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Application of the Caputo–Fabrizio derivative without singular kernel to fractional Schrödinger equations

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Abstract. In this work, we solve time, space and time-space fractional Schrödinger equations based on the non-singular Caputo–Fabrizio derivative definition for 1D infinite-potential well problem. To achieve this, we first work out the fractional differential equations defined in terms of Caputo–Fabrizio derivative. Then, the eigenvalues and the eigenfunctions of the three kinds of fractional Schrödinger equations are deduced. In contrast to Laskin's results which are based on Riesz derivative, both the obtained wave number and wave function are different from the standard ones. Moreover, the number of solutions is finite and dependent on the space derivative order. When the fractional orders of derivatives become integer numbers (one for time derivative or/and two for space), our findings collapse to the standard results.

Keywords. Caputo–Fabrizio fractional derivative; fractional differential equation; fractional Schrödinger equation; 1D infinite-potential well.

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1. Introduction

The fractional calculus is a generalisation of the classical derivatives and integrals to an arbitrary non-integer order. In contrast to the ordinary calculus, fractional calculus has several definitions [1–3], such as Riemann–Liouville, Grünwald–Letnikov, Caputo and Riesz, etc. In general, by performing an adequate limit, these different approaches lead to the standard theory. This notion has received much attention and has been widely applied in various fields of physics, including classical and quantum physics, plasma physics, thermodynamics, statistical mechanics, heat transfer, etc. [2–14]. Unlike the integer-order differential equations, the fractional differential equations can successfully describe the anomalous kinetics [15] and transport [16]. Moreover, this kind of equations are considered as a powerful mathematical tool to interpret complex experimental results, for instance, anomalous diffusion [17] or predict new effects [18,19] or describe some phenomena [20–22]. The main advantage of the Caputo–Fabrizio approach is that the boundary conditions of the fractional differential equations with Caputo–Fabrizio derivatives admit the same form as for the integer-order differential equations. Another advantage is that Caputo–Fabrizio's derivative of a constant is zero.

In order to develop the space fractional Schrödinger equation (FSE), Laskin has generalised the Feynman path integral to the Lévy path integral in his seminal paper [5]. Using the Riesz fractional derivative (FD), Laskin has applied the space FSE to treat many quantum mechanical systems, for example, 1D free particle in an infinite-potential well, the fractional Bohr atom and 1D fractional oscillator [23,24]. This work has been extended to involve other quantum problems [25–27]. Moreover, by considering a non-Markovian evolution, Naber [28] has invented the time FSE which is solved for a free particle and for a potential well using Caputo

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derivative. Based on Laskin's and Naber's works, Wang and Xu [29] have constructed the generalised FSE with time and space FDs. On the one hand, it is possible to use regular boundary and initial conditions to solve a fractional differential equation using the Riesz or Caputo approach. On the other hand, fractional differential equation with Riemann–Liouville derivative requires fractional conditions which are physically unacceptable [1]. Consequently, Riesz and Caputo FDs have been very popular among physicists.

Caputo and Fabrizio have subsequently introduced a new definition of the fractional derivative without singular kernel [30]. This recent presentation of the fractional derivative has quickly been used to treat many problems in fundamental and applied sciences [31–35]. To the best of our knowledge, no major study has applied this definition in quantum mechanics.

In this paper, we deal with the new definition of fractional Caputo–Fabrizio derivatives to determine the eigenvalues and the eigenfunctions of the time, space and time-space FSEs for a trapped particle in 1D infinite-potential well. Our paper is organised as follows: Section 2 solves fractional differential equations with Caputo–Fabrizio derivatives of order γ and 2γ. In §3, the solutions are highlighted to study the problem for the three types of FSEs. The conclusion is then presented in §4.

