{cases}$$

For simplicity, we denote the R–L and Caputo fractional derivative operators respectively as $\frac{{}^{\mathrm{RL}}\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$ and $\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}$.

Note 3 [1,2]. The Laplace transformation of Caputo fractional derivative of order $\alpha \in (n-1,n], n\in\mathbb{N}$, is

$$L\left\{\frac{\mathrm{d}^\alpha g(t)}{\mathrm{d}t^\alpha}\right\} = s^\alpha\overline{g}(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1}g^{(k)}(0), \quad \mathcal{R}e(s)>0.$$

DEFINITION 3 [1,2]

Two-parametric Mittag–Leffler function is defined as

$$E_{\beta,\gamma}(z) = \sum_{r=0}^{\infty}\frac{z^r}{\Gamma(\beta r+\gamma)}, \quad \beta,\gamma,z\in\mathbb{C},\ \mathcal{R}e(\beta)>0,\ \mathcal{R}e(\gamma)>0.$$

Some properties of the Mittag–Leffler function are as follows:

$$E_{1,1}(z) = \mathrm{e}^z,$$

$$E_{1,k}(z) = \frac{1}{z^{k-1}}\left[\mathrm{e}^z - \sum_{r=0}^{k-2}\frac{z^r}{r!}\right], \quad k\in\mathbb{N}.$$

Note that $E_{\beta,1}(z) \equiv E_\beta(z)$.

Note 4 [1,2]. The Laplace transformation of $t^{\gamma-1}E_{\beta,\gamma}(\pm kt^\beta)$ is

$$L\{t^{\gamma-1}E_{\beta,\gamma}(\pm kt^\beta)\} = \frac{s^{\beta-\gamma}}{(s^\beta \mp k)}, \quad \mathcal{R}e(s)>|k|^{1/\beta}.$$

Caputo fractional derivative of the Mittag–Leffler functions are

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}[t^{\gamma-1}E_{\beta,\gamma}(kt^\beta)] = t^{\gamma-\alpha-1}E_{\beta,\gamma-\alpha}(kt^\beta),$$
$$\frac{\mathrm{d}^\alpha}{\mathrm{d}t^\alpha}\left[E_\alpha(kt^\alpha)\right] = kE_\alpha(kt^\alpha),\ \alpha,\beta,\gamma>0,\ k\in\mathbb{R}.$$

Theorem 1. *If* $L\{\varphi(t)\} = \bar{\varphi}(s)$ *and* $L\{\phi(t)\} = \bar{\phi}(s)$, *then*

$$\varphi(t)\star\phi(t) = \int_0^t \varphi(t-\tau)\phi(\tau)\,\mathrm{d}\tau = L^{-1}\left\{\bar{\varphi}(s)\bar{\phi}(s)\right\},$$

where $\varphi(t)\star\phi(t)$ *is called a convolution of* $\varphi(t)$ *and* $\phi(t)$.

2.1 *Invariant subspace method for scalar and coupled FPDEs*

2.1.1 *Scalar time-space FPDE.*

We consider the following generalised scalar time-space FPDE:

$$\sum_{i=0}^m \lambda_i \frac{\partial^{\alpha+i}u(x,t)}{\partial t^{\alpha+i}} = \hat{G}[u(x,t)]$$
$$= G\left[x, u, \frac{\partial^\beta u}{\partial x^\beta}, \ldots, \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right), \frac{\partial^{r\beta}u}{\partial x^{r\beta}}, \frac{\partial^{\beta+k}u}{\partial x^{\beta+k}}\right], \quad (2)$$

where $\alpha,\beta>0$, $k,r\in\mathbb{N}$, $\lambda_i\in\mathbb{R}$, and $\hat{G}[u]$ is a linear or nonlinear fractional differential operator. Here, $(\partial^\beta/\partial x^\beta)(\cdot)$ and $(\partial^\alpha/\partial t^\alpha)(\cdot)$ are space- and time-fractional derivatives in the R–L or Caputo sense and $\hat{G}(\cdot)$ is a sufficiently given smooth function. First, we define the linear space

$$\mathcal{W}_n = \mathfrak{L}\{\varphi_1(x),\ldots,\varphi_n(x)\}$$
$$= \left\{\sum_{j=1}^n a_j\varphi_j(x) | a_j\in\mathbb{R},\ j=1,2,\ldots,n\right\},$$

where $\mathfrak{L}$ denotes the linear span and the functions $\varphi_1(x),\ldots,\varphi_n(x)$ are linearly independent. The linear space $\mathcal{W}_n$ is said to be invariant with respect to the fractional differential operator $\hat{G}[u]$ if $\hat{G}:\mathcal{W}_n\to\mathcal{W}_n$, that is, $\hat{G}[\mathcal{W}_n]\subseteq\mathcal{W}_n$ or $\hat{G}[u]\in\mathcal{W}_n$, for all $u\in\mathcal{W}_n$. This means that there exist n-functions $\Psi_1, \Psi_2,\ldots,\Psi_n$ such that

$$\hat{G}\left[\sum_{j=1}^n a_j\varphi_j(x)\right] = \sum_{j=1}^n \Psi_j(a_1,a_2,\ldots,a_n)\varphi_j(x),$$

for $a_j\in\mathbb{R}$.

Theorem 2. *Let* $\mathcal{W}_n$ *be an* n*-dimensional linear space over* $\mathbb{R}$. *If* $\mathcal{W}_n$ *is invariant under the fractional differential operator* $\hat{G}[u]$, *then the time-space FPDE (2) admits the following exact solution*:

$$u(x,t) = A_1(t)\varphi_1(x)+A_2(t)\varphi_2(x)+\cdots+A_n(t)\varphi_n(x), \quad (3)$$

where the coefficients $A_j(t)$ $(j=1,2,\ldots,n)$ *satisfy the following system of fractional ordinary differential equations (FODEs)*:

$$\sum_{i=0}^m \lambda_i\frac{\mathrm{d}^{\alpha+i}A_j(t)}{\mathrm{d}t^{\alpha+i}} = \Phi_j(A_1(t),A_2(t),\ldots,A_n(t)),$$
$$j=1,\ldots,n. \quad (4)$$

Proof. Using the linearity of the fractional derivative with eq. (3), we obtain

$$\sum_{i=0}^m \lambda_i\frac{\partial^{\alpha+i}u(x,t)}{\partial t^{\alpha+i}} = \sum_{j=1}^n\left[\sum_{i=0}^m \lambda_i\frac{\mathrm{d}^{\alpha+i}A_j(t)}{\mathrm{d}t^{\alpha+i}}\right]\varphi_j(x). \quad (5)$$

Let $\mathcal{W}_n$ be an invariant subspace with respect to the fractional differential operator $\hat{G}[u]$. Then there exist n functions $\Phi_1,\Phi_2,\ldots,\Phi_n$ such that

$$\hat{G}\left[\sum_{j=1}^n a_j\varphi_j(x)\right] = \sum_{j=1}^n \Phi_j(a_1,a_2,\ldots,a_n)\varphi_j(x), \quad (6)$$

where $a_j\in\mathbb{R}$ and Φ_j's are expansion coefficients of $\hat{G}[u]\in\mathcal{W}_n$ corresponding to φ_j's. From eqs (3) and (6), we have

$$\hat{G}[u(x,t)] = \hat{G}\left[\sum_{j=1}^n A_j(t)\varphi_j(x)\right]$$
$$= \sum_{j=1}^n \Phi_j(A_1(t),\ldots,A_n(t))\varphi_j(x). \quad (7)$$

Substituting eqs (7) and (5) in eq. (2), we have

$$\sum_{j=1}^n\left[\sum_{i=0}^m\lambda_i\frac{\mathrm{d}^{\alpha+i}A_j}{\mathrm{d}t^{\alpha+i}} - \Phi_j(A_1(t),A_2(t),\ldots,A_n(t))\right]$$
$$\times\varphi_j(x) = 0. \quad (8)$$

From eq. (8) and using their linear independence of $\{\varphi_j(x),\ j=1,2,\ldots,n\}$, we yield the system of FODEs

$$\sum_{i=0}^m \lambda_i\frac{\mathrm{d}^{\alpha+i}A_j(t)}{\mathrm{d}t^{\alpha+i}} = \Phi_j(A_1(t),A_2(t),\ldots,A_n(t)),$$
$$j=1,2,\ldots,n. \quad (9)$$

2.1.2 *Two-coupled system of time-space FPDEs.* Consider the following two-coupled system of time-space FPDEs:

$$\frac{\partial^{\alpha_1}u_1}{\partial t^{\alpha_1}} = G_1\left(x, u_1, u_2, \frac{\partial^{\beta}u_1}{\partial x^{\beta}}, \frac{\partial^{\beta}u_2}{\partial x^{\beta}}, \ldots, \frac{\partial^{r\beta}u_1}{\partial x^{r\beta}}, \frac{\partial^{r\beta}u_2}{\partial x^{r\beta}}, \frac{\partial^{\beta+k_1}u_1}{\partial x^{\beta+k_1}}, \frac{\partial^{\beta+k_1}u_2}{\partial x^{\beta+k_1}}\right),$$
$$\frac{\partial^{\alpha_2}u_2}{\partial t^{\alpha_2}} = G_2\left(x, u_1, u_2, \frac{\partial^{\beta}u_1}{\partial x^{\beta}}, \frac{\partial^{\beta}u_2}{\partial x^{\beta}}, \ldots, \frac{\partial^{r\beta}u_1}{\partial x^{r\beta}}, \frac{\partial^{r\beta}u_2}{\partial x^{r\beta}}, \frac{\partial^{\beta+k_2}u_1}{\partial x^{\beta+k_2}}, \frac{\partial^{\beta+k_2}u_2}{\partial x^{\beta+k_2}}\right), \quad (10)$$

where $\alpha_1, \alpha_2, \beta > 0$, $k_1, k_2, r \in \mathbb{N}$, and G_1, G_2 are generalised linear/nonlinear fractional differential operators and it can be considered as the given sufficient smooth functions, and $(\partial^{\alpha}/\partial t^{\alpha})(\cdot)$ and $(\partial^{\beta}/\partial x^{\beta})(\cdot)$ are time- and space-fractional derivatives in R–L/Caputo sense. Hereafter, we shall use the following notations throughout the article:

$$\hat{G}_p[u_1, u_2] = G_p\left(x, u_1, u_2, \frac{\partial^{\beta}u_1}{\partial x^{\beta}}, \frac{\partial^{\beta}u_2}{\partial x^{\beta}}, \ldots, \frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\beta}u_1}{\partial x^{\beta}}\right), \frac{\partial^{r\beta}u_1}{\partial x^{r\beta}}, \frac{\partial^{r\beta}u_2}{\partial x^{r\beta}}, \frac{\partial^{\beta+k_p}u_1}{\partial x^{\beta+k_p}}, \frac{\partial^{\beta+k_p}u_2}{\partial x^{\beta+k_p}}\right),$$

$$u_p = u_p(x, t), \quad p = 1, 2.$$

Estimation of invariant subspace: Following the procedure same as above for scalar time-space FPDEs, we develop the following result for the two-coupled system of time-space FPDEs. First, we define the linear spaces

$$\mathcal{W}^p_{n_p} = \mathfrak{L}\{\varphi^p_1(x), \ldots, \varphi^p_{n_p}(x)\} \equiv \left\{\sum_{j=1}^{n_p} a^p_j\varphi^p_j(x) \,\middle|\, a^p_j \in \mathbb{R},\ j = 1, \ldots, n_p\right\},$$

where $p = 1, 2$ and the functions $\varphi^p_1(x), \ldots, \varphi^p_{n_p}(x)$ are linearly independent. The linear spaces $\mathcal{W}^p_{n_p}$, $p = 1, 2$, are called invariant under the vector fractional differential operator $\hat{\mathbb{G}} = (G_1, G_2)$ if $\hat{\mathbb{G}} : \mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2} \to \mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2}$, which means that $\hat{G}_p : \mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2} \to \mathcal{W}^p_{n_p}$, $p = 1, 2$, that is, $\hat{G}_p[\mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2}] \subseteq \mathcal{W}^p_{n_p}$ or $\hat{G}_p[u_1, u_2] \in \mathcal{W}^p_{n_p}$, for all $(u_1, u_2) \in \mathcal{W}^1_{n_1} \times \mathcal{W}^2_{n_2}$, $p = 1, 2$. Then, we have

$$\hat{G}_p\left[\sum_{j=1}^{n_1} a^1_j\varphi^1_j(x), \sum_{j=1}^{n_2} a^2_j\varphi^2_j(x)\right] = \sum_{j=1}^{n_p} \Psi^p_j(a^1_1, \ldots, a^1_{n_1}, a^2_1, \ldots, a^2_{n_2})\varphi^p_j(x), \quad p = 1, 2.$$

