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Analysis of solution trajectories of fractional-order systems

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Abstract. The behavior of solution trajectories usually changes if we replace the classical derivative in a system with a fractional one. In this article, we throw light on the relation between two trajectories $X(t)$ and $Y(t)$ of such a system, where the initial point $Y(0)$ is at some point $X(t_1)$ of the trajectory $X(t)$. In contrast with classical systems, these trajectories X and Y do not follow the same path. Further, we provide a Frenet apparatus for both trajectories in various cases and discuss their effect.

Keywords. Fractional derivative; Mittag–Leffler functions; Orthogonal transformation; Frenet apparatus.

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1. Introduction and preliminaries

In the recent past, fractional differential equations (FDE) became a popular topic among the researchers working in pure as well as applied mathematics. Applications of FDEs are found in various fields ranging from physics to biology (see refs [1–12]).

Mathematical analysis of FDEs is also an interesting and equally important topic of research. Existence and uniqueness [13–17], stability [18–25] and positivity [26–32] of these equations are studied by the researchers in detail. As FDEs are generalisations of classical differential dynamical systems, we cannot expect the same properties from these models as the classical ones.

In [33], we have shown that the planar linear FDE system ${}_0^C\mathrm{D}_t^\alpha X = AX$ may produce self-intersecting trajectories. Such singular points are not shown by their classical counterparts. We continue our investigations in the present paper and discuss the trajectories of FDE systems whose initial point is on a different trajectory of the same system.

Now, we discuss some basic definitions and results given in the literature.

DEFINITION 1

Let $\alpha \geq 0$ $(\alpha \in \mathbb{R})$ [34]. Then Riemann–Liouville (RL) fractional integral of function $f \in C[0, b]$, $t > 0$ of order α is defined as

$$ {}_0\mathrm{I}_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau. \tag{1}$$

DEFINITION 2

The Caputo fractional derivative of order $\alpha > 0, n-1 < \alpha < n, n \in \mathbb{N}$ is defined for $f \in C^n[0, b]$, $t > 0$ as [34]

$$ {}_0^C\mathrm{D}_t^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\,\mathrm{d}\tau & \text{if } n-1<\alpha<n \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) & \text{if } \alpha = n. \end{cases} \tag{2}$$

Note that ${}_0^C\mathrm{D}_t^\alpha c = 0$, where c is a constant.

DEFINITION 3

The one-parameter Mittag–Leffler function is defined as [34]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}\ (\alpha > 0). \tag{3}$$

The two-parameter Mittag–Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}\ (\alpha > 0,\ \beta > 0). \tag{4}$$

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DEFINITION 4

Let $\alpha : I \to \mathbb{R}^n$ be a curve. The speed of α is defined as [35]

$$\nu(t) = \| \alpha'(t) \| . \tag{5}$$

DEFINITION 5

An isometry of $\mathbb{R}^n$ is a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$ such that [35]

$$d(F(p), F(q)) = d(p, q) \tag{6}$$

for all points p, q in $\mathbb{R}^n$. $d(x, y)$ is the Euclidean distance.

DEFINITION 6

Two curves $\alpha, \beta : I \to \mathbb{R}^n$ are congruent provided there exists an isometry F of $\mathbb{R}^n$ such that $\beta = F(\alpha)$; that is, $\beta(t) = F(\alpha(t))$ for all t in I [35].

DEFINITION 7

A transformation $C : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation if it preserves dot products in the sense that [35]

$$C(p) \cdot C(q) = p \cdot q \quad \text{for all } p, q. \tag{7}$$

Every orthogonal transformation is an isometry.

Theorem 1. *Solution of homogeneous system of fractional order differential equation* [36]

$$^C_0\mathrm{D}^\alpha_t X(t) = AX(t), \quad 0 < \alpha < 1, \tag{8}$$

where A is $n \times n$ matrix, is given by

$$X(t) = E_\alpha(At^\alpha)X(0), \tag{9}$$

where $E_\alpha(At^\alpha)$ is the matrix variate Mittag–Leffler function.