2. The fractional differential equations with Caputo–Fabrizio derivatives of order γ and 2γ

The common definition of Caputo fractional time derivative of order γ, such that $0 < \gamma \leq 1$, is given by [1]

$$ {}^{\mathrm{C}}D_t^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} f'(\tau)\mathrm{d}\tau, \quad t > 0. \tag{1} $$

Replacing the kernel $(t-\tau)^{-\gamma}$ by the function $\exp\left[-\gamma(t-\tau)/(1-\gamma)\right]$ yields the new Caputo–Fabrizio FD which has been recently introduced by Caputo and Fabrizio in [30] as follows:

$$ \mathcal{D}_t^{\gamma} f(t) = \frac{(2-\gamma)M(\gamma)}{2(1-\gamma)} \times \int_0^t \mathrm{e}^{-\frac{\gamma}{1-\gamma}(t-\tau)} f'(\tau)\,\mathrm{d}\tau, \quad t \geq 0, \tag{2} $$

where $M(\gamma)$ is a constant depending on γ defined by $M(\gamma) = 2/(2-\gamma)$ [31]. Thus, we have

$$ \mathcal{D}_t^{\gamma} f(t) = \frac{1}{(1-\gamma)} \int_0^t \mathrm{e}^{-\frac{\gamma}{1-\gamma}(t-\tau)} f'(\tau)\,\mathrm{d}\tau. \tag{3} $$

Obviously, the Caputo–Fabrizio derivative of a constant function is identically zero as in the Caputo derivative. When $\gamma \to 1$, the two definitions become a conventional first derivative of the function $f(t)$. The essential difference between the two formalisms is that, unlike the old formalism, the new kernel has no singularity for $t = \tau$.

According to [31] the solution of the equation

$$ \mathcal{D}_t^{\gamma} f(t) = u(t) \tag{4} $$

is deduced from

$$ f(t) = (1-\gamma)\,(u(t) - u(0)) + \gamma \int_0^t u(\tau)\,\mathrm{d}\tau + f(0). \tag{5} $$

Deriving the above equation over time allows us to directly establish the solution by solving the resulting ordinary differential equation

$$ f'(t) = (1-\gamma)u'(t) + \gamma u(t). \tag{6} $$

If we consider

$$ \mathcal{D}_t^{\gamma} f(t) = \sigma f(t), \tag{7} $$

where σ is time-independent, we find

$$ f'(t) = \frac{\gamma\sigma}{1-(1-\gamma)\sigma} f(t). \tag{8} $$

Hence, the solution of (7) is

$$ f(t) = f(0) \exp\left(\frac{\gamma\sigma}{1-(1-\gamma)\sigma} t\right). \tag{9} $$

Now, we seek to solve a fractional differential equation of order 2γ like

$$ \mathcal{D}_t^{2\gamma} f(t) = g(t), \quad 0 < \gamma \leq 1. \tag{10} $$

First, let us take $\mathcal{D}_t^{\gamma} f(t) = u(t)$ and $\mathcal{D}_t^{2\gamma} f(t) = \mathcal{D}_t^{\gamma} u(t) \equiv g(t)$. Then, according to eq. (5), we have

$$ u(t) = (1-\gamma)\,(g(t) - g(0)) + \gamma \int_0^t g(\tau)\,\mathrm{d}\tau + u(0) \tag{11} $$

and

$$ f(t) = (1-\gamma)\,(u(t) - u(0)) + \gamma \int_0^t u(\tau)\,\mathrm{d}\tau + f(0). \tag{12} $$

When we substitute (11) into (12), we obtain

$$ \begin{aligned} f(t) = {} & (1-\gamma)^2\,(g(t) - g(0)) \\ & + 2\gamma(1-\gamma) \int_0^t g(\tau)\,\mathrm{d}\tau \\ & + \gamma^2 \int_0^t \mathrm{d}\tau \int_0^{\tau} g(\tau)\,\mathrm{d}\tau \\ & + \left[u(0) - (1-\gamma)g(0)\right]\gamma t + f(0). \end{aligned} \tag{13} $$

By deriving twice, the last equation yields

$$f''(t) = (1-\gamma)^2 g''(t) + 2\gamma(1-\gamma)g'(t) + \gamma^2 g(t). \tag{14}$$

In the case,

$$\mathcal{D}_t^{2\gamma} f(t) = \sigma f(t), \tag{15}$$

we find

$$f''(t) + 2\Lambda f'(t) + \Theta^2 f(t) = 0 \tag{16}$$

where

$$\Lambda = \frac{-\gamma(1-\gamma)\sigma}{1-(1-\gamma)^2\sigma} \quad \text{and} \quad \Theta^2 = \frac{-\gamma^2\sigma}{1-(1-\gamma)^2\sigma}. \tag{17}$$