Theorem 3. *Let $\mathcal{W}^p_{n_p}$ be a finite-dimensional linear space over $\mathbb{R}$. If $\mathcal{W}^p_{n_p}$ is invariant with respect to the fractional differential operator $\hat{G}_p[u_1, u_2]$, then the two-coupled system of time-space FPDEs (10) admit the following exact solution*:

$$u_p(x, t) = \sum_{j=1}^{n_p} A^p_j(t)\varphi^p_j(x), \quad p = 1, 2, \quad (11)$$

where the coefficients $A^p_j(t)$ satisfy the following system of FODEs:

$$\frac{\mathrm{d}^{\alpha_p}A_j(t)}{\mathrm{d}t^{\alpha_p}} = \Phi^p_j(A^1_1(t), A^1_2(t), \ldots, A^1_{n_1}(t), A^2_1(t), \ldots, A^2_{n_2}(t)), \quad j = 1, \ldots, n_p,\ p = 1, 2.$$

Proof. Using the linearity of the fractional derivative with eq. (11), we obtain

$$\frac{\partial^{\alpha_p}u_p(x, t)}{\partial t^{\alpha_p}} = \sum_{j=1}^{n_p} \frac{\mathrm{d}^{\alpha_p}A_j(t)}{\mathrm{d}t^{\alpha_p}}\varphi^p_j(x), \quad p = 1, 2. \quad (12)$$

Let $\mathcal{W}^p_{n_p}$ be an invariant subspace under the fractional differential operator $\hat{G}_p[u_1, u_2]$. Then there exists the functions $\Phi^p_1, \Phi^p_2, \ldots, \Phi^p_{n_p}$ $(p = 1, 2)$ such that

$$\hat{G}_p\left[\sum_{j=1}^{n_1} a^1_j\varphi^1_j(x), \sum_{j=1}^{n_2} a^2_j\varphi^2_j(x)\right] = \sum_{j=1}^{n_p} \Phi^p_j(a^1_1, \ldots, a^1_{n_1}, a^2_1, \ldots, a^2_{n_2})\varphi^p_j(x), \quad (13)$$

where $a^p_j \in \mathbb{R}$, $p = 1, 2$, and Φ^p_j's are expansion coefficients of $\hat{G}[u_1, u_2] \in \mathcal{W}^p_{n_p}$ corresponding to φ^p_j's. From eqs (11) and (13), we have

$$\hat{G}_p[u_1(x,t),u_2(x,t)] = \hat{G}_p\left[\sum_{j=1}^{n_1} A_j^1(t)\varphi_j^1(x), \sum_{j=1}^{n_2} A_j^2(t)\varphi_j^2(x)\right] = \sum_{j=1}^{n_p} \Phi_j^p(A_1^1(t),\ldots,A_{n_1}^1(t),A_1^2(t),\ldots,A_{n_2}^2)\varphi_j(x), \quad p=1,2. \tag{14}$$

Substituting eqs (14) and (12) into eq. (10), we have

$$\sum_{j=1}^{n_p}\left(\frac{\mathrm{d}^{\alpha_p}A_j^p}{\mathrm{d}t^{\alpha_p}} - \Phi_j^p(A_1^1(t),A_2^1(t),\ldots,A_{n_1}^1(t),A_1^2(t), A_2^2(t),\ldots,A_{n_2}^2(t))\right)\varphi_j^p(x)=0, \quad p=1,2. \tag{15}$$

By their linear independence of $\{\varphi_j^p,\ j=1,2,\ldots,n_p,\ p=1,2\}$, we have

$$\frac{\mathrm{d}^{\alpha_p}A_j^p(t)}{\mathrm{d}t^{\alpha_p}} = \Phi(A_1^1(t),A_2^1(t),\ldots,A_n^1(t),A_1^2(t),A_2^2(t),\ldots,A_n^2(t)), \quad j=1,2,\ldots,n_p,\ p=1,2. \tag{16}$$

2.1.3 *m-coupled system of time-space FPDEs.* Consider the following m-coupled system of time and space FPDEs

$$\frac{\partial^{\alpha_p}\mathbb{U}}{\partial t^{\alpha_p}} = \hat{\mathbb{G}}(\mathbb{U}) \equiv (G_1(\mathbb{U}),\ldots,G_m(\mathbb{U}))\in\mathbb{R}^m, \tag{17}$$

where $\alpha_p > 0$, $p=1,2,\ldots,m$, and the operators $G_q(\cdot)/(q=1,2,\ldots,m)$ are generalised linear/non-linear fractional differential operators and can be considered as sufficient smooth functions, and $(\partial^\alpha/\partial t^\alpha)(\cdot)$ and $(\partial^\beta/\partial x^\beta)(\cdot)$ are time- and space-fractional derivatives in R–L/Caputo sense, and $\mathbb{U}=(u_1,u_2,\ldots,u_m)\in\mathbb{R}^m$, $u_p=u_p(x,t)$,

$$\hat{G}_p[\mathbb{U}] = G_p\left(x,u_1,\ldots,u_m,\frac{\partial^\beta u_1}{\partial x^\beta},\ldots,\frac{\partial^\beta u_m}{\partial x^\beta}, \frac{\partial^{r\beta}u_1}{\partial x^{r\beta}},\ldots,\frac{\partial^{r\beta}u_m}{\partial x^{r\beta}},\ldots,\frac{\partial^{\beta+k_p}u_p}{\partial x^{\beta+k_p}},\frac{\partial^{\beta+k_p}u_m}{\partial x^{\beta+k_p}}\right),$$

$r,k_p\in\mathbb{N}$, $\beta>0$, $p=1,2,\ldots,m$.

Estimation of invariant subspace: Proceeding as above, we can develop the following result for an m-coupled system of time-space FPDEs. Here, we define the linear spaces as

$$\mathcal{W}_{n_p}^p = \mathfrak{L}\{\varphi_1^p(x),\ldots,\varphi_{n_p}^p(x)\} \equiv \left\{\sum_{j=1}^{n_p} a_j^p\varphi_j^p(x)\ \middle|\ (a_j^p,\ldots,a_{n_p}^p)\in\mathbb{R}^{n_p}\right\},$$

where $p=1,2,\ldots,m$, and the functions $\varphi_1^p(x),\ldots,\varphi_{n_p}^p(x)$ $(n_p\ge 1)$ are linearly independent. The linear spaces $\mathcal{W}_{n_p}^p$, $p=1,2,\ldots,m$, are called invariant with respect to the vector fractional differential operator $\hat{\mathbb{G}}=(G_1,G_2,\ldots,G_m)$ if $\hat{\mathbb{G}}:\mathcal{W}_{n_1}^1\times\cdots\times\mathcal{W}_{n_m}^m\to\mathcal{W}_{n_1}^1\times\cdots\times\mathcal{W}_{n_m}^m$, which means that $\hat{G}_p:\mathcal{W}_{n_1}^1\times\cdots\times\mathcal{W}_{n_m}^m\to\mathcal{W}_{n_p}^p$, $p=1,2,\ldots,m$, that is, $\hat{G}_p[\mathcal{W}_{n_1}^1\times\cdots\times\mathcal{W}_{n_m}^m]\subseteq\mathcal{W}_{n_p}^p$ or $\hat{G}_p[u_1,\ldots,u_m]\in\mathcal{W}_{n_p}^p$, for all $(u_1,\ldots,u_m)\in\mathcal{W}_{n_1}^1\times\cdots\times\mathcal{W}_{n_m}^m$, $p=1,\ldots,m$. Then there exists Φ_j^p, $j=1,2,\ldots,n_p$, $p=1,2,\ldots,m$, such that

$$\hat{G}_p\left[\sum_{j=1}^{n_1}a_j^1\varphi_j^1(x),\ldots,\sum_{j=1}^{n_m}a_j^m\varphi_j^m(x)\right] = \sum_{j=1}^{n_p}\Phi_j^p(a_1^1,\ldots,a_{n_1}^1,\ldots,a_1^m,\ldots,a_{n_m}^m)\varphi_j^p(x),$$

for all $(a_1^p,\ldots,a_{n_p}^p)\in\mathbb{R}^{n_p}$, $p=1,2,\ldots,m$.

Theorem 4. *Let $\mathcal{W}_{n_p}^p$ be a finite-dimensional linear space over $\mathbb{R}$ and if $\mathcal{W}_{n_p}^p$ is invariant under the fractional differential operator $\hat{G}_p[\mathbb{U}]$, then the m-coupled system (17) has a solution of the form*

$$u_p(x,t)=\sum_{j=1}^{n_p}A_j^p(t)\varphi_j^p(x), \quad p=1,2,\ldots,m, \tag{18}$$

where the coefficients $A_j^p(t)$ satisfy the following system of FODEs:

$$\frac{\mathrm{d}^{\alpha_p}A_j(t)}{\mathrm{d}t^{\alpha_p}} = \Phi_j(A_1^1(t),A_2^2(t),\ldots,A_{n_2}^2(t)), \quad j=1,\ldots,n_p,\ p=1,2,\ldots,m. \tag{19}$$

Proof. Similar to the proof of Theorem 3. Let us assume that the invariant subspace

$$\mathcal{W}_{n_p}^p=\mathfrak{L}\{\varphi_1^p,\ldots,\varphi_{n_p}^p\}$$

is defined as space generated by solutions of the following linear fractional order ODEs:

$$L_p[y_p]=y_p^{(\alpha)}+c_{n_p-1}^p(x)y_p^{(\alpha-1)}+\cdots+c_0^p(x)y_p=0,$$

where $\alpha\in(n_p-1,n_p]$, $n_p\in\mathbb{N}$, $p=1,2,\ldots,m$, and $y_p^{(\alpha)}=\mathrm{d}^\alpha y_p/\mathrm{d}x^\alpha$. Thus, the invariant condition reads as

$$L_p[\hat{G}_p[\mathbb{U}]]|_{[H_1]\cap\cdots\cap[H_m]}=0, \quad p=1,2,\ldots,m,$$

where $[H_p]$ denotes the equation $L_p[u_p] = 0$ and its differential consequences with respect to x.

