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Control-based verification of multiatoms in a cavity

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Abstract. In this paper, we study a model of two two-level atoms interacting with a quantum field. An analytical solution is obtained which is used to study the information entropy of the system. It is shown that the nonlinear term plays a significant role in the behaviour of the minimum uncertainty (MU) compared with the concurrence (C). Our extensive study of information entropy of atoms–field interaction demonstrates that using the coupling strength between the atoms and the field as a controller parameter, one can control the dynamics of the system by increasing the lower bound of the entropic uncertainty relation or decreasing the entanglement.

Keywords. Entanglement; minimum uncertainty; intensity-dependent coupling.

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1. Introduction

The uncertainty principle has important significance because it states that the known quantum information stored in the quantum memory can reduce or eliminate the uncertainty about the measurement outcomes of another particle that is entangled with the quantum memory and is confirmed in recent experiments [1,2]. The entropic uncertainty relations have been widely used in quantum entanglement witness [3,4], security analysis of quantum cryptographic protocols [5], locking of classical correlation in quantum state [6], quantum phase transitions [7] and quantum information processing [8].

On the other hand, to generate an entangled state, the non-distinguishability of the two subsystems in an interference phenomenon must be achieved [9], and stabilised, where quantum entanglement is probably the most fascinating manifestation of quantum theory in which the beauty of quantum mechanics is truly realised. Quantum entanglement plays a significant role in the context of quantum computation [10], information and teleportation theory [11], quantum error correction [12,13], etc. There are many applications of quantum entanglement in various contexts [14–42]. In interferometry, the process of entanglement is used to achieve the Heisenberg limit [43]. In the multielectron atomic system, the electronic shells always consist of electrons in the entangled state [44].

In this work, we discuss the effect of coupling strength and effective photon number on the entanglement and the minimum uncertainty. Our results show that we have identified new ways to control the strength of the degree of entanglement and angleme minimum uncertainty in two two-level atoms system.

This paper is organised as follows. In §2, we introduce a physical model. In §3, we review entropic uncertainty relations. We present numerical results in §4 and discuss the different effects on minimum uncertainty and entanglement. Finally, a summary is given in §5.

2. The model and its solution

We consider a system of two atoms trapped in an optical cavity. The two atoms were driven by two independent thermal fields and separated by a large enough distance. The two eigenstates of the individual atom ($|g\rangle$, $|e\rangle$) constitute the qubit states. The Hamiltonian of the system can be written as [45,46]

$$H = \omega a^{\dagger} a + \frac{\omega_0}{2} \sum_{j=1}^{2} \sigma_j^z + \eta \sum_{j=1}^{2} (B^{\dagger} \sigma_j^- + B \sigma_j^+), \quad (1)$$

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where $a(a^\dagger)$ is the annihilation(creation) operator, $B = aF(a^\dagger a)$, $F(a^\dagger a)$ represents the arbitrary function of intensity-dependent coupling (IDC), ω is the cavity field frequency, ω_0 is the transition frequency of the atoms and η is the atom–cavity coupling constant. In the large detuning limit, $\Delta = \omega_0 - \omega \gg \eta\sqrt{\bar{n}+1}$ with $\bar{n}$ being the mean photon number of the cavity field. $\sigma_j^+ = |e_j\rangle\langle g_j|$ and $\sigma_j^- = |g_j\rangle\langle e_j|$ are the raising and lowering operators of the jth atom and $\sigma_j^z = |e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|$ is a Pauli operator for the jth atom. The effective Hamiltonian can be obtained as

$$\begin{aligned}H_{\text{eff}} = \Omega\big[&aa^\dagger F^2(a^\dagger a)(|e_1\rangle\langle e_1| + |e_2\rangle\langle e_2|) \\ &-a^\dagger a F^2(a^\dagger a)(|g_1\rangle\langle g_1| + |g_2\rangle\langle g_2|)\big] \\ &+F^2(a^\dagger a)(\sigma_1^+\sigma_2^- + \sigma_1^-\sigma_2^+),\end{aligned} \tag{2}$$