Therefore, the solution has the form

$$f(t) = A\mathrm{e}^{r_1 t} + B\mathrm{e}^{r_2 t}, \tag{18}$$

with

$$r_1 = -\Lambda + \sqrt{\Lambda^2 - \Theta^2} \quad \text{and} \quad r_2 = -\Lambda - \sqrt{\Lambda^2 - \Theta^2}.$$

3. Application to 1D infinite-potential well

To preserve the units in Schrödinger equation after fractionalisation of its derivatives, one can express it in Planck units [28] as

$$\mathrm{i}T_p \frac{\partial}{\partial t}\psi(x,t) = -\frac{L_p^2 M_p}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t) + \frac{V}{E_p}\psi(x,t) \tag{19}$$

or

$$\mathrm{i}T_p \frac{\partial}{\partial t}\psi(x,t) = -\frac{L_p^2}{2N_m}\frac{\partial^2}{\partial x^2}\psi(x,t) + N_v\psi(x,t), \tag{20}$$

where T_p, L_p, M_p and E_p are Planck time, length, mass and energy, respectively. These units are defined as follows:

$$T_p = \sqrt{\frac{G\hbar}{c^5}}, \quad L_p = \sqrt{\frac{G\hbar}{c^3}}, \quad M_p = \sqrt{\frac{\hbar c}{G}}$$

and

$$E_p = M_p c^2. \tag{21}$$

Here G and c are the gravitational constant and speed of light, respectively. $N_m = m/M_p$ denotes the number of Planck masses in m and $N_v = V/E_p$ is the number of Planck energies in V. For a free particle confined in infinite-potential well

$$V(x) = \begin{cases} 0, & 0 < x < a \\ \infty & \text{elsewhere}, \end{cases} \tag{22}$$

eq. (20) becomes

$$\mathrm{i}T_p \frac{\partial}{\partial t}\psi(x,t) = -\frac{L_p^2}{2N_m}\frac{\partial^2}{\partial x^2}\psi(x,t). \tag{23}$$

In the following subsections, we shall investigate three types of FSEs.

3.1 *Time fractional Schrödinger equation*

The order of the time derivative in this type of FSE is the fractional number γ rather than one, as shown in [28]

$$(\mathrm{i}T_p)^\gamma \partial_t^\gamma \psi(x,t) = -\frac{L_p^2}{2N_m}\partial_x^2\psi(x), \quad 0 < \gamma \le 1. \tag{24}$$

Before presenting our study, we must emphasise a point here: the right-hand side of the last equation is an auto-adjoint operator (operating on the function $\psi(x)$). The left-hand side operator $(\mathrm{i}T_p)^\gamma \partial_t^\gamma$ must be auto-adjoint too. Using the definition of Caputo–Fabrizio (3), the last equation can be written as

$$(\mathrm{i}T_p)^\gamma \mathcal{D}_t^\gamma \psi = -\frac{L_p^2}{2N_m}\partial_x^2\psi. \tag{25}$$

The last equation admits solutions by the separation of variables $\psi(x,t) = \phi(t)\varphi(x)$, and we obtain

$$(\mathrm{i}T_p)^\gamma \frac{\mathcal{D}_t^\gamma \phi}{\phi} = -\frac{L_p^2}{2N_m}\frac{\partial_x^2\varphi}{\varphi} = \lambda. \tag{26}$$

Therefore, we have two independent equations

$$\varphi'' + \lambda\frac{2N_m}{L_p^2}\varphi = 0 \tag{27}$$

and

$$\mathcal{D}_t^\gamma \phi = \frac{\lambda}{(\mathrm{i}T_p)^\gamma}\phi. \tag{28}$$

The solutions of spatial equation (27) are well-known:

$$\varphi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right) \tag{29}$$

and

$$\lambda_n = \frac{1}{2N_m}\left(\frac{n\pi L_p}{a}\right)^2, \quad n \in \mathbb{N}, \tag{30}$$

under the following boundary conditions:

$$\varphi(0) = 0, \quad \varphi(a) = 0. \tag{31}$$

As we have seen, the Caputo–Fabrizio definition (3) does not pay attention to the dimensionalisation. Thus, we must change the time coordinate t in (28) to the dimensionless coordinate $t' = t/T_p$

$$D^{\gamma}_{t'}\phi(t') = (-\mathrm{i})^{\gamma}\,\lambda_n\phi(t'). \tag{32}$$