3. Construction of invariant subspaces and exact solutions

3.1 *Time-space fractional diffusion-convection equation*

Consider the following time-space fractional diffusion-convection equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \hat{G}[u] = \left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 \left(\frac{\partial p}{\partial u}\right) + p(u)\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) - \frac{\partial^\beta u}{\partial x^\beta}\left(\frac{\partial q}{\partial u}\right), \tag{20}$$

where $t > 0$, $\alpha, \beta \in (0, 1]$, and the functions $p(u)$ and $q(u)$ represent the phenomenon of diffusion and convection respectively. Equation (20) with $\alpha = 1$ and $\beta = 1$ is a combination of the diffusion and convection (advection) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes: diffusion and convection [8]. It was discussed with $\alpha = 1$ and $\beta = 1$ through the invariant subspace method in [33,42]. We would like to point out that the operator $\hat{G}[u]$ admits no invariant subspace for arbitrary functions $p(u)$ and $q(u)$. Hence, we choose $p(u) = a_n u^n + a_{n-1}u^{n-1} + \cdots + a_1 u + a_0$ and $q(u) = b_{n+1}u^{n+1} + b_n u^n + \cdots + b_1 u + b_0$, $n \in \mathbb{N}$, where $a_n, a_{n-1}, \ldots, a_0, b_{n+1}, \ldots, b_1, b_0$ are arbitrary constants.

Then, eq. (20) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \hat{G}[u] = [na_n u^{n-1} + (n-1)a_{n-1}u^{n-2} + \cdots + a_1]\left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 + [a_n u^n + a_{n-1}u^{n-1} + \cdots + a_1 u + a_0] \times \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) - [(n+1)b_{n+1}u^n + nb_n u^{n-1} + \cdots + b_1] \times \frac{\partial^\beta u}{\partial x^\beta}, \quad t > 0, \tag{21}$$

where $\alpha, \beta \in (0, 1]$. It is easy to find that the differential operator $\hat{G}[u]$ admits a one-dimensional invariant subspace $\mathcal{W}_1 = \mathfrak{L}\{E_\beta(kx^\beta)\}$, $k \in \mathbb{R}$, if $a_r k = b_{r+1}$, $r = 1, 2 \ldots n$, $n \in \mathbb{N}$, because

$\hat{G}[A_1 E_\beta(kx^\beta)] = (a_0k^2 - b_1k)A_1 E_\beta(kx^\beta) \in \mathcal{W}_1$.

Thus, we can write the exact solution in the form

$$u(x, t) = A_1(t)E_\beta(kx^\beta), \tag{22}$$

where $A_1(t)$ is an unknown function to be determined. Substituting (22) in (21), we get

$$\frac{\mathrm{d}^\alpha A_1}{\mathrm{d}t^\alpha} = (a_0k^2 - b_1k)A_1(t). \tag{23}$$

First, we consider $\alpha = \beta = 1$. In this case, we have

$$u(x, t) = k_0 \mathrm{e}^{(a_0k^2 - b_1k)t + kx}, \quad k_0, k, a_0, b_1 \in \mathbb{R}. \tag{24}$$

Next, we consider $\alpha,\ \beta \in (0, 1]$. Applying Laplace transformation technique on both sides of eq. (23), we get

$$s^\alpha \bar{A}_1(s) - s^{\alpha-1}A_1(0) = (a_0k^2 - b_1k)\bar{A}_1(s)$$

which can be written as

$$\bar{A}_1(s) = \frac{k_0 s^{\alpha-1}}{s^\alpha - (a_0k^2 - b_1k)},$$

where $k_0 = A_1(0)$. Applying inverse Laplace transformation to the above equation, we get

$$A_1(t) = k_0 E_\alpha((a_0k^2 - b_1k)t^\alpha).$$

Hence, we obtain an exact solution for time-space fractional diffusion-convection equation (21) as follows:

$$u(x, t) = k_0 E_\alpha((a_0k^2 - b_1k)t^\alpha)E_\beta(kx^\beta), \tag{25}$$

where $\alpha, \beta \in (0, 1]$ and $k_0, k, a_0, b_1 \in \mathbb{R}$. Note that, for $\alpha = \beta = 1$, eq. (25) is exactly the same as (24). The graphical representation of solution (25) for $a_0 = 2$, $k = k_0 = b_1 = 1$, $t = 1$, and different values of α and β is shown in figure 1.

Case 1. Let $p(u) = a_1u + a_0$, $q(u) = -ka_1u^2 + b_1u + b_0$, $k, a_0, a_1, b_1, b_0 \in \mathbb{R}$ and $k (\neq 0)$. Then, eq. (20) can be written as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a_1\left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 + (a_1u + a_0)\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) + (2ka_1u - b_1)\frac{\partial^\beta u}{\partial x^\beta}. \tag{26}$$

It is easy to find that eq. (26) admits the following two-dimensional invariant subspaces:

(i) $\mathcal{W}_2 = \mathfrak{L}\{1, E_\beta(-kx^\beta)\}$.
(ii) $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$.

Here we consider the invariant subspace (i). Thus, we can write the exact solution of eq. (26) as follows:

$$u(x, t) = A_1(t) + A_2(t)E_\beta(-kx^\beta), \tag{27}$$

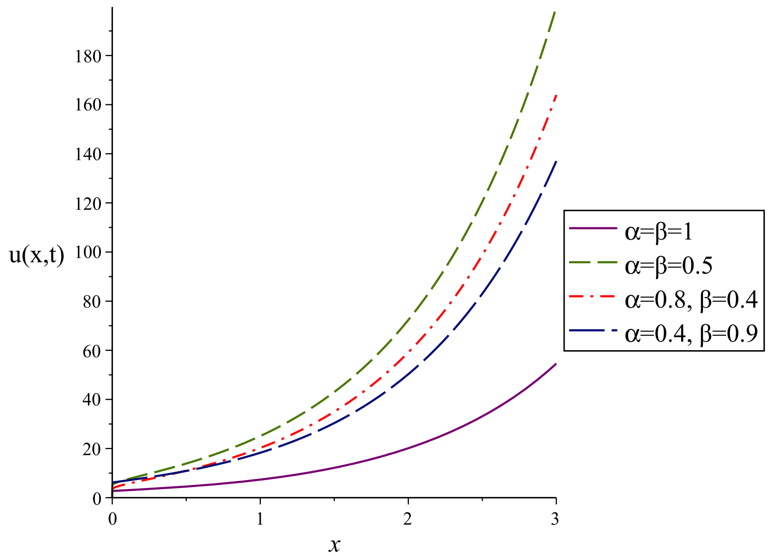

Figure 1. Graphical representation of solution (25) for $a_0 = 2, k = k_0 = b_1 = 1, t = 1$, and different values of α and β.

where the coefficients $A_1(t)$ and $A_2(t)$ satisfy the following system of FODEs:

$$\frac{d^\alpha A_1}{dt^\alpha} = 0,$$
$$\frac{d^\alpha A_2}{dt^\alpha} = (-k^2 a_1 A_1 + a_0 k^2 + k b_1) A_2. \tag{28}$$

First, we consider $\alpha = \beta = 1$. Solving system (28), we have

$$u(x,t) = k_1 + k_2 e^{k((-kk_1a_1 + a_0k + b_1)t - x)}, \tag{29}$$

where $k, k_1, k_2, a_0, a_1, b_1 \in \mathbb{R}$. Next, we consider $\alpha, \beta \in (0, 1]$. Applying Laplace transformation technique to system (28), we obtain an exact solution of eq. (26) as follows:

$$u(x,t) = k_1 + k_2 E_\alpha((-k^2 a_1 k_1 + a_0 k^2 + k b_1) t^\alpha) E_\beta(-k x^\beta), \tag{30}$$

where $k, k_1, k_2, a_0, a_1, b_1 \in \mathbb{R}$. Observe that, for $\alpha = \beta = 1$, eq. (30) is exactly the same as (29). It is also noted that when $\alpha = \beta = 1, a_1 = b_1, a_0 = b_0, k_1 = d_1, k_2 = d_2, k = a_1, b_1 = c_1$ and $b_0 = c_0$, the exact solution (30) is exactly the same as given in [33]. The graphical representation of the exact solution (30) for $k_1 = -1$, $a_1 = 2, a_0 = 0, k_2 = k = b_1 = 1, t = 2$, and different values of α and β is shown in figure 2.

Case 2. Let $p(u) = a_0$ and $q(u) = b_1 u + b_0, a_0, b_1, b_0 \in \mathbb{R}$. Then, eq. (20) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a_0 \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) - b_1 \frac{\partial^\beta u}{\partial x^\beta} \tag{31}$$

which admits the following distinct invariant subspaces:

(i) $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$.
(ii) $\mathcal{W}_n = \mathfrak{L}\{E_\beta(k_1 x^\beta), \ldots, E_\beta(k_n x^\beta)\}, n \in \mathbb{N}$,
(iii) $\mathcal{W}_{n+1} = \mathfrak{L}\{1, E_\beta(k_1 x^\beta), \ldots, E_\beta(k_n x^\beta)\}, n \in \mathbb{N}$,
(iv) $\mathcal{W}_{n+2} = \mathfrak{L}\{1, x^\beta, E_\beta(k_1 x^\beta), \ldots, E_\beta(k_n x^\beta)\}, n \in \mathbb{N}$,

where $k_i \in \mathbb{R},\ i = 1, \ldots, n$.