where $\Omega = \eta^2/\Delta$. The master equation for the system density operator is ($\hbar = 1$)

$$\begin{aligned}\frac{d\rho_s}{dt} = &-i[H_{\text{eff}}, \rho_s] \\ &+\sum_{j=1}^{2} n_T^{(j)}\Gamma^{(j)}[2|e_j\rangle\langle g_j|\rho_s|g_j\rangle\langle e_j| \\ &- |g_j\rangle\langle g_j|\rho_s - \rho_s|g_j\rangle\langle g_j|\,] \\ &+\sum_{j=1}^{2}(n_T^{(j)}+1)\Gamma^{(j)}\big[\,2|g_j\rangle\langle e_j|\rho_s|e_j\rangle\langle g_j| \\ &-|e_j\rangle\langle e_j|\rho_s - \rho_s|e_j\rangle\langle e_j|\,\big],\end{aligned} \tag{3}$$

where ρ_s is the density matrix describing the subsystem containing only two atoms, $\Gamma^{(j)}$ describes the coupling of the jth atom to external field, $n_T^{(j)}\Gamma^{(j)}$ is the transition rate due to the thermal field and $n_T^{(j)}$ can be interpreted as an effective photon number and the spontaneous decay of the atom out of the cavities is included in this scenario via the $n_T^{(j)}+1$ term.

Assume that atoms 1 and 2 are initially in the pure product state $|e_1\rangle \otimes |g_2\rangle$ with $n_T^{(1)} = n_T^{(2)} = n_T$ and $\Gamma^{(1)} = \Gamma^{(2)} = \Gamma$, i.e., two atoms are driven by two independent thermal fields with the same intensity. And the field is initially prepared in the Fock state $|n\rangle$ with $a^\dagger a|n\rangle = n|n\rangle$. The analytical solution of the master equation (3) can be obtained as

$$\rho_{AB} = \begin{pmatrix} \rho_{11} & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{32} & \rho_{33} & 0 \\ 0 & 0 & 0 & \rho_{44} \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned}\rho_{11} &= \frac{n_T}{(2n_T+1)^2}\big[n_T + e^{-(4n_T+2)\Gamma t} \\ &\quad -(n_T+1)e^{-(8n_T+4)\Gamma t}\big], \\ \rho_{22} &= \frac{n_T^2+n_T}{(2n_T+1)^2}[1+e^{-(8n_T+4)\Gamma t}] \\ &\quad +\frac{e^{-(4n_T+2)\Gamma t}}{2}\left[\frac{1}{(2n_T+1)^2} + \cos(2F^2(n)\Omega t)\right], \\ \rho_{33} &= \frac{n_T^2+n_T}{(2n_T+1)^2}[1+e^{-(8n_T+4)\Gamma t}] \\ &\quad +\frac{e^{-(4n_T+2)\Gamma t}}{2}\left[\frac{1}{(2n_T+1)^2} - \cos\left(2F^2(n)\Omega t\right)\right], \\ \rho_{44} &= \frac{n_T}{(2n_T+1)^2}\big[n_T + 1 - e^{-(4n_T+2)\Gamma t} \\ &\quad - n_T e^{-(8n_T+4)\Gamma t}\big], \\ \rho_{23} &= \rho_{32}^\star = \frac{i}{2}e^{-(4n_T+2)\Gamma t}\sin(2F^2(n)\Omega t).\end{aligned}$$

3. The entropic uncertainty relation in the presence of quantum memory

The uncertainty relation is the core of quantum mechanics and illustrates the differences between classical and quantum mechanics. The uncertainty relation, first quantified by Heisenberg [47] for the position and momentum observables, measure the quantum fluctuations of two non-commuting observables P and Q by standard deviations,