As the previous equation has the same form of eq. (7), its solution will be

$$\phi_n(t) = \exp\left[(-\mathrm{i})^{\gamma}\,\frac{\gamma\lambda_n/T_p}{1-(-\mathrm{i})^{\gamma}\,(1-\gamma)\lambda_n}t\right]. \tag{33}$$

Here $\phi(0) = 1$. If $(-\mathrm{i})^{\gamma} = \mathrm{e}^{-\mathrm{i}(\frac{\pi}{2}\gamma+2k\pi\gamma)}$,

$$\phi_n(t) = \exp\left[\gamma\lambda_n\frac{\cos(\frac{\pi}{2}\gamma+2k\gamma\pi)-(1-\gamma)\lambda_n-\mathrm{i}\sin(\frac{\pi}{2}\gamma+2k\gamma\pi)}{1-2(1-\gamma)\lambda_n\cos(\frac{\pi}{2}\gamma+2k\pi\gamma)+(1-\gamma)^2\lambda_n^2}\frac{t}{T_p}\right]. \tag{34}$$

Keeping in mind the remark stated above, the exponent in formula (33), depending on γ, can be a pure imaginary number or a complex number. In the first case, the operator of the left-hand side of eq. (24) is an auto-adjoint operator for which the energy is a real quantity. In the second case, the exponent in (33) is a complex number and the operator of the left-hand side of eq. (24) is not an auto-adjoint operator for which the energy, as we shall show, is a complex number with real and imaginary parts. If $\gamma = 1$, we recover the usual case

$$\phi(t) = \mathrm{e}^{-\mathrm{i}\frac{\lambda_n}{T_p}t} = \mathrm{e}^{-\mathrm{i}\omega_n t}, \tag{35}$$

where $\omega_n = \lambda_n/T_p$ represents the frequency.

Now, we can write the eigenfunctions and the eigenvalues of the particle as

$$\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\exp\left[\gamma\lambda_n\frac{\cos(\frac{\pi}{2}\gamma+2k\gamma\pi)-(1-\gamma)\lambda_n-\mathrm{i}\sin(\frac{\pi}{2}\gamma+2k\gamma\pi)}{1-2(1-\gamma)\lambda_n\cos(\frac{\pi}{2}\gamma+2k\pi\gamma)+(1-\gamma)^2\lambda_n^2}\frac{t}{T_p}\right] \tag{36}$$

and

$$E_n = \int_0^a \psi^*\mathrm{i}\hbar\partial_t\psi\,\mathrm{d}x = \frac{\gamma\hbar\lambda_n}{T_p}\frac{\sin(\frac{\pi}{2}\gamma+2k\pi\gamma)+\mathrm{i}[\cos(\frac{\pi}{2}\gamma+2k\pi\gamma)-(1-\gamma)\lambda_n]}{1-2(1-\gamma)\lambda_n\cos(\frac{\pi}{2}\gamma+2k\pi\gamma)+(1-\gamma)^2\lambda_n^2}$$
$$\times\exp\left[2\gamma\lambda_n\frac{\cos(\frac{\pi}{2}\gamma+2k\gamma\pi)-(1-\gamma)\lambda_n}{1-2(1-\gamma)\lambda_n\cos(\frac{\pi}{2}\gamma+2k\pi\gamma)+(1-\gamma)^2\lambda_n^2}\frac{t}{T_p}\right]. \tag{37}$$

Interestingly, our qualitative calculations for $\gamma = 1$ show that the eigenfunctions and eigenvalues agree with the standard ones

$$\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)\mathrm{e}^{-\mathrm{i}\omega_n t} \quad \text{and} \quad E_n = \hbar\omega_n. \tag{38}$$

3.2 *Space fractional Schrödinger equation*

In the space FSE, the order of the space derivative is β instead of 2, i.e.

$$\mathrm{i}T_P\partial_t\psi = \frac{(-\mathrm{i}L_P)^{\beta}}{2N_m}\partial_x^{\beta}\psi, \quad 1<\beta\le 2. \tag{39}$$