Theorem 2. *For planar regular curve $\alpha : I \to \mathbb{R}^2$ given by $\alpha(t) = (x(t), y(t))$, $t \in I$, the Frenet apparatus is given by* [35]

$$T = \frac{(\dot{x}, \dot{y})}{\sqrt{(\dot{x})^2 + (\dot{y})^2}}$$

$$N = \frac{(-\dot{y}, \dot{x})}{\sqrt{(\dot{x})^2 + (\dot{y})^2}}$$

$$k = \frac{\det \begin{vmatrix} \dot{x} & \dot{y} \\ \ddot{x} & \ddot{y} \end{vmatrix}}{((\dot{x})^2 + (\dot{y})^2)^{3/2}}. \tag{10}$$

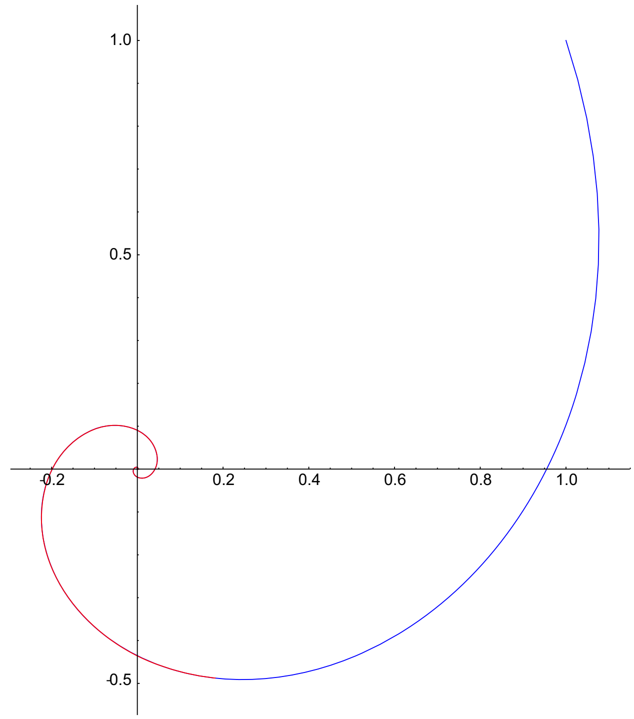


Figure 1. Solution trajectory of system (11) starting at some point on the original trajectory.

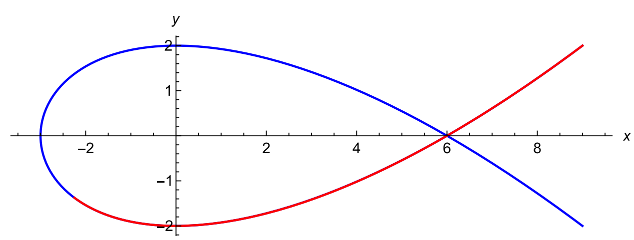


Figure 2. Self-intersecting trajectory of system (12).

2. Observations

We have the following observations.
(1) Consider the system

$$\dot{X}(t) = \begin{bmatrix} -2 & 4 \\ -4 & -2 \end{bmatrix} X(t). \tag{11}$$

We have sketched the solution trajectory

$$X(t) = \mathrm{e}^{-2t} \begin{bmatrix} \cos(4t) + \sin(4t) \\ -\sin(4t) + \cos(4t) \end{bmatrix}$$

of system (11) with initial condition $X(0) = [1, 1]^T$ in figure 1 and it is shown in blue.

Now, consider the same system (11) with initial condition $X(0) = [\mathrm{e}^{-1}(\cos 2 + \sin 2), e^{-1}(-\sin 2 + \cos 2)]^T$ on the original trajectory, discussed above. Solution of this system is shown in the same figure in red. It can be observed that both the trajectories follow the same path.

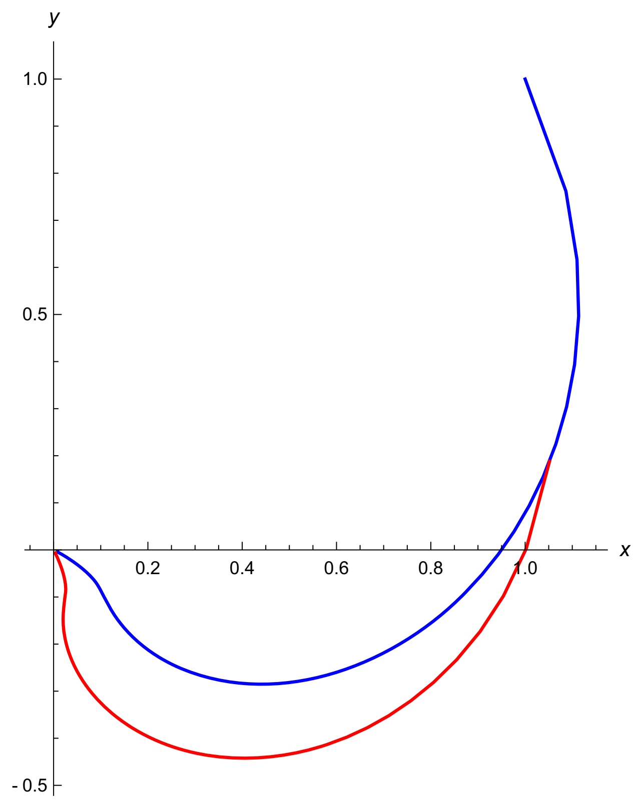


Figure 3. Solution trajectory of system (13).