$$\Delta P \cdot \Delta Q \geq \frac{1}{2}|\langle [P, Q]\rangle|, \tag{5}$$

where $\Delta P \equiv \sqrt{\langle P^2\rangle - \langle P\rangle^2}$, $\Delta Q \equiv \sqrt{\langle Q^2\rangle - \langle Q\rangle^2}$, and the commutator $[P, Q] = PQ - QP$. Afterwards, Bialynicki-Birula *et al* [48,49] derived the entropic uncertainty relations for position and momentum in terms of entropy rather than the standard deviation. Later, Deutsch [50] developed a relation that holds for any pair of observables. An improvement of this relation was subsequently conjectured by Kraus [51] and then proved by Maassen and Uffink [52], leading to the following entropic uncertainty relation:

$$H(P) + H(Q) \geq \log_2 \frac{1}{c}, \tag{6}$$

where $H(X) = -\Sigma_k p_k(x)\log_2 p_k(x)$ denotes the Shannon entropy and $p(x)$ is the probability distribution for the measurement outcome x for the random variable X. The term $1/c$ quantifies the complementary

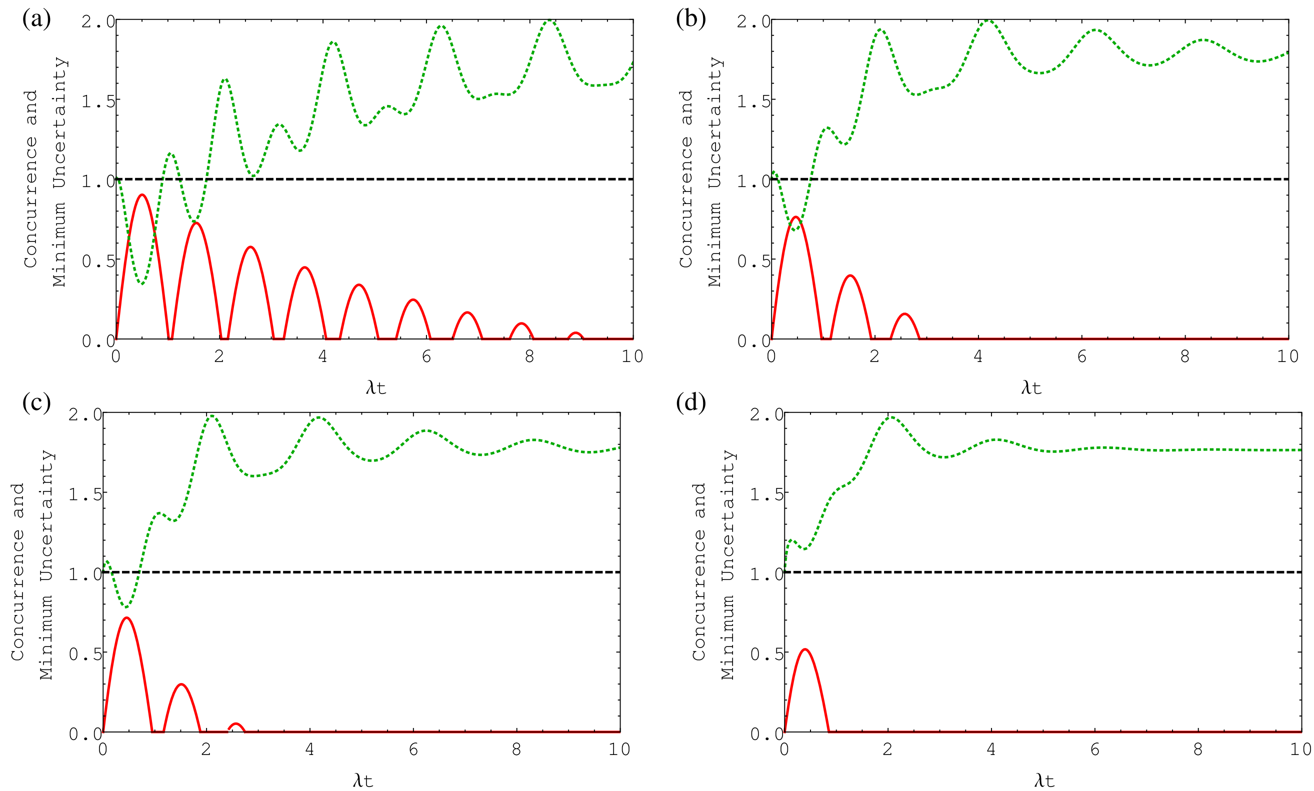

Figure 1. Concurrence (C) (red line) and the minimum uncertainty (MU) (green line) of two atoms are plotted as functions of scaled time, λt, with $\Omega = 0.3$, $n_T = 0.4$, $F(n) = \sqrt{n}$ and $n = 5$. (**a**) $\Gamma = 0.03$, (**b**) $\Gamma = 0.08$, (**c**) $\Gamma = 0.1$ and (**d**) $\Gamma = 0.2$

term of the observables. For non-degenerate observables, $c := \max_{j,k} |\langle\psi_j|\varphi_k\rangle|^2$, where $|\psi_j\rangle$ and $|\varphi_k\rangle$ are the eigenvectors of P and Q, respectively. Recently, Berta *et al* [53] and Renes and Boilean [54] put forward a stronger entropy uncertainty relation, in the presence of quantum memory in the following form:

$$H(P|B) + H(Q|B) \geq \log_2 \frac{1}{c} + H(A|B), \tag{7}$$