In this case, we can use the same definition of the time derivative (3) for space coordinate x with $x \ge 0$, which means

$$\mathcal{D}_x^{\beta}\psi = \mathcal{D}_x^{2\mu}\psi, \quad \beta = 2\mu \text{ and } \frac{1}{2}<\mu\le 1. \tag{40}$$

As $(-\mathrm{i})^{\beta} = \mathrm{e}^{-\mathrm{i}\beta(\frac{\pi}{2}+2k\pi)}$, $k \in \mathbb{N}$, we can choose the values of β as $2q/(1+4k)$, where q is an odd number that fulfills $(1+4k)/2 < q \le 1+4k$. This leaves $(-\mathrm{i})^{\beta} = -1$. Implicitly, these conditions imply that $(-\mathrm{i}\partial_x)^{\beta}$ is a Hermitian operator. Under these circumstances, the space FSE becomes

$$\mathrm{i}T_p D_t\psi = -\frac{L_p^{\beta}}{2N_m}\mathcal{D}_x^{\beta}\psi. \tag{41}$$

Taking $\psi(x,t) = \varphi(x)\phi(t)$ yields

$$\mathrm{i}T_p\frac{D_t\phi}{\phi} = -\frac{L_p^{\beta}}{2N_m}\frac{\mathcal{D}_x^{\beta}\varphi}{\varphi} = \frac{E}{E_p} = N_e \tag{42}$$

or

$$D_t\phi = -\mathrm{i}\frac{E}{\hbar}\phi, \quad \frac{E}{\hbar} = \frac{N_e}{T_p} \tag{43}$$

and

$$\mathcal{D}_x^{\beta}\varphi = -k^{\beta}\varphi, \quad k = \frac{(2N_mN_e)^{1/\beta}}{L_p}. \tag{44}$$

The solution of eq. (43) is given by

$$\phi(t) = \mathrm{e}^{-\mathrm{i}\frac{E}{\hbar}t}. \tag{45}$$

To work out eq. (44), we first rewrite it by introducing scaled position as

$$\mathcal{D}_{\xi}^{\beta}\varphi(\xi) = -(L_p k)^{\beta}\varphi(\xi), \quad \xi = \frac{x}{L_p}. \tag{46}$$

From §2, one can write the solution of the above equation as

$$\varphi(\xi) = \mathrm{e}^{-\Lambda\xi}(A\mathrm{e}^{\mathrm{i}R\xi} + B\mathrm{e}^{-\mathrm{i}R\xi}), \tag{47}$$

with

$$\Lambda = \frac{\beta(2-\beta)(L_p k)^{\beta}}{4+(2-\beta)^2(L_p k)^{\beta}} \tag{48}$$

and

$$R = \frac{2\beta(L_p k)^{\beta/2}}{4+(2-\beta)^2(L_p k)^{\beta}}. \tag{49}$$

Using the boundary conditions $B(0) = B(a/L_p) = 0$, the eigenfunction reads as

$$\varphi(\xi) = 2\mathrm{i}A\mathrm{e}^{-\Lambda\xi}\sin\left(\frac{\pi n L_p}{a}\xi\right) = C\mathrm{e}^{-\frac{\Lambda}{L_p}x}\sin\left(\frac{\pi n}{a}x\right), \quad C = 2\mathrm{i}A \tag{50}$$

and

$$R_n = \frac{\pi L_p}{a}n, \quad n \in \mathbb{N}, \tag{51}$$

where C is determined by the normalisation condition

$$\int_0^a |\varphi(x)|^2 \mathrm{d}x = 1.$$

From (49) and (51), we have

$$(2-\beta)^2 R_n (L_p k)^{\beta} - 2\beta(L_p k)^{\beta/2} + 4R_n = 0. \tag{52}$$

This equation has two real solutions only if

$$0 < R_n < \frac{\beta}{2(2-\beta)}. \tag{53}$$

In other words, the number n will be limited.

$$0 < n < \frac{\beta a}{2\pi(2-\beta)L_p}. \tag{54}$$

The first solution is

$$(L_p k_n)^{\beta/2} = \beta\frac{1-\sqrt{1-4(\frac{2-\beta}{\beta})^2 R_n^2}}{(2-\beta)^2 R_n}. \tag{55}$$