First, we consider the invariant subspace

$$\mathcal{W}_n = \mathfrak{L}\{E_\beta(k_1 x^\beta), \ldots, E_\beta(k_n x^\beta)\}, \quad n \in \mathbb{N}.$$

Thus, we can write the exact solution in the form

$$u(x,t) = A_1(t) E_\beta(k_1 x^\beta) + \cdots + A_n(t) E_\beta(k_n x^\beta), \tag{32}$$

where $A_i(t), i = 1, \ldots, n$ satisfy the following system of n-FODEs:

$$\frac{d^\alpha A_1}{dt^\alpha} = (a_0 k_1^2 - b_1 k_1) A_1,$$
$$\vdots$$
$$\frac{d^\alpha A_n}{dt^\alpha} = (a_0 k_n^2 - b_1 k_n) A_n. \tag{33}$$

Applying Laplace transformation technique to system (33), we obtain an exact solution of eq. (31) with $\alpha = \beta = 1$, which reads as

$$u(x,t) = \sum_{s=1}^{n} r_s e^{((a_0 k_s - b_1) k_s t + k_s x)}. \tag{34}$$

When $\alpha \in (0, 1]$

$$u(x,t) = \sum_{s=1}^{n} r_s E_\alpha\left((a_0 k_s - b_1) k_s t^\alpha\right) E_\beta(k_s x^\beta), \tag{35}$$

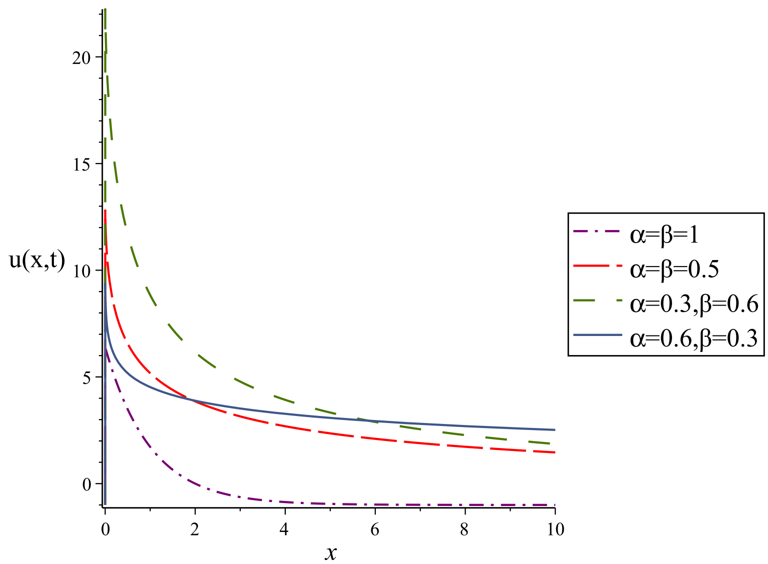

Figure 2. Graphical representation of solution (30) for $k_1 = -1$, $a_1 = 2$, $a_0 = 0$, $k_2 = k = b_1 = 1$, $t = 2$, and different values of α and β.

where $r_s, k_s, a_0, b_1 \in \mathbb{R}$ $(s = 1, 2, \ldots, n)$. We observe that for $\alpha = \beta = 1$, eq. (35) is exactly the same as (34).

Proceeding the same way as before, we can derive another more general exact solution associated with the more general invariant subspace $\mathcal{W}_{n+2} = \mathfrak{L}\{1, x^\beta, E_\beta(k_1x^\beta), \ldots, E_\beta(k_nx^\beta)\}$, $n \in \mathbb{N}$. For this case, we obtain the more general exact solution of (31) as follows:

$$u(x,t) = c_1 - c_2b_1\frac{\Gamma(\beta+1)}{\Gamma(\alpha+1)}t^\alpha + c_2x^\beta + \sum_{s=1}^{n} r_sE_\alpha\left((a_0k_s - b_1)k_st^\alpha\right)E_\beta(k_sx^\beta), \tag{36}$$

where $\alpha, \beta \in (0, 1]$ and c_1, c_2, r_s, k_s $(s = 1, 2, \ldots, n)$, a_0, b_1 are arbitrary constants. Observe that, for $c_1 = c_2 = 0$, eq. (36) is exactly the same as (35). Similarly, we can derive different types of exact solutions for time-space fractional diffusion-convection equation (31) using the afore-mentioned invariant subspaces.

Case 3. Let $p(u) = a_1u$ and $q(u) = (b_2/2)u^2$, b_2 and a_1 are constants. Then, eq. (20) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a_1\left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 + a_1u\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) - b_2u\frac{\partial^\beta u}{\partial x^\beta}. \tag{37}$$

It is easy to find that eq. (37) admits the following two-dimensional invariant subspaces:

(i) $\mathcal{W}_2 = \mathfrak{L}\{1, E_\beta(kx^\beta)\}$ if $a_1 = b_2/2k$.
(ii) $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$.

Here we consider the invariant subspace (i). Thus, we obtain an exact solution of (37) with $\alpha = \beta = 1$, which reads as

$$u(x,t) = k_0 + k_1\mathrm{e}^{k(-\frac{b_2}{2}k_0t + x)}, \tag{38}$$

and when $\alpha \in (0, 1]$,

$$u(x,t) = k_0 + k_1E_\alpha\left(-\frac{b_2}{2}kk_0t^\alpha\right)E_\beta(kx^\beta), \tag{39}$$

where $k_0,\ k_1,\ b_2,\ k \in \mathbb{R}$. Note that, for $\alpha = \beta = 1$, eq. (39) is exactly the same as (38). We would like to mention that when $\beta = 1$, solution (39) is exactly the same as given in [42].

Case 4. Let $p(u) = u$ and $q(u) = b_0$, $b_0 \in \mathbb{R}$. Then, eq. (20) describes only the time-space fractional diffusion equation as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 + u\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) \tag{40}$$

which admits an invariant subspace $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$. In this case, we have

$$u(x,t) = k_0 + k_1^2\frac{(\Gamma(\beta+1))^2}{\Gamma(\alpha+1)}t^\alpha + k_1x^\beta, \tag{41}$$

where $k_0, k_1 \in \mathbb{R}$ and $\alpha, \beta \in (0, 1]$. We would like to point out that when $\beta = 1$, solution (41) is exactly the same as given in [42].

Case 5. We note that if $p(u) = 1$ and $q(u) = -\frac{1}{2}u^2$, then the diffusion-convection equation (20) reduces to the time-space fractional Burgers equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) + u\frac{\partial^\beta u}{\partial x^\beta}, \tag{42}$$

which admits a two-dimensional polynomial invariant subspace $\mathcal{W}_2 = \mathfrak{L}\{1, x^{\beta}\}$.

3.2 *Time-space fractional reaction-diffusion equation*

Consider the following time-space fractional reaction-diffusion equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \hat{G}[u] = p(u)\frac{\partial^{\beta+1} u}{\partial x^{\beta+1}} + q(u), \quad \alpha, \beta \in (0, 1], \tag{43}$$

where the functions $p(u)$ and $q(u)$ represent the phenomenon of diffusion and reaction respectively. The above PDE with $\beta = 1$ was discussed using the generalised differential transform method in [47]. We would like to point out that the differential operator $\hat{G}[u]$ admits no invariant subspace for arbitrary functions $p(u)$ and $q(u)$. Hence, we discuss the following specific cases:

Case 1. Let $p(u) = a_n u^n + a_{n-1}u^{n-1} + \cdots + a_1 u + a_0$, $q(u) = b_{n+1}u^{n+1} + b_n u^n + \cdots + b_1 u + b_0$, $n \in \mathbb{N}$, where $k, a_n, b_{n+1}, \ldots, a_1, b_1, b_0, a_0$ are non-zero arbitrary constants. Then, the time-space fractional reaction-diffusion equation (43) reduces to

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \hat{G}[u] = [a_n u^n + a_{n-1}u^{n-1} + \cdots + a_1 u + a_0]\frac{\partial^{\beta+1} u}{\partial x^{\beta+1}} + (b_{n+1}u^{n+1} + b_n u^n + \cdots + b_1 u + b_0). \tag{44}$$

It is easy to find that eq. (44) admits a one-dimensional invariant subspace $\mathcal{W}_1 = \mathfrak{L}\{E_{\beta+1}(-kx^{\beta+1})\}$ if $a_i k = b_{i+1}$, $i = 1, 2, \ldots, n$, $n \in \mathbb{N}$ and $b_0 = 0$, because

$$\hat{G}[A_1 E_{\beta+1}(-kx^{\beta+1})] = (-ka_0 + b_1)A_1 E_{\beta+1}(-kx^{\beta+1}) \in \mathcal{W}_1.$$

Following similar procedure, first, we consider $\alpha = \beta = 1$. In this case, we have

$$u(x, t) = k_0 \mathrm{e}^{(-ka_0+b_1)t - x^2}, \tag{45}$$

where k, k_0, b_1 and a_0 are arbitrary constants. Next, we assume $\alpha, \beta \in (0, 1]$. Thus, we obtain an exact solution of eq. (44) as follows:

$$u(x, t) = k_0 E_{\alpha}((-ka_0 + b_1)t^{\alpha})E_{\beta+1}(-kx^{\beta+1}), \tag{46}$$

where $\alpha, \beta \in (0, 1]$ and k_0, k, b_1, a_0 are non-zero arbitrary constants. We observe that for $\alpha = \beta = 1$, eq. (46) is exactly the same as (45). It is also observed that when $\alpha = 1$, $\beta = 0$, $k_0 = k$, $k = -a$, $a_0 = c_0$ and $b_1 = k_1$, solution (46) is exactly the same as the solution given in [32] with $b_0 = \cdots = b_n = 0$. The graphical representation of (46) for $k = b_1 = 1$, $a_0 = -1$, $k_0 = 2$, $t = 2$ and different values of α and β is shown in figure 3.

Subcase 1. Let $a_1, b_2, b_1 \in \mathbb{R}$ and $a_n = a_{n-1} = \cdots = a_0 = b_{n+1} = b_n = \cdots = b_1 = b_0 = 0$, that is, $p(u) = a_1 u$ and $q(u) = b_2 u^2 + b_1 u$. Then, eq. (43) reduces to

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = a_1 u \frac{\partial^{\beta+1} u}{\partial x^{\beta+1}} + b_2 u^2 + b_1 u \tag{47}$$

which admits a one-dimensional invariant subspace $\mathcal{W}_2 = \mathfrak{L}\{E_{\beta+1}(-kx^{\beta+1})\}$ if $b_2 = a_1 k$. In this case, we obtain the exact solution

$$u(x, t) = k_1 E_{\alpha}(b_1 t^{\alpha})E_{\beta+1}(-kx^{\beta+1}), \ \alpha, \beta \in (0, 1], \tag{48}$$

where $k, k_1, b_1 \in \mathbb{R}$. The graphical representation of (48) for $k_1 = 1$, $b_1 = -4$, $k = 2$, $t = 2$, and different values of α and β is shown in figure 4.

Subcase 2. Let $a_1 = 1$, $b_1 = -k \neq 0$, $b_0, k \in \mathbb{R}$ and $a_n = a_{n-1} = \cdots = a_0 = b_{n+1} = b_n = \cdots = b_2 = 0$, that is, $p(u) = u$ and $q(u) = -ku + b_0$. Then, eq. (44) reduces to

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = u \frac{\partial^{\beta+1} u}{\partial x^{\beta+1}} - ku + b_0 \tag{49}$$

which admits the following distinct invariant subspaces:

(i) $\mathcal{W}_2 = \mathfrak{L}\{1, x^{\beta}\}$,
(ii) $\mathcal{W}_2 = \mathfrak{L}\{1, x^{\beta+1}\}$,
(iii) $\mathcal{W}_3 = \mathfrak{L}\{1, x^{\beta}, x^{\beta+1}\}$.

Now, we consider the invariant subspace $\mathcal{W}_2 = \mathfrak{L}\{1, x^{\beta}\}$. In this case, we get the exact solution of eq. (49) as follows:

$$u(x, t) = (k_1 + k_2 x^{\beta})E_{\alpha}(-kt^{\alpha}) + b_0 t^{\alpha} E_{\alpha,\alpha+1}(-kt^{\alpha}), \tag{50}$$

where $\alpha, \beta \in (0, 1]$ and $k, k_1, k_2, b_0 \in \mathbb{R}$. The graphical representation of solution (50) for $k_1 = k_2 = 1$, $b_0 = 0$, $k = 2$, $x = 2$, and different values of α and β is shown in figure 5.