where $H(A|B) = H(\rho_{AB}) - H(\rho_B)$ is the conditional von Neumann entropy of the state ρ_{AB} with $H(\rho) = -\sum_j \lambda_j \log_2 \lambda_j$ is the von Neumann entropy of the state ρ [55] and λ_j denotes the eigenvalues of the state ρ. Also, $H(Y|B)$ with $Y \in (P, Q)$ is the conditional von Neumann entropy of the post-measurement state

$$\rho_{YB} = \sum_k (|\phi_k\rangle\langle\phi_k| \otimes I_B)\rho_{AB}(|\phi_k\rangle\langle\phi_k| \otimes I_B),$$

where $|\phi_k\rangle$ are the eigenvectors of Y.

The results in [56–58] show that a negative conditional entropy is a signature of entanglement, i.e., ρ_{AB} is entangled when $H(A|B) < 0$. Hence, the entanglement can be witnessed by the right-hand side of the inequality in eq. (7). That is, A is entangled with B if $\log_2 \frac{1}{c} + H(A|B) < 1$.

4. Discussion

In this section, we analyse numerically the lower bound of the entropic uncertainty relation and witness entanglement according to this lower bound. We introduce the minimum uncertainty (MU) which represents the lower bound of the entropic uncertainty relation in eq. (7), i.e., MU$= \log_2 \frac{1}{c} + H(A|B)$. We choose $R = \sigma_x$ and $Q = \sigma_z$ to be two observables, and hence $c = 1/2$. By using the definition of conditional von Neumann entropy, MU can be obtained as

$$\begin{aligned} \text{MU} &= 1 + H(\rho_{AB}) - H(\rho_B) \\ &= 1 - \sum_{i=1}^{4} \lambda_i \log_2 \lambda_i \\ &\quad + (\rho_{11} + \rho_{33}) \log_2 (\rho_{11} + \rho_{33}) \\ &\quad + (\rho_{22} + \rho_{44}) \log_2 (\rho_{22} + \rho_{44}), \end{aligned} \tag{8}$$

where λ_i are the eigenvalues of ρ_{AB} given by

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}(\rho_{22} + \rho_{33} \pm \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2}) \\ \lambda_3 &= \rho_{11}, \quad \lambda_4 = \rho_{44}. \end{aligned} \tag{9}$$

Also, we employ Wootter's concurrence [59], quantifying the entanglement between the two atoms, which