In the limit $\beta \to 2$, solution (55) tends to R_n, and then, $k_n = \pi n/a$ which is the standard wave number. We discard the second solution because it is physically unacceptable. From eq. (55) we have

$$k_n = \frac{\beta^{2/\beta}}{L_p}\left[\frac{1-\sqrt{1-4\left(\frac{2-\beta}{\beta}\right)^2\left(\frac{n\pi L_p}{a}\right)^2}}{(2-\beta)^2\left(\frac{n\pi L_p}{a}\right)}\right]^{2/\beta}. \tag{56}$$

Consequently

$$E_n = \frac{E_p L_p^{\beta}}{2N_m}k_n^{\beta} = \frac{\beta^2 E_p}{2N_m}\left[\frac{1-\sqrt{1-4\left(\frac{2-\beta}{\beta}\right)^2\left(\frac{n\pi L_p}{a}\right)^2}}{(2-\beta)^2\left(\frac{n\pi L_p}{a}\right)}\right]^{2} \tag{57}$$

and

$$\psi_n(x,t) = C\mathrm{e}^{-\frac{\Lambda}{L_p}x}\sin\left(\frac{\pi n}{a}x\right)\mathrm{e}^{-\mathrm{i}\frac{E_n}{\hbar}t}. \tag{58}$$

By comparing with previous results [23], we can see that, in the two approaches, the dependence of the energy on momentum is non-quadratic ($E \propto k^{\beta}$). According to Laskin's findings, both the wave number and the wave function coincide with the standard ones for all β values, whereas, in our case, they have different form. Furthermore, the effect of Caputo–Fabrizio definition also seems to limit the number of solutions and relate it to the derivative order β throughout condition (54). For $\beta = 2$, all the results go to the standard case (38).

3.3 *Generalised FSE with space-time fractional derivatives*

In this subsection, we consider that each of the space and time derivative in the Schrödinger equation has a fractional order. The resulting equation is the so-called generalised FSE with space-time FDs [29]

$$(\mathrm{i}T_P)^{\gamma}\,\mathcal{D}_t^{\gamma}\psi = \frac{(-\mathrm{i}L_p)^{\beta}}{2N_m}\mathcal{D}_x^{\beta}\psi, \quad 1 < \beta \leq 2 \ \text{and} \ 0 < \gamma \leq 1. \tag{59}$$

We consider a class of systems such that γ and β fulfill $\gamma = \beta - 1$. In this case, the above equation reads as

$$\mathrm{i}T_P^{\gamma}\mathcal{D}_t^{\gamma}\psi = -\frac{(-L_p)^{\beta}}{2N_m}\mathcal{D}_x^{\beta}\psi. \tag{60}$$

Let $(-1)^{\beta} = \exp(\mathrm{i}\pi k\beta)$, where k is an odd number. We can take $\beta = q/k$, where q is an even number with

$k < q \leq 2k$. In this case, $(-1)^{\beta} = 1$, and

$$\mathrm{i}T_p^{\gamma}\mathcal{D}_t^{\gamma}\psi = -\frac{L_p^{\beta}}{2N_m}\mathcal{D}_x^{\beta}\psi. \tag{61}$$

As usual, the separation of variables $\psi(x,t) = \varphi(x)\phi(t)$ leads to

$$\mathrm{i}T_p^{\gamma}\frac{\mathcal{D}_t^{\gamma}\phi}{\phi} = -\frac{L_p^{\beta}}{2N_m}\frac{\mathcal{D}_x^{\beta}\varphi}{\varphi} = \lambda \tag{62}$$

or

$$\mathcal{D}_t^{\gamma}\phi = -\mathrm{i}\frac{\lambda}{T_p^{\gamma}}\phi \tag{63}$$

and

$$\mathcal{D}_x^{\beta}\varphi = -\frac{2N_m\lambda}{L_p^{\beta}}\varphi. \tag{64}$$

To solve the last two equations, we show that by non-dimensionalisation the above equations can be expressed as

$$\mathcal{D}_{t'}^{\gamma}\phi(t') = -\mathrm{i}\lambda\phi(t') \tag{65}$$

and

$$\mathcal{D}_{\xi}^{\beta}\varphi(\xi) = -2N_m\lambda\varphi(\xi) = -\eta\varphi(\xi), \quad \eta = 2N_m\lambda. \tag{66}$$