Subcase 3. Let $a_0 = c \in \mathbb{R}$, $b_1 = -k$ and $a_n = a_{n-1} = \cdots = a_1 = b_{n+1} = b_n = \cdots = b_2 = b_0 = 0$, that is, $p(u) = c$ and $q(u) = -ku$. Then, eq. (44) reduces to

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = c \frac{\partial^{\beta+1} u}{\partial x^{\beta+1}} - ku \tag{51}$$

which admits distinct invariant subspaces, that is,

(i) $\mathcal{W}_2 = \mathfrak{L}\{1, x^{\beta}\}$,
(ii) $\mathcal{W}_3 = \mathfrak{L}\{1, x^{\beta}, x^{\beta+1}\}$,
(iii) $\mathcal{W}_n = \mathfrak{L}\{E_{\beta+1}(k_1 x^{\beta+1}), \ldots, E_{\beta+1}(k_n x^{\beta+1})\}$,

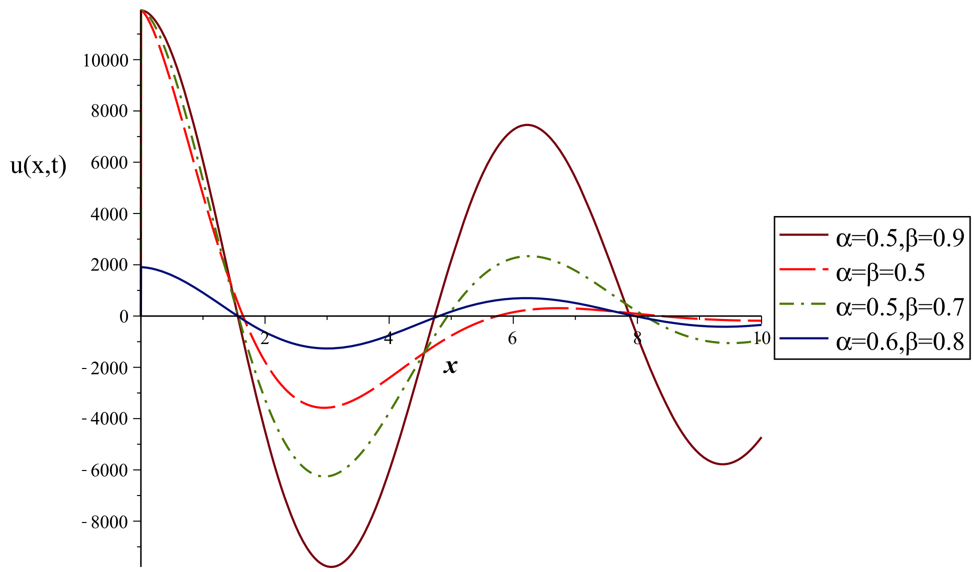

Figure 3. Graphical representation of solution (46) for $k = b_1 = 1$, $k_0 = 2$, $a_0 = -1$, $t = 2$, and different values of α and β.

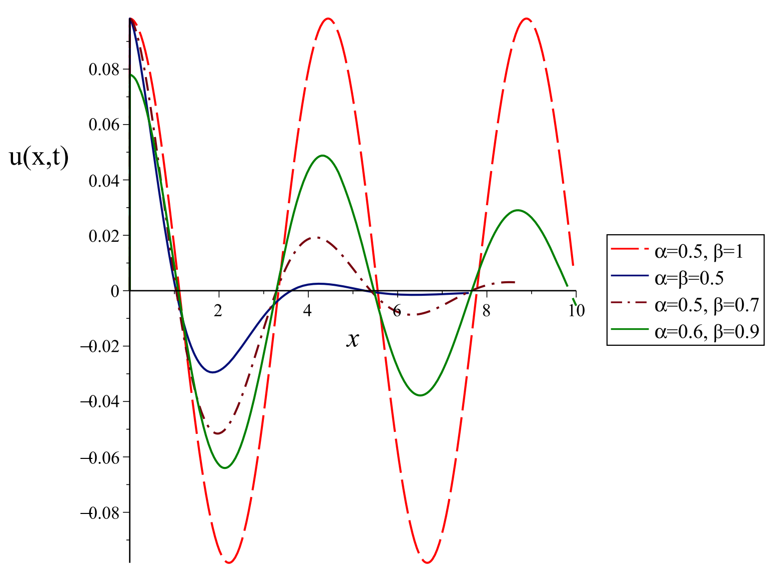

Figure 4. Graphical representation of solution (48) for $k_1 = 1$, $b_1 = -4$, $k = 2$, $t = 2$, and different values of α and β.

(iv) $\mathcal{W}_{n+1}=\mathfrak{L}\{1,\, E_{\beta+1}(k_1x^{\beta+1}),\ldots, E_{\beta+1}\,(k_nx^{\beta+1})\}$,

(v) $\mathcal{W}_{n+2} = \mathfrak{L}\{1,\, x^{\beta+1},\, E_{\beta+}(k_1x^{\beta+1}),\ldots, E_{\beta+1}\,(k_nx^{\beta+1})\}$,

(vi) $\mathcal{W}_{n+3}=\mathfrak{L}\{1,\, x^{\beta},\, x^{\beta+1},\, E_{\beta+1}(k_1x^{\beta+1}),\ldots, E_{\beta+1}\,(k_nx^{\beta+1})\}$,

where $n \in \mathbb{N}$ and $k_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. Now, we consider the more general invariant subspace $\mathcal{W}_{n+3} = \mathfrak{L}\{1, x^{\beta}, x^{\beta+1}, E_{\beta+1}(k_1x^{\beta+1}), \ldots, E_{\beta+1}(k_nx^{\beta+1})\}$, which suggests that eq. (51) possesses the more general exact solution

$$\begin{aligned} u(x,t) &= (\lambda_1 + \lambda_2 x^{\beta} + \lambda_3 x^{\beta+1})E_{\alpha}(-kt^{\alpha}) \\ &\quad + \sum_{r=1}^{n} \lambda_{r+3} E_{\alpha}((k_r c - k)t^{\alpha}) E_{\beta+1}(k_r x^{\beta+1}) \\ &\quad + c\lambda_3 \Gamma(\beta+2) \int_0^t E_{\alpha}(-k(t-\tau)^{\alpha})(\tau)^{\alpha-1} \\ &\quad \times E_{\alpha,\alpha}(-k\tau^{\alpha})\,\mathrm{d}\tau, \end{aligned} \tag{52}$$

where k, c, k_i $(i = 1, 2, \ldots, n)$ and λ_s $(s = 1, 2, \ldots, n+3)$ are arbitrary constants.

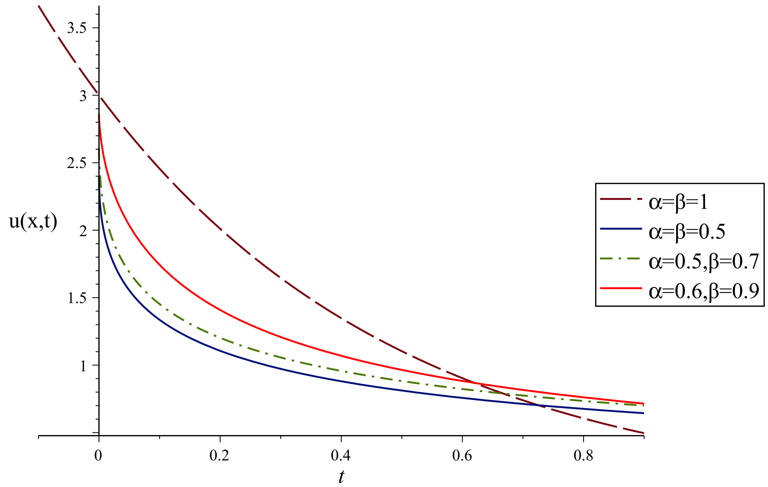

Figure 5. Graphical representation of solution (50) for $k_1 = k_2 = 1$, $b_0 = 0$, $k = 2$, $x = 2$, and different values of α and β.

Subcase 4. Let $a_0 = c$, $c \in \mathbb{R}$, and $a_n = a_{n-1} = \cdots = a_1 = b_{n+1} = b_n = \cdots = b_1 = b_0 = 0$, that is, $p(u) = c$ and $q(u) = 0$. Then, fractional reaction-diffusion equation (44) reduces to the fractional linear sub-diffusion equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = c\frac{\partial^{\beta+1} u}{\partial x^{\beta+1}} \quad (53)$$

which admits the following distinct invariant subspaces:

(i) $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$,
(ii) $\mathcal{W}_3 = \mathfrak{L}\{1, x^\beta, x^{\beta+1}\}$,
(iii) $\mathcal{W}_n = \mathfrak{L}\{E_{\beta+1}(k_1 x^{\beta+1}), \ldots, E_{\beta+1}(k_n x^{\beta+1})\}$,
(iv) $\mathcal{W}_{n+1} = \mathfrak{L}\{1, E_{\beta+1}(k_1 x^{\beta+1}), \ldots, E_{\beta+1}(k_n x^{\beta+1})\}$,
(v) $\mathcal{W}_{n+2} = \mathfrak{L}\{1, x^{\beta+1}, E_{\beta+}(k_1 x^{\beta+1}), \ldots, E_{\beta+1}\ (k_n x^{\beta+1})\}$,
(vi) $\mathcal{W}_{n+3} = \mathfrak{L}\{1, x^\beta, x^{\beta+1}, E_{\beta+1}(k_1 x^{\beta+1}), \ldots, E_{\beta+1}\ (k_n x^{\beta+1})\}$,

where $n \in \mathbb{N}$ and $k_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$. Here, we consider the more general invariant subspace (vi), which possesses the more general exact solution of fractional sub-diffusion equation (53) which reads as

$$u(x,t) = (\lambda_1 + \lambda_2 x^\beta + \lambda_3 x^{\beta+1}) + c\lambda_3 \frac{\Gamma(\beta+2)}{\Gamma(\alpha+1)} t^\alpha + \sum_{r=1}^{n} \lambda_{r+3} E_\alpha(k_r c t^\alpha) E_{\beta+1}(k_r x^{\beta+1}),$$
$$\alpha, \beta \in (0, 1], \quad (54)$$

where c, k_i $(i = 1, 2, \ldots, n)$ and λ_s $(s = 1, 2, \ldots, n+3)$, are arbitrary constants. Equation (53) with R–L sense was thoroughly studied in [10,14] using the Lie symmetry analysis method. We would like to point out that the obtained solution (54) is a more generalised solution compared to the given solution in [10,14]. Note that for $k = 0$, eq. (52) is exactly the same as eq. (54). The graphical representation of (54) for $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = k_1 = c = n = 1$, $t = 2$, and different values of α and β is shown in figure 6.

Case 2. Let $p(u)$ be an arbitrary function of u and $q(u) = b_1 u + b_0$, $b_0, b_1 \in \mathbb{R}$. Then, fractional reaction-diffusion equation (44) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = p(u)\frac{\partial^{\beta+1} u}{\partial x^{\beta+1}} + b_1 u + b_0 \quad (55)$$

which admits a two-dimensional invariant subspace $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$. Thus, we obtain the exact solution of (55) as follows:

$$u(u,t) = (k_1 + k_2 x^\beta) E_\alpha(b_1 t^\alpha) + b_0 t^\alpha E_{\alpha,\alpha}(b_1 t^\alpha), \quad (56)$$

where $\alpha, \beta \in (0, 1]$.