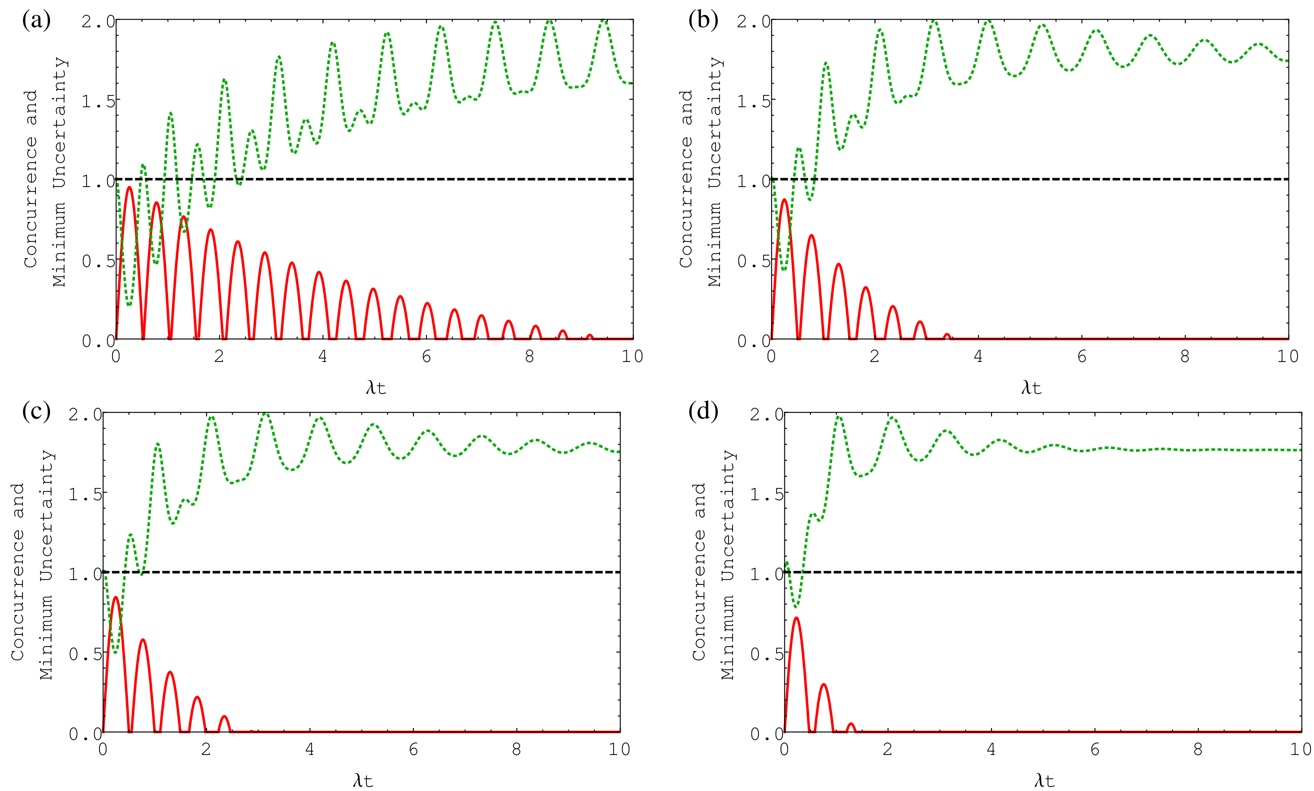

Figure 2. Concurrence C (red line) and MU (green line) of two atoms are plotted as functions of scaled time, λt, with $\Omega = 0.3$, $n_T = 0.4$, $F(n) = \sqrt{n}$ and $n = 10$. (**a**) $\Gamma = 0.03$, (**b**) $\Gamma = 0.08$, (**c**) $\Gamma = 0.1$ and (**d**) $\Gamma = 0.2$