(2) Consider the non-autonomous system of differential equations

$$\dot{X}(t) = \begin{bmatrix} 6t \\ 3t^2 - 3 \end{bmatrix}. \quad (12)$$

The solution trajectories of system (12) with initial conditions $X(-2) = [9, -2]^T$ (blue curve) and $X(0) = [-2.25, -1.375]^T$ (red curve) are shown in figure 2.

It can be observed that, the loop in the original trajectory can be eliminated by choosing the initial condition of a new trajectory at a point on the original trajectory after self-intersection.

(3) Consider the fractional-order system

$$_0^C\mathrm{D}_t^{0.7} X(t) = \begin{bmatrix} -1 & 3 \\ -3 & -1 \end{bmatrix} X(t). \quad (13)$$

In figure 3, we have sketched the solution trajectory of system (13) with initial condition $X(0) = [1, 1]^T$ (blue). Further, the solution trajectory of the same system (13) with initial condition

$$\begin{aligned} X(0) = [&\mathrm{Re}[E_{0.7}((-1+3i)(0.1)^{0.7})] \\ &+\mathrm{Im}[E_{0.7}((-1+3i)(0.1)^{0.7})] \\ &-\mathrm{Im}[E_{0.7}((-1+3i)(0.1)^{0.7})] \\ &+\mathrm{Re}[E_{0.7}((-1+3i)(0.1)^{0.7})]]^T \end{aligned}$$

at the original trajectory, is shown in the same figure by red. However, the paths followed by these two trajectories are different, unlike in classical model (11).

(4) In ref. [33], we have observed self-intersecting trajectories of some linear fractional-order systems. Consider the system

$$_0^C\mathrm{D}_t^{\alpha} X(t) = \begin{bmatrix} 0.983469 & 0.181075 \\ -0.181075 & 0.983469 \end{bmatrix} X(t) = AX(t). \quad (14)$$

If $\alpha = 0.1$ and $X(0) = [1, 1]^T$, the solution trajectory shows self-intersection (see figure 4a). Let us consider this system with initial condition

$$\begin{aligned} X(0) = [&\mathrm{Re}(E_{0.1}(0.983469 + 0.181075i)50^{0.1}) \\ &+\mathrm{Im}(E_{0.1}(0.983469+0.181075i)50^{0.1}) \\ &-\mathrm{Im}(E_{0.1}(0.983469+0.181075i)50^{0.1}) \\ &+\mathrm{Re}(E_{0.1}(0.983469+0.181075i)50^{0.1})]^T \end{aligned}$$

on the original trajectory.

Though we have taken new initial condition on the original trajectory at a point after self-intersection, the singular points cannot be removed unlike in classical case (2). Further, it seems that the new trajectory is some linear transformation of the original one.

(5) Time used to complete the loop:
Consider system (14) with $\alpha = 0.1$ and initial condition $X(0) = [1, 1]^T$. Consider a node formed by solution trajectory in the time interval (12.35, 34). If we solve the same system (14) with $\alpha = 0.1$ and initial condition