3.3 *Time-space fractional diffusion equation with the source term*

Consider a time-space fractional nonlinear diffusion equation with the source term

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \hat{G}[u] = \left(\frac{\partial p}{\partial u}\right)\left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 + p(u)\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) + q(u), \quad (57)$$

where $\alpha, \beta \in (0, 1]$ and the terms $p(u)$ and $q(u)$ represent the phenomenon of diffusion and reaction respectively. We would like to point out that the operator $\hat{G}[u]$ admits no invariant subspace for arbitrary functions $p(u)$ and $q(u)$. Hence, we choose $p(u) = a_n u^n + a_{n-1} u^{n-1} + \cdots + a_1 u + a_0$ and $q(u) = b_{n+1} u^{n+1} +$

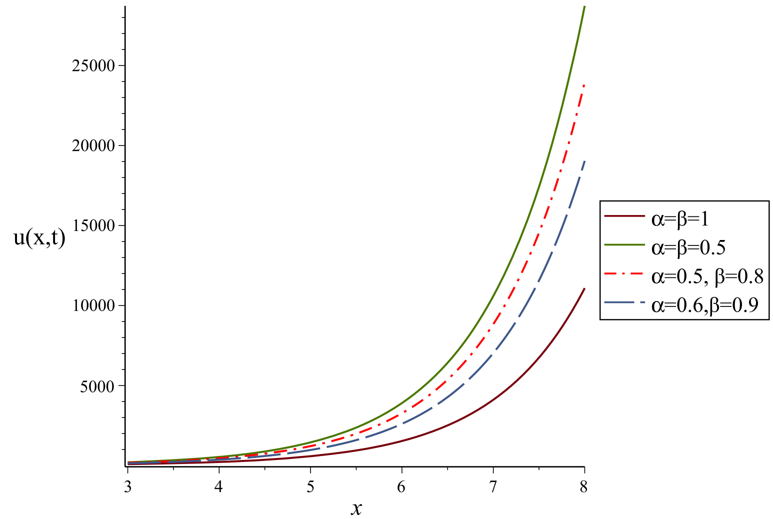

Figure 6. Graphical representation of solution (54) for $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = k_1 = c = n = 1$, $t = 2$, and different values of α and β.

$b_n u^n + \cdots + b_1 u$, where $a_n, b_{n+1}, \ldots, a_1, b_1, a_0$ are arbitrary constants. Then, eq. (57) describes the time-space diffusion equation with the source term

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \hat{G}[u] = (na_n u^{n-1} + (n-1)a_{n-1}u^{n-2} + \cdots + a_1)\left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 + (a_n u^n + a_{n-1}u^{n-1} + \cdots + a_1 u + a_0)\frac{\partial^\beta}{\partial x^\beta} \times \left(\frac{\partial^\beta u}{\partial x^\beta}\right) + b_{n+1}u^{n+1} + b_n u^n + \cdots + b_1 u. \tag{58}$$

It is easy to check that eq. (58) admits a one-dimensional invariant subspace $\mathcal{W}_1 = \mathfrak{L}\{E_\beta(kx^\beta)\}$ if $(i+1)a_i k^2 = -b_{i+1}$, $i = 1, 2 \ldots, n$, $n \in \mathbb{N}$. Thus, for $\alpha = \beta = 1$, we have

$$u(x,t) = k_0 \mathrm{e}^{(a_0 k^2 + b_1)t + kx}, \quad b_1, k_0, k, a_0 \in \mathbb{R}. \tag{59}$$

Now, we consider $\alpha, \beta \in (0, 1]$. In this case, we obtain an exact solution of eq. (58) as

$$u(x,t) = k_0 E_\alpha((a_0 k^2 + b_1)t^\alpha) E_\beta(kx^\beta), \tag{60}$$

where $b_1, k_0, k, a_0 \in \mathbb{R}$. Note that if $\alpha = \beta = 1$, then eq. (60) is exactly the same as eq. (59). We also would like to point out that when $\alpha = \beta = 1$, $a_i = b_i$, $b_{i+1} = k_{i+1}$ $(i = 1, \ldots, n)$, $k_0 = k$ and $k = a$, solution (60) is exactly the same as discussed in [32]. Here we consider the following specific cases for discussion:

Case 1. Let $p(u) = a_0$ and $q(u) = b_1 u + b_0$, a_0, b_0 and b_1 are arbitrary constants. Then, eq. (57) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a_0 \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) + b_1 u + b_0 \tag{61}$$

which admits distinct invariant subspaces, that is,

(i) $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$,
(ii) $\mathcal{W}_{n+1} = \mathfrak{L}\{1, E_\beta(k_1 x^\beta), E_\beta(k_2 x^\beta), \ldots, E_\beta(k_n x^\beta)\}$,
(iii) $\mathcal{W}_{n+2} = \mathfrak{L}\{1, x^\beta, E_\beta(k_1 x^\beta), E_\beta(k_2 x^\beta), \ldots, E_\beta(k_n x^\beta)\}$, $k_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

Let us consider the more general invariant subspace (iii), that is,

$$\mathcal{W}_{n+2} = \mathfrak{L}\{1, x^\beta, E_\beta(k_1 x^\beta), E_\beta(k_2 x^\beta), \ldots, E_\beta(k_n x^\beta)\}.$$

Thus, we obtain the exact solution of (61) as

$$u(x,t) = (c_1 + c_2 x^\beta)E_\alpha(b_1 t^\alpha) + b_0 t^\alpha E_{\alpha,\alpha+1}(b_1 t^\alpha) + \sum_{r=1}^{n} c_{r+2} E_\alpha((b_1 + a_0 k_r^2)t^\alpha) E_\beta(k_r x^\beta), \tag{62}$$

where a_0, b_0, b_1, k_i $(i=1, 2, \ldots, n)$ and c_s $(s=1, 2, \ldots, n+2)$ are arbitrary constants.

Case 2. Let $p(u) = a_0$ and $q(u) = b_1 u$, a_0 and b_1 are arbitrary constants. Then, eq. (57) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a_0 \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) + b_1 u \tag{63}$$

which admits the following distinct invariant subspaces:

(i) $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$,

(ii) $\mathcal{W}_n = \mathfrak{L}\{E_\beta(k_1x^\beta), E_\beta(k_2x^\beta), \ldots, E_\beta(k_nx^\beta)\}$,
(iii) $\mathcal{W}_{n+1}=\mathfrak{L}\{1, E_\beta(k_1x^\beta), E_\beta(k_2x^\beta), \ldots, E_\beta(k_nx^\beta)\}$,
(iv) $\mathcal{W}_{n+2}=\mathfrak{L}\{1, x^\beta, E_\beta(k_1x^\beta), E_\beta(k_2x^\beta), \ldots, E_\beta(k_nx^\beta)\}$,

where $n \in \mathbb{N}$ and $k_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$.

Let us consider the more general invariant subspace (iv), that is, $\mathcal{W}_{n+2} = \mathfrak{L}\{1, x^\beta, E_\beta(k_1x^\beta), E_\beta(k_2x^\beta), \ldots, E_\beta(k_nx^\beta)\}$. Thus, we obtain the more general exact solution of (63) as

$$u(x,t) = (c_1 + c_2x^\beta)E_\alpha(b_1t^\alpha) + \sum_{r=1}^{n} c_{r+2}E_\alpha((b_1 + a_0k_r^2)t^\alpha)E_\beta(k_rx^\beta), \tag{64}$$

where a_0, b_1, k_i $(i = 1, 2, \ldots, n)$ and c_s $(s = 1, 2, \ldots, n+2)$ are arbitrary constants. Note that for $b_0 = 0$, eq. (62) is exactly the same as eq. (64).

Case 3. Let $p(u) = a_1u + a_0$ and $q(u) = b_1u + b_0$, a_0, a_1, b_0 and b_1 are arbitrary constants. Then, eq. (57) reduces to

$$\frac{\partial^\alpha u}{\partial t^\alpha} = a_1\left(\frac{\partial^\beta u}{\partial x^\beta}\right)^2 + (a_1u + a_0)\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u}{\partial x^\beta}\right) + b_1u + b_0 \tag{65}$$

which admits a two-dimensional invariant subspace $\mathcal{W}_2 = \mathfrak{L}\{1, x^\beta\}$.

3.4 *Two-coupled system of time-space fractional diffusion equation*

Consider the following two-coupled system of time-space fractional diffusion equation:

$$\begin{aligned}\frac{\partial^\alpha u_1}{\partial t^\alpha} &= \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u_1}{\partial x^\beta}\right) + \mu u_2\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u_1}{\partial x^\beta}\right) \\ &\quad + (\mu+\rho)\left(\frac{\partial^\beta u_1}{\partial x^\beta}\right)\left(\frac{\partial^\beta u_2}{\partial x^\beta}\right) \\ &\quad + \rho u_1\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u_2}{\partial x^\beta}\right), \\ \frac{\partial^\alpha u_2}{\partial t^\alpha} &= \frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u_2}{\partial x^\beta}\right) + \lambda\frac{\partial^\beta}{\partial x^\beta}\left(\frac{\partial^\beta u_1}{\partial x^\beta}\right) \\ &\quad + \gamma u_1 + \delta u_2,\end{aligned} \tag{66}$$

where $\alpha, \beta \in (0, 1]$ and $\mu, \rho, \lambda, \gamma, \delta \in \mathbb{R}$. It is easy to find that coupled system (66) admits the following two-dimensional distinct invariant subspaces:

(i) $\mathcal{W}_2^1 \times \mathcal{W}_2^2 = \mathfrak{L}\{E_\beta(kx^\beta), E_\beta(-kx^\beta)\}\mathfrak{L}\{E_\beta(kx^\beta), E_\beta(-kx^\beta)\}$ if $\mu = -\rho$,

(ii) $\mathcal{W}_2^1 \times \mathcal{W}_2^2 = \mathfrak{L}\{1, x^\beta\} \times \mathfrak{L}\{1, x^\beta\}$, because

$$\begin{aligned}&\hat{G}_1[A_1E_\beta(kx^\beta) + A_2E_\beta(-kx^\beta), A_3E_\beta(kx^\beta) + A_4E_\beta(-kx^\beta)] \\ &\quad = k^2[A_1E_\beta(kx^\beta) + A_2E_\beta(-kx^\beta)] \in \mathcal{W}_2^1, \\ &\quad \text{if } \mu = -\rho, \\ &\hat{G}_2[A_1E_\beta(kx^\beta) + A_2E_\beta(-kx^\beta), A_3E_\beta(kx^\beta) + A_4E_\beta(-kx^\beta)] \\ &\quad = [(k^2+\delta)A_3 + (\lambda k^2+\gamma)A_1]E_\beta(kx^\beta) \\ &\quad\quad + [(k^2+\delta)A_4 + (\lambda k^2+\gamma)A_2] E_\beta(-kx^\beta) \in \mathcal{W}_2^2\end{aligned}$$

and

$$\begin{aligned}&\hat{G}_1[A_1 + A_2x^\beta, A_3 + A_4x^\beta] \\ &\quad = (\rho+\mu)\,(\Gamma(\beta+1))^2\,A_2A_4 \in \mathcal{W}_2^1, \\ &\hat{G}_2[A_1 + A_2x^\beta, A_3 + A_4x^\beta] \\ &\quad = (\gamma A_1 + \delta A_3) + (\gamma A_2 + \delta A_4)\,x^\beta \in \mathcal{W}_2^2.\end{aligned}$$