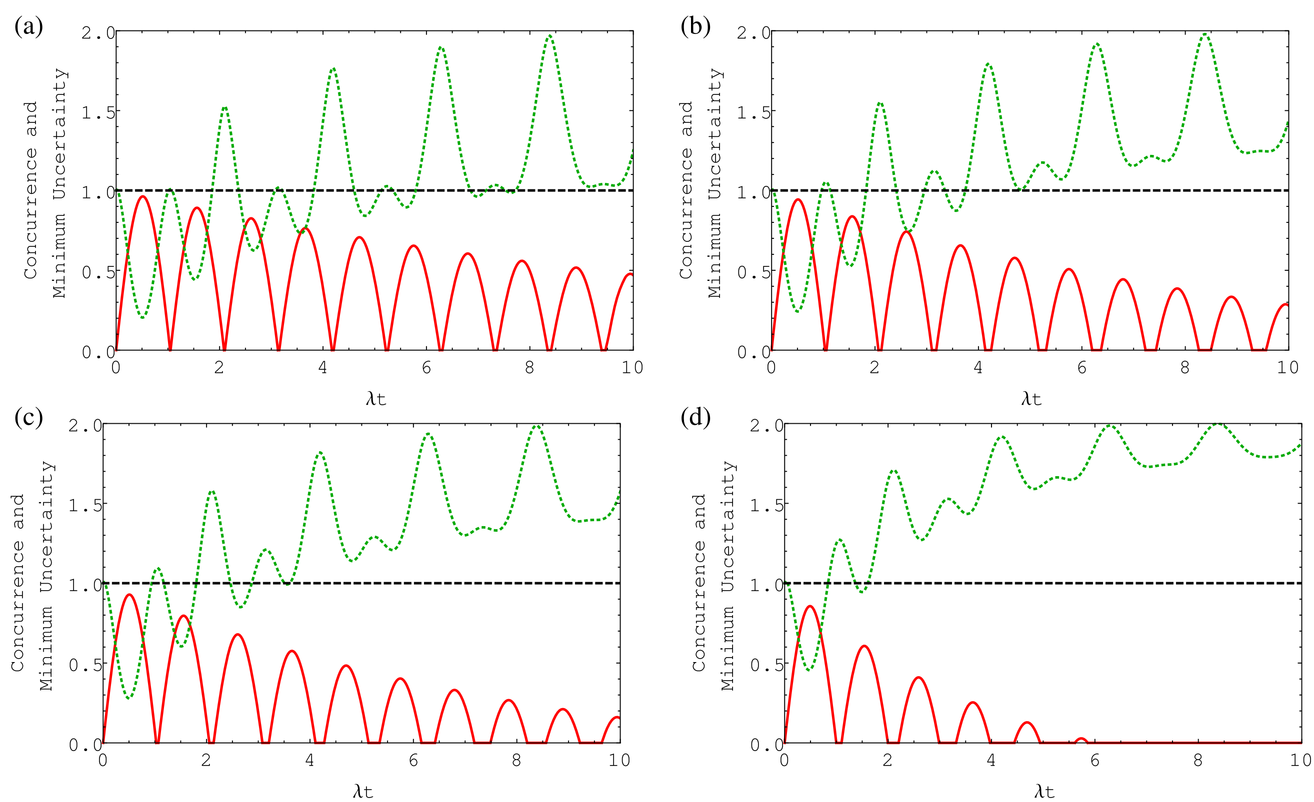

Figure 3. Concurrence (C) (red line) and MU (green line) of two atoms are plotted as functions of scaled time, λt, with $\Omega = 0.3$, $\Gamma = 0.03$, $F(n) = \sqrt{n}$ and $n = 5$. (**a**) $n_T = 0.01$, (**b**) $n_T = 0.1$, (**c**) $n_T = 0.2$ and (**d**) $n_T = 0.8$.

is defined as

$$C_{AB} = \max(0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}), \tag{10}$$

where λ_i are the eigenvalues, organised in descending order, of the matrix $\tilde{\rho} = \rho_{AB}(\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$. Based on the definition of concurrence, the entanglement of two two-level atoms can be obtained as

$$C = \max[0\,,\, 2(\sqrt{\rho_{23}\rho_{32}} \;-\; \sqrt{\rho_{11}\rho_{44}})]. \tag{11}$$

In the following, we plot MU and concurrence (C) as functions of scaled time (λt) for different values of the parameters of the system under consideration.

In figure 1, we investigate the effect of coupling constant (Γ) on MU and C for a fixed value of the intensity-dependent coupling ($n = 5$). From figure 1, we observe that there is a positive relationship between Γ and MU. When Γ increases, MU increases. On the other hand, there is an inverse relationship between Γ and the entanglement. When Γ increases, the degree of entanglement tends to zero.