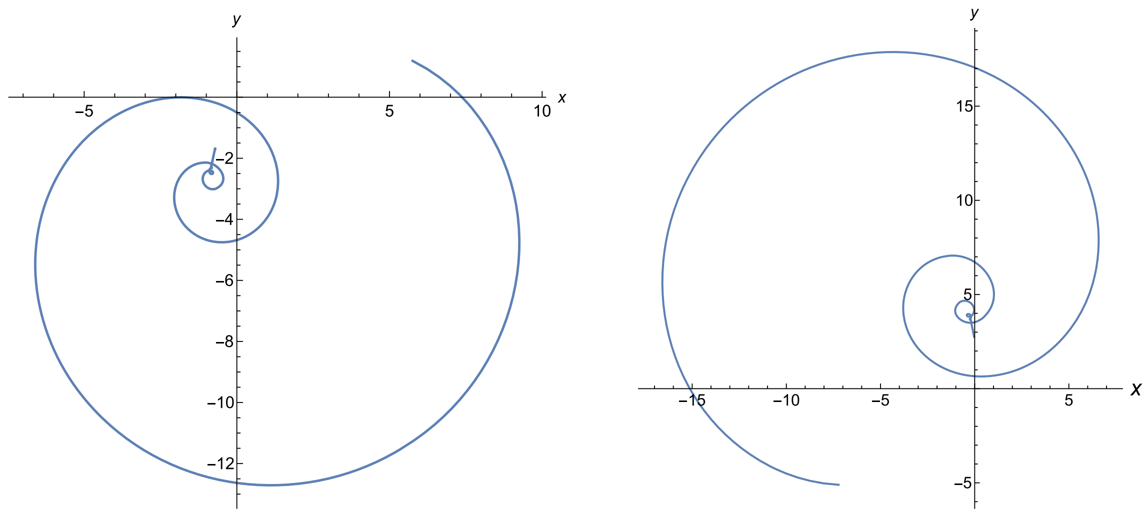
$$\begin{aligned} X(0) = [&\mathrm{Re}(E_{0.1}(0.983469 + 0.181075i)50^{0.1}) \\ &+\mathrm{Im}(E_{0.1}(0.983469 + 0.181075i)50^{0.1}) \\ &-\mathrm{Im}(E_{0.1}(0.983469 + 0.181075i)50^{0.1}) \\ &+\mathrm{Re}(E_{0.1}(0.983469+0.181075i)50^{0.1})]^T \end{aligned}$$

on the original trajectory, then we get a node in the new trajectory in the same time interval (12.35, 34). However, it seems that the new node has different size and is obtained by rotating the original node as shown in figure 5. Further, the time taken to complete the loop in both the nodes is the same but the speed is different.

Our motivation for the present study is to find the linear transformation between original trajectory and the new trajectory of the fractional-order system.

(6) Bifurcation in (14):
The stability analysis [18] shows system (14) is stable if $\alpha < (2/\pi)\mathrm{Arg}(\lambda)$, where $\lambda = 0.983469 \pm 0.181075i$ are eigenvalues of A, i.e. if $\alpha < 0.1159$.

We have considered various values of $\alpha \in (0, 1)$ and observed solution trajectories of (14).



(a) $x(0) = y(0) = 1$. (b) x(0) = -0.777226, y(0) = -1.98038.

Figure 4. Transformation of self-intersecting trajectory of system (14) with $\alpha = 0.1$.

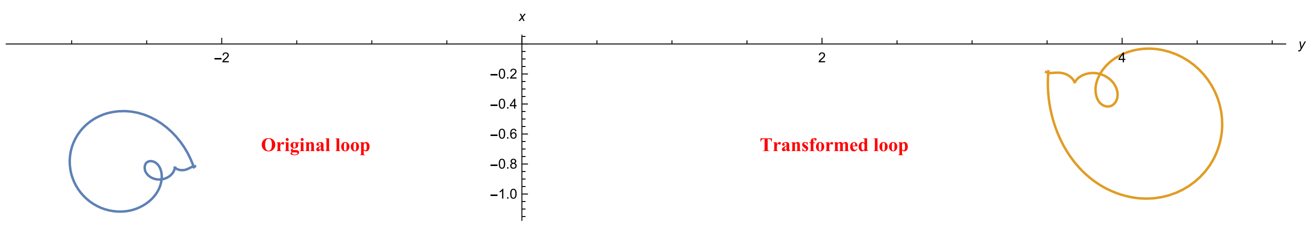


Figure 5. Transformed loops in system (14).

- For $0 < \alpha < 0.08$, the trajectories produce spiral sink without singularities.
- For $0.08 < \alpha < 0.1159$, we get spiral sink with singular points.
- For $0.1159 < \alpha < 0.117$, there is spiral source with singular points (see figure 6a).
- For $0.117 < \alpha \leq 1$, the trajectories form spiral source and they are smooth (see figure 6b).

3. Analysis

In this section, we consider linear system of integer and fractional differential equations in $\mathbb{R}^2$ and $\mathbb{R}^3$. First we solve the system with initial condition $X(0) = X_0$ to obtain the solution $X(t)$. Then we solve the same system with initial condition at $Y(0) = X(t_1)$ for some $t_1 > 0$ and call the new solution as $Y(t)$. We show that $Y(t) = TX(t)$, where T is some linear transformation.