Case 1. First, we consider the invariant subspace $\mathcal{W}_2^1\mathcal{W}_2^2 = \mathfrak{L}\{E_\beta(kx^\beta), E_\beta(-kx^\beta)\}\mathfrak{L}\{E_\beta(kx^\beta), E_\beta(-kx^\beta)\}$ with $\mu = -\rho$, which suggests that eq. (66) admits an exact solution in the form

$$\begin{aligned}u_1(x,t) &= A_1(t)E_\beta(kx^\beta) + A_2(t)E_\beta(-kx^\beta), \\ u_2(x,t) &= A_3(t)E_\beta(kx^\beta) + A_4(t)E_\beta(-kx^\beta),\end{aligned} \tag{67}$$

where $A_1(t)$, $A_2(t)$, $A_3(t)$ and $A_4(t)$ are unknown functions to be determined. Substituting (67) in (66), we get

$$\begin{aligned}\frac{d^\alpha A_1}{dt^\alpha} &= k^2A_1, \\ \frac{d^\alpha A_2}{dt^\alpha} &= k^2A_2, \\ \frac{d^\alpha A_3}{dt^\alpha} &= (k^2+\delta)A_3 + (\lambda k^2+\gamma)A_1, \\ \frac{d^\alpha A_4}{dt^\alpha} &= (k^2+\delta)A_4 + (\lambda k^2+\gamma)A_2.\end{aligned} \tag{68}$$

To obtain a non-zero solution, we assume that $A_i(0) = a_i \neq 0$, $i = 1, 2, 3, 4$. Let us first consider $\alpha = \beta = 1$. In this case, we have

$$\begin{aligned}u_1(x,t) &= e^{k^2t}(a_1e^{kx} + a_2e^{-kx}), \\ u_2(x,t) &= \left[a_3e^{\delta t} + (\lambda k^2+\gamma)a_1\left(\frac{e^{\delta t}-1}{\delta}\right)\right]e^{k(kt+x)} \\ &\quad + \left[a_4e^{\delta t} + (\lambda k^2+\gamma)a_2\left(\frac{e^{\delta t}-1}{\delta}\right)\right]e^{k(kt-x)},\end{aligned} \tag{69}$$

where $a_i, k, \lambda, \gamma, \delta \in \mathbb{R}$ $(i = 1, 2, 3, 4)$ and $\delta \neq 0$. Next, we consider $\alpha \in (0, 1]$. Applying the Laplace transformation technique to the linear system (68), we obtain

$$A_1(t) = a_1 E_\alpha(k^2 t^\alpha),$$
$$A_2(t) = a_2 E_\alpha(k^2 t^\alpha),$$
$$A_3(t) = a_3 E_\alpha((k^2+\delta)t^\alpha) + (\lambda k^2+\gamma)a_1 \times \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}((k^2+\delta)\tau^\alpha) E_\alpha(k^2(t-\tau)^\alpha)\,\mathrm{d}\tau,$$
$$A_4(t) = a_4 E_\alpha((k^2+\delta)t^\alpha) + (\lambda k^2+\gamma)a_2 \times \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}((k^2+\delta)\tau^\alpha) E_\alpha(k^2(t-\tau)^\alpha)\,\mathrm{d}\tau.$$

In this case, we obtain an exact solution of the time-space fractional coupled diffusion system (66) associated with the invariant subspace $\mathcal{W}_2^1 \times \mathcal{W}_2^2 = \mathfrak{L}\{E_\beta(kx^\beta), E_\beta(-kx^\beta)\} \times \mathfrak{L}\{E_\beta(kx^\beta), E_\beta(-kx^\beta)\}$ if $\rho = -\mu$, as follows:

$$u_1(x,t) = [a_1 E_\beta(kx^\beta) + a_2 E_\beta(-kx^\beta)] E_\alpha(k^2 t^\alpha),$$
$$u_2(x,t) = [a_3 E_\beta(kx^\beta) + a_4 E_\beta(-kx^\beta)] E_\alpha((k^2+\delta)t^\alpha) + (\lambda k^2+\gamma)[a_1 E_\beta(kx^\beta) + a_2 E_\beta(-kx^\beta)] \times \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}((k^2+\delta)\tau^\alpha) \times E_\alpha(k^2(t-\tau)^\alpha)\,\mathrm{d}\tau, \quad (70)$$

where $a_i, k, \lambda, \gamma, \delta \in \mathbb{R}$ $(i = 1,2,3,4)$. Observe that if $\alpha = \beta = 1$, then eq. (70) is exactly the same as (69).

Case 2. Following similar procedure, we can derive another exact solution associated with the invariant subspace $\mathcal{W}_2^1 \times \mathcal{W}_2^2 = \mathfrak{L}\{1, x^\beta\} \times \mathfrak{L}\{1, x^\beta\}$. Let $\alpha = \beta = 1$. In this case, we have

$$u_1(x,t) = \left[\frac{k_4}{\delta}(\mathrm{e}^{\delta t}-1) + \frac{\gamma k_2}{\delta^2}(\mathrm{e}^{\delta t}-1-\delta t)\right] \times (\rho+\mu)k_2 + k_1 + k_2 x,$$
$$u_2(x,t) = \frac{k_1\gamma}{\delta}(\mathrm{e}^{\delta t}-1) + \left[k_4 \mathrm{e}^{\delta t} + \frac{\gamma k_2}{\delta}(\mathrm{e}^{\delta t}-1)\right]x + k_3 \mathrm{e}^{\delta t} + \frac{\gamma(\rho+\mu)k_2}{\delta^2}[k_4(\delta t \mathrm{e}^{\delta t} - \mathrm{e}^{\delta t} + 1) + \frac{\gamma k_2}{\delta}(2 - 2\mathrm{e}^{\delta t} + \delta t + \delta t \mathrm{e}^{\delta t})], \quad (71)$$

where k_i $(i = 1,2,3,4)$, γ and $\delta(\neq 0)$ are arbitrary constants. While $\alpha, \beta \in (0,1]$, it takes

$$u_1(x,t) = k_1 + k_2 x^\beta + (\Gamma(\beta+1))^2(\rho+\mu)k_2 \times [k_4 t^\alpha E_{\alpha,\alpha+1}(\delta t^\alpha) + \gamma k_2 t^{2\alpha} E_{\alpha,2\alpha+1}(\delta t^\alpha)],$$
$$u_2(x,t) = \left[k_4 E_\alpha(\delta t^\alpha) + \gamma k_2 t^\alpha E_{\alpha,\alpha+1}(\delta t^\alpha)\right] x^\beta + k_3 E_\alpha(\delta t^\alpha) + \gamma k_1 t^\alpha E_{\alpha,\alpha+1}(\delta t^\alpha) + (\Gamma(\beta+1))^2 k_2 k_4 \gamma(\rho+\mu) \times \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(\delta\tau^\alpha)(t-\tau)^\alpha \times E_{\alpha,\alpha+1}(\delta(t-\tau)^\alpha)\,\mathrm{d}\tau + (\Gamma(\beta+1))^2(\rho+\mu)k_2^2\gamma^2 \times \int_0^t \tau^{\alpha-1} E_{\alpha,\alpha}(\delta\tau^\alpha)(t-\tau)^{2\alpha} \times E_{\alpha,2\alpha+1}(\delta(t-\tau)^\alpha)\,\mathrm{d}\tau, \quad (72)$$

where $k_i, \gamma, \delta(\neq 0) \in \mathbb{R}$ $(i = 1,2,3,4)$. Note that for $\alpha = \beta = 1$, eqs (72) are exactly the same as (71).

3.5 *Two-coupled system of time-space fractional nonlinear model of stationary transonic plane-parallel gas flow*

Consider the following two-coupled system of time-space fractional PDEs

$$\frac{\partial^\alpha u_1}{\partial t^\alpha} = \frac{\partial^\beta u_2}{\partial x^\beta},$$
$$\frac{\partial^\alpha u_2}{\partial t^\alpha} = -u_1 \frac{\partial^\beta u_1}{\partial x^\beta},\ t > 0,\ \alpha, \beta \in (0,1], \quad (73)$$

which describes the model of stationary transonic plane-parallel gas flow [11,24]. The symmetries of the coupled system (73) with $\alpha = 1$ and $\beta = 1$ are presented in [11,48]. System (73) with $\alpha = \beta = 1$ is discussed through the invariant subspace method by Galaktionov and Svirshchevskii [24]. It is easy to find that (73) admits an invariant subspace $\mathcal{W}_2^1 \times \mathcal{W}_2^2 = \mathfrak{L}\{1, x^\beta\} \times \mathfrak{L}\{1, x^\beta\}$. Let us first consider $\alpha = \beta = 1$. In this case, we get the exact solution of system (73) associated with $\mathcal{W}_2^1 \times \mathcal{W}_2^2 = \mathfrak{L}\{1, x^\beta\} \times \mathfrak{L}\{1, x^\beta\}$ as follows:

$$u_1(x,t) = k_1 + k_4 t - k_2^2 \frac{t^2}{2} + k_2 x,$$
$$u_2(x,t) = k_3 - k_2\left(k_1 + \frac{k_4}{2}t - \frac{k_2^2}{6}t^2\right)t + (k_4 - k_2^2 t)x, \quad (74)$$

where $k_i \in \mathbb{R}$, $i = 1,2,3,4$. Next, we consider $\alpha, \beta \in (0,1]$. In this case, we have

$$u_1(x,t) = k_1 + k_4\Gamma(\beta+1)\frac{t^\alpha}{\Gamma(\alpha+1)} - k_2^2(\Gamma(\beta+1))^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + k_2 x^\beta,$$
$$u_2(x,t) = -k_1 k_2 \Gamma(\beta+1)\frac{t^\alpha}{\Gamma(\alpha+1)} - k_2 k_4(\Gamma(\beta+1))^2 \times \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + k_2^3(\Gamma(\beta+1))^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