The solution of (65) has the same form as (9)

$$\phi(t') = \exp\left(\frac{-\mathrm{i}\gamma\lambda}{1+\mathrm{i}(1-\gamma)\lambda}t'\right), \tag{67}$$

where $\phi(0) = 1$. Accordingly, the solution of (63) reads as

$$\phi(t) = \exp\left(-\frac{\gamma(1-\gamma)\lambda^2 + \mathrm{i}\gamma\lambda}{1+(1-\gamma)^2\lambda^2}\frac{t}{T_p}\right). \tag{68}$$

The solution of (66) is

$$\varphi(\xi) = \mathrm{e}^{-\Lambda\xi}(C\mathrm{e}^{\mathrm{i}R\xi} + D\mathrm{e}^{-\mathrm{i}R\xi}). \tag{69}$$

Thus, we have

$$\varphi(x) = \mathrm{e}^{-\Lambda\frac{x}{L_p}}(A\mathrm{e}^{\mathrm{i}R\frac{x}{L_p}} + B\mathrm{e}^{-\mathrm{i}R\frac{x}{L_p}}), \tag{70}$$

where

$$\Lambda = \frac{\beta(2-\beta)\eta}{4+(2-\beta)^2\eta} \tag{71}$$

and

$$R = \frac{2\beta\sqrt{\eta}}{4+(2-\beta)^2\eta}. \tag{72}$$

Using the boundary conditions, we find

$$\varphi_n(x) = C\mathrm{e}^{-\Lambda\frac{x}{L_p}}\sin\left(\frac{\pi n}{a}x\right), \tag{73}$$

where

$$R_n = \frac{\pi n}{a}L_p. \tag{74}$$

From (72) and (74), we can deduce that

$$\lambda_n = \frac{\beta^2}{2N_m}\left[\frac{1-\sqrt{1-4\left(\frac{2-\beta}{\beta}\right)^2\left(\frac{n\pi L_p}{a}\right)^2}}{(2-\beta)^2\left(\frac{n\pi L_p}{a}\right)}\right]^2, \tag{75}$$

where $0 < n < \beta a/2\pi(2-\beta)L_p$. If $\beta = 2$, we immediately obtain λ_n which we have previously found in § 3.1. The solutions of the generalised FSE (61) are

$$\psi_n(x,t) = C\mathrm{e}^{-\Lambda_n\frac{x}{L_p}}\sin\left(\frac{\pi n}{a}x\right) \times \exp\left(-\frac{\gamma(1-\gamma)\lambda_n^2 + \mathrm{i}\gamma\lambda_n}{1+(1-\gamma)^2\lambda_n^2}\frac{t}{T_p}\right) \tag{76}$$

and

$$E_n = \int_0^a \psi^*\mathrm{i}\hbar\partial_t\psi\,\mathrm{d}x = \frac{\hbar}{T_p}\frac{\gamma\lambda_n - \mathrm{i}\gamma(1-\gamma)\lambda_n^2}{1+(1-\gamma)^2\lambda_n^2} \times \exp\left[\frac{-2\gamma(1-\gamma)\lambda_n^2}{1+(1-\gamma)^2\lambda_n^2}\frac{t}{T_p}\right]. \tag{77}$$

When $\beta = 2$, the solutions coincide with the standard ones (38).

4. Conclusion

In this work, we have applied the Caputo–Fabrizio fractional derivative to solve three types of partial differential equations: the time, space, and time-space FSEs for a free particle in 1D infinite potential well. The comparison of the results with those found using Caputo and Riesz derivatives reveals the different effects of the fractional derivative types. The solutions of time FSE with Caputo–Fabrizio FDs have a form which is different from the solutions obtained using Caputo derivative. For the space FSE, we have found that both the wave number and the wave function are different from the standard ones. This finding is not stated in Laskin's approach which is based on Riesz's definition. Moreover, the number of solutions is limited and dependent on the space derivative order. If we put the fractional orders $\gamma = 1$ and $\beta = 2$, our results will be in complete agreement with those in the usual case. Furthermore, we believe that the fractional derivative in the Caputo–Fabrizio formalism, due to the non-singularity of the kernel, may provide an alternative way to solve many problems. In our future work, we hope to use this approach to investigate the superconductivity phenomenon and the quantum transport theory in semiconductor nanostructures.

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