The time evolution of MU and C for various values of Γ for a fixed value of intensity-dependent coupling ($n = 10$) is plotted in figure 2. We conclude that it has the same effect that is seen in figure 1, but number of oscillations of minimum uncertainty and concurrence increase.

In figure 3, we study the effect of effective photon number n_T on MU and C for a fixed value of the intensity-dependent coupling. We observe that as n_T increases the number of oscillations decreases. Furthermore, MU increases and the entanglement tends to zero as n_T increases. Overall, it is clear from figures 1–3 that the entanglement goes to the maximum value and MU tends to a minimum value. Moreover, our results lead us to conclude that when the value of the degree of entanglement decreases, MU increases. It is interesting to note here that, this model can be considered as a promising model for different applications as many different parameters can be used as controller parameters for the dynamics of such a system.

5. Conclusion

In this work, we have discussed the effect of system parameters, namely, the coupling strength for the two atoms, the cavity field and effective photon number on the entanglement as well as minimum uncertainty. It is shown that the coupling parameter can be used as a control parameter of the system. Also, the opposite role played by the effective photon number is shown, where when n_T increases, the entanglement decreases and MU increases. Finally, this model can be used in different applications such as quantum memory and quantum information processing.

References

[1] R Prevedel, D R Hamel, R Colbeck, K Fisher and K J Resch, *Nat. Phys.* **7**, 757 (2011)
[2] C F Li, J S Xu, X Y Xu, K Li and G C Guo, *Nat. Phys.* **7**, 752 (2011)
[3] J Shi, Z Ding, T Wu, J He, L Yu, W Sun, D Wang and L Ye, *Laser Phys. Lett.* **14**, 125208 (2017)
[4] H M Zou, M F Fang, B Y Yang, Y N Guo, W He and S Y Zhang, *Phys. Scr.* **89**, 115101 (2014)
[5] M Tomamichel, C C W Lim, N Gisin and R Renner, *Nat. Commun.* **3**, 634 (2012)
[6] D P DiVincenzo, M Horodecki, D W Leung, J A Smolin and B M Terhal, *Phys. Rev. Lett.* **92**, 067902 (2004)
[7] P Nataf, M Dogan and K L Hur, *Phys. Rev. A* **86**, 043807 (2012)
[8] M O Hu and H Fan, *Phys. Rev. A* **86**, 032338 (2012)
[9] R Horodecki, P Horodecki, M Horodecki and K Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009)
[10] C H Bennett, G Brassard, C Crepeau, R Jozsa, A Peres and W K Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993)
[11] S Sazim, S Adhikari, S Banerjee and T Pramanik, *Quantum Inf. Process.* **13**, 863 (2014)
[12] S J Devitt, W J Munro and K Nemoto, *Rep. Prog. Phys.* **76**, 076001 (2013)
[13] S Omkar, R Srikanth and S Banerjee, *Phys. Rev. A* **91**, 012324 (2015)
[14] M Abdel-Aty, *J. Phys. A* **41**, 185304 (2008)
[15] M Abdel-Aty, *Eur. Phys. J. D* **46**, 537 (2008)
[16] N Zidan, M Abdel-Aty and A-S F Obada, *Chaos Solitons Fractals* **13**, 1421 (2002)
[17] M Abdel-Aty, *Laser Phys.* **11**, 781 (2001)
[18] T M El-shahat, S Abdel-Khalek, M Abdel-Aty and A-S F Obada, *J. Mod. Opt.* **50**, 2013 (2003)
[19] M Abdel-Aty, *J. Phys. A* **38**, 8589 (2005)
[20] M Abdel-Aty, *J. Mod. Opt.* **50**, 161 (2003)
[21] M Abdel-Aty, *J. Math. Phys.* **44**, 1457 (2003)
[22] S Abdel-Khalek, Y S El-Saman and M Abdel-Aty, *Opt. Commun.* **283**, 1826 (2010)
[23] M Abdel-Aty, *Progr. Quant. Electr.* **31**, 1 (2007)
[24] T M El-Shahat, S Abdel-Khalek, M Abdel-Aty and A-S F Obada, *Chaos Solitons Fractals* **18**, 289 (2003)
[25] N Zidan, *Physica A* **391**, 401 (2012)
[26] M Abdel-Aty, M S Abdalla, and A-S F Obada, *J. Opt. B: Quantum Semiclass. Opt.* **4**, S133 (2002)
[27] H Bakry, A S Mohamed and N Zidan, *Int. J. Theor. Phys.* **57**, 539 (2018)
[28] M S Abdalla, S S Hassan and M Abdel-Aty, *Opt. Commun.* **244**, 431 (2005)
[29] M Abdel-Aty, M S Abdalla and A-S F Obada, *J. Phys. B* **35**, 4773 (2002)
[30] N Zidan, S Abdel-Khalek and M Abdel-Aty, *Int. J. Quantum Inf.* **10**, 1250007 (2012)
[31] S Abdel-Khalek, Y S El-Saman and M Abdel-Aty, *Pramana – J. Phys.* **90**: 1 (2018)
[32] M Q Lone, *Pramana – J. Phys.* **87**: 16 (2016)
[33] S Ahmed, P N Wasnik, S Singh and P A Lakshmi, *Pramana – J. Phys.* **87**: 85 (2016)