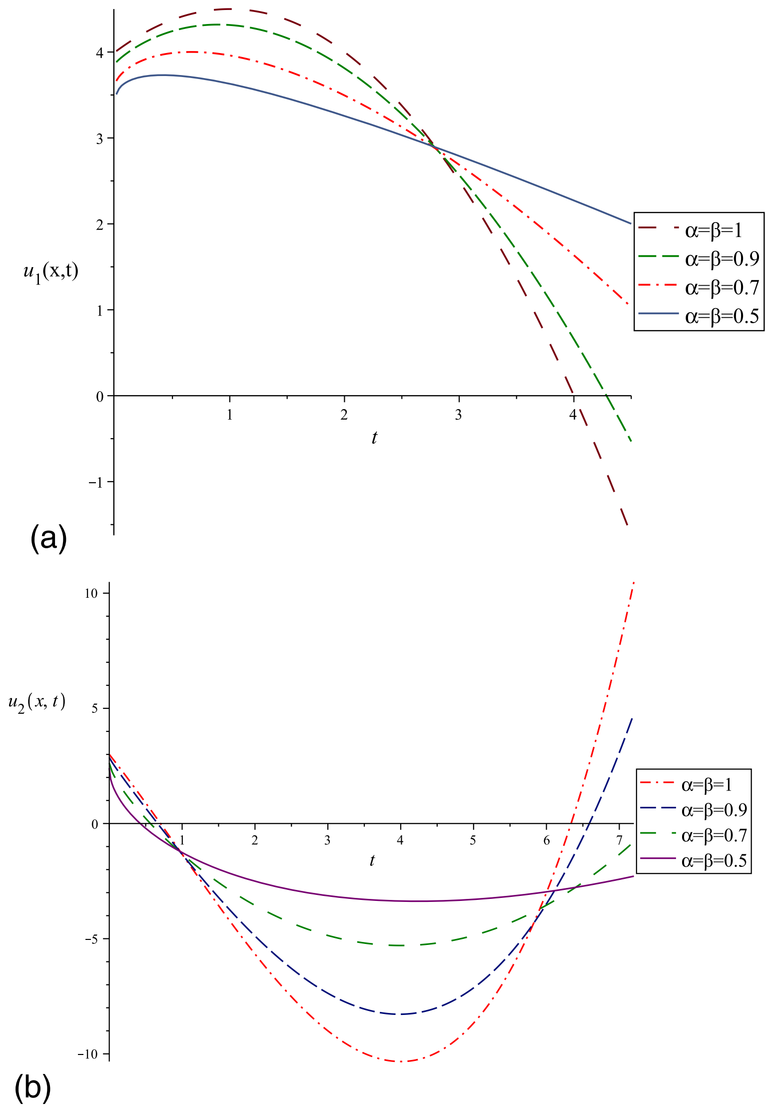

Figure 7. Graphical representations of the solution (**a**) $u_1(x, t)$ and (**b**) $u_2(x, t)$ for eq. (73) with $k_1 = 2$, $k_2 = k_3 = k_4 = 1$, $x = 2$, and different values of α and β.

$$+k_3 + \left[k_4 - k_2^2 \Gamma(\beta+1)\frac{t^\alpha}{\Gamma(\alpha+1)}\right] x^\beta,$$
$$\alpha, \beta \in (0, 1], \tag{75}$$

where k_1, k_2, k_3 and k_4 are non-zero arbitrary constants. Observe that for $\alpha = \beta = 1$, eqs (75) are exactly the same as eqs (74). The graphical representation of (75) for $k_1 = 2$, $k_2 = k_3 = k_4 = 1$, $x = 2$, and different values of α and β are shown in figure 7.

3.6 *Three-coupled system of time-space fractional generalised Hirota–Satsuma KdV equation*

Consider the three-coupled system of time-space fractional generalised Hirota–Satsuma KdV equation

$$\begin{aligned}
\frac{{}^{\mathrm{RL}}\partial^\alpha u_1}{\partial t^\alpha} &= \frac{1}{2}\frac{\partial^{\beta+2} u_1}{\partial x^{\beta+2}} - 3u_1\frac{\partial^\beta u_1}{\partial x^\beta} \\
&\quad + 3u_3\frac{\partial^\beta u_2}{\partial x^\beta} + 3u_2\frac{\partial^\beta u_3}{\partial x^\beta}, \\
\frac{{}^{\mathrm{RL}}\partial^\alpha u_2}{\partial t^\alpha} &= -\frac{\partial^{\beta+2} u_2}{\partial x^{\beta+2}} + 3u_1\frac{\partial^\beta u_2}{\partial x^\beta}, \\
\frac{{}^{\mathrm{RL}}\partial^\alpha u_3}{\partial t^\alpha} &= -\frac{\partial^{\beta+2} u_3}{\partial x^{\beta+2}} + 3u_1\frac{\partial^\beta u_3}{\partial x^\beta}, \quad t > 0,
\end{aligned} \tag{76}$$

where $\alpha, \beta \in (0, 1]$. This system with $\alpha = \beta = 1$ describes the interaction of two long waves with different dispersion relations [49]. System (76) with $\beta = 1$ is discussed using the invariant subspace method in [12]. Let us consider the dimensions $n_1 = n_2 = n_3 = 2$. For this case, eq. (76) admits the invariant subspace

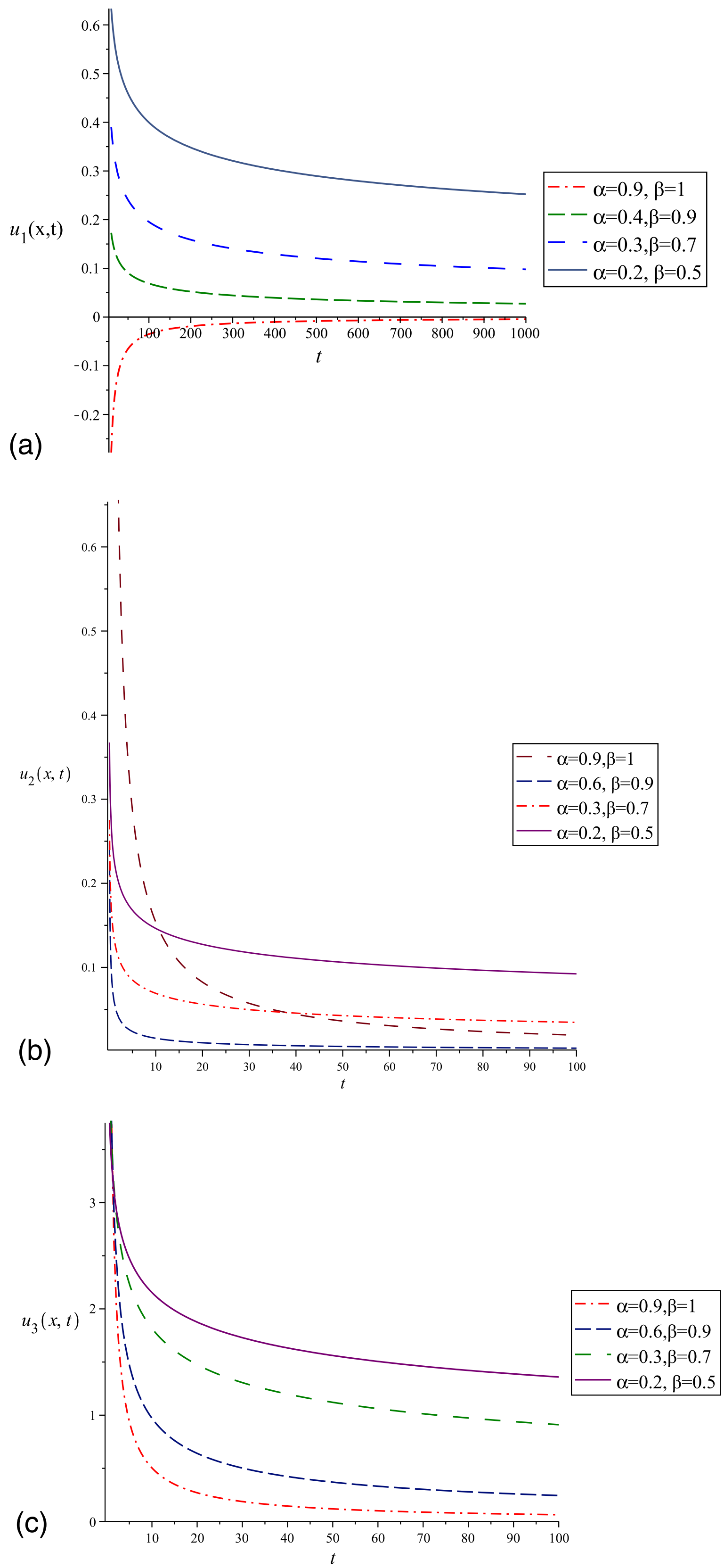

Figure 8. Graphical representations of the solution (**a**) $u_1(x, t)$, (**b**) $u_2(x, t)$ and (**c**) $u_3(x, t)$ for eq. (76) with $k_1 = 2$, $k_2 = 1$, $x = 2$, and different values of α and β.

$\mathcal{W}_{2,2,2} = \mathcal{W}_2^1 \times \mathcal{W}_2^2 \times \mathcal{W}_2^3 = \mathfrak{L}\{1, x^\beta\} \times \mathfrak{L}\{1, x^\beta\} \times \mathfrak{L}\{1, x^\beta\}$. Thus, we obtain the exact solution of system (76) as follows:

$$u_1(x,t) = \frac{1}{3\Gamma(\beta+1)}\left(\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\right)\left[\frac{k_1}{k_2} + x^\beta\right]t^{-\alpha},$$
$$u_2(x,t) = \frac{1}{9k_2(\Gamma(\beta+1))^2}\left(\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\right)^2 \times \left[\frac{k_1}{k_2} + x^\beta\right]t^{-\alpha},$$
$$u_3(x,t) = k_1 t^{-\alpha} + k_2 x^\beta t^{-\alpha}, \quad k_2 \neq 0, \tag{77}$$

where $\alpha \in (0,1) - \left\{\frac{1}{2}\right\}$, $\beta \in (0,1]$ and $k_1, k_2 \in \mathbb{R}$. The graphical representations of the exact solution (77) for $k_1 = 2$, $k_2 = 1$, $x = 2$, and different values of α and β are shown in figure 8. We would like to mention that when $\beta = 1$, solution (77) is exactly the same as given in [12].

Note 5. We would like to point out that the result in Note 1 holds only for $\mu > n - 1$. Hence

$$\frac{\mathrm{d}^\alpha(t^{-\alpha})}{\mathrm{d}t^\alpha} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}t^{-2\alpha}$$

is not valid in Caputo fractional derivative of order $\alpha \in (0,1)$, but it holds for R–L fractional derivative of order $\alpha \in (0,1)$:

$$\frac{{}^{\mathrm{RL}}\mathrm{d}^\alpha(t^{-\alpha})}{\mathrm{d}t^\alpha} = \frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}t^{-2\alpha}, \quad \mu = -\alpha > -1.$$

4. Conclusion

In this article, we have presented how the invariant subspace method can be extended to a scalar and coupled system of time-space FPDEs. Also, we have explicitly presented how the scalar and coupled system of time-space FPDEs admit more than one invariant subspace which in turn helps to derive more than one exact solution. The applicability of the method was illustrated through the scalar and coupled system of time-space FPDEs given in (20), (43), (57), (66), (73) and (76). Using the invariant subspace method, the scalar and coupled system of time-space FPDEs is reduced to the system of FODEs. The obtained reduced system of FODEs can be solved by well-known analytical methods. The obtained exact solutions can be expressed in terms of the polynomial and well-known Mittag–Leffler functions. It is also noted that the derived solutions were compared to the existing literature wherever possible. These investigations show that the invariant subspace method is a very effective tool to derive the exact solutions for the scalar and coupled system of nonlinear time-space FPDEs.

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