[34] N Zidan, *Can. J. Phys.* **92**, 406 (2014)
[35] N H Abdel-Wahab and A Salah, *Pramana – J. Phys.* **89**: 87 (2017)
[36] M Abdel-Aty and N Zidan, *Pramana – J. Phys.* **61**, 553 (2003)
[37] A-S F Obada, D A M Abo-Kahla, N Metwally and M Abdel-Aty, *Physica E* **43**, 1792 (2011)
[38] N Zidan, *Appl. Math.* **5**, 2485 (2014)
[39] N Metwally, M Abdel-Aty and A-S F Obada, *Opt. Commun.* **250**, 148 (2005)
[40] M Abdel-Aty, *J. Opt. B: Quantum Semiclass. Opt.* **6**, 201 (2004)
[41] N Zidan, *Int. J. Quantum Inf.* **8**, 1121 (2010)
[42] M Abdel-Aty, *Physica A* **313**, 471 (2002)
[43] L Pezze and A Smerzi, *Phys. Rev. Lett.* **102**,100401 (2009)
[44] F Jensen, *Introduction to computational chemistry* (Wiley, Hoboken, 2017)
[45] S B Zheng and G C Guo, *Phys. Rev. Lett.* **85**, 2392 (2000)
[46] E Solano, G S Agarwal and H Walther, *Phys. Rev. Lett.* **90**, 027903 (2003)
[47] W Heisenberg, *Z. Phys.* **43**, 172 (1927)
[48] I Bialynicki-Birula and J Mycielski, *Commun. Math. Phys.* **44**, 129 (1975)
[49] I Bialynicki-Birula, *Phys. Lett. A* **103**, 253 (1984)
[50] D Deutsch, *Phys. Rev. Lett.* **50**, 631 (1983)
[51] K Kraus, *Phys. Rev. D* **35**, 3070 (1987)
[52] H Maassen and J B Uffink, *Phys. Rev. Lett.* **60**, 1103 (1988)
[53] M Berta, M Christandl, R Colbeck, J M Renes and B Benner, *Nat. Phys.* **6**, 659 (2010)
[54] J M Renes and J C Boilean, *Phys. Rev. Lett.* **103**, 020402 (2009)
[55] M A Nielsen and I L Chuang, *Quantum computation and quantum information* (Cambridge University Press, Cambridge, 2000)
[56] N J Cerf and C Adami, *Phys. Rev. Lett.* **79**, 5194 (1997)
[57] K G H Vollbrecht and M M Wolf, *J. Math. Phys.* **43**, 4299 (2002)
[58] I Devetak and A Winter, *Proc. R. Soc. Lond. Ser. A* **461**, 207 (2005)
[59] W K Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998)