Lemma 1. *Consider a planar system $\dot{X}(t) = AX(t)$. New trajectory $Y(t)$ starting at some point $X(t_1)$ on original trajectory $X(t)$ is given by the linear transformation*

$$Y(t) = TX(t), \tag{15}$$

where $T = \mathrm{e}^{At_1}$.

(i) *If A has real distinct eigenvalues, then T represents scaling (only).*
(ii) *If A has complex conjugate eigenvalues $a \pm ib$, then T represents both scaling and rotation.*

Proof. Solution of a system of ODEs $\dot{X}(t) = AX(t)$, $X(0) = X_0$ is given by

$$X(t) = \mathrm{e}^{At}X_0.$$

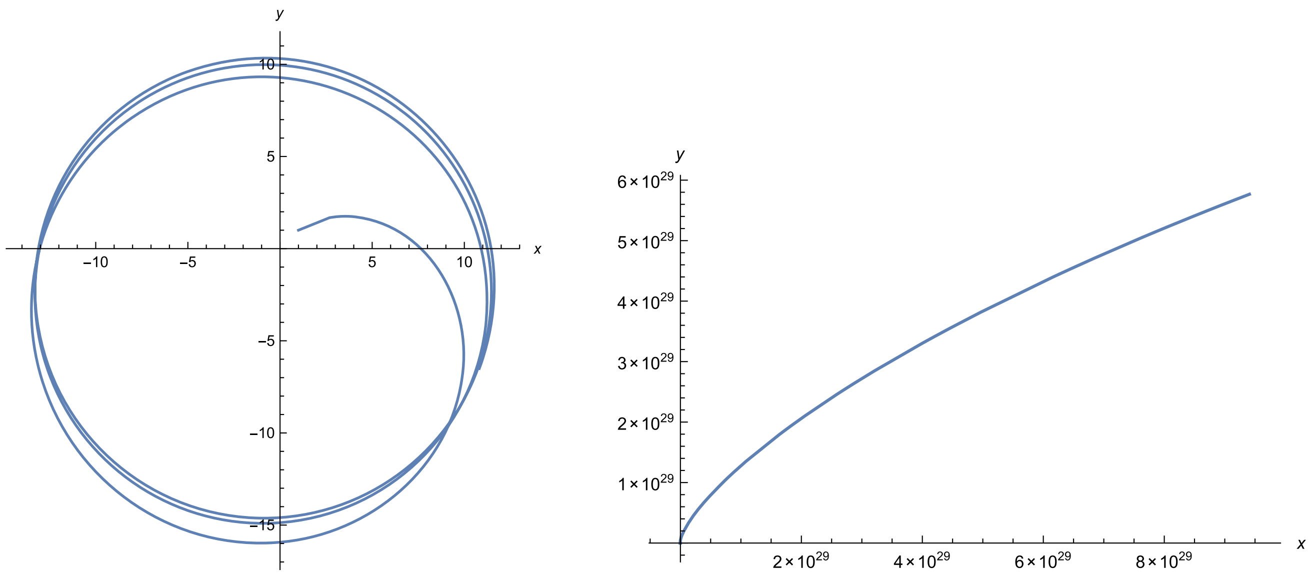
Now, let us consider the system $\dot{Y}(t) = AY(t)$, $Y(0) = X_1$, where $X_1 = X(t_1) = \mathrm{e}^{At_1}X_0$. Then its solution is given by $Y(t) = \mathrm{e}^{At}X_1 = \mathrm{e}^{At_1}\mathrm{e}^{At}X_0 = TX(t)$, where $T = \mathrm{e}^{At_1}$. The qualitative behaviour of the system $\dot{X}(t) = AX(t)$ does not change if we replace A by its canonical form.

(i) If

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

then

$$T = \mathrm{e}^{At_1} = \begin{bmatrix} \mathrm{e}^{\lambda_1 t_1} & 0 \\ 0 & \mathrm{e}^{\lambda_2 t_1} \end{bmatrix}.$$



(a) $\alpha = 0.116$. (b) $\alpha = 0.99$

Figure 6. Bifurcation in (14).

Here T represents scaling only. The type of scaling depends on the sign of λ_j, $j = 1, 2$.

(ii) If

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

then

$$\begin{aligned} T = e^{At_1} &= \begin{bmatrix} e^{at_1}\cos(bt_1) & e^{at_1}\sin(bt_1) \\ -e^{at_1}\sin(bt_1) & e^{at_1}\cos(bt_1) \end{bmatrix} \\ &= \begin{bmatrix} e^{at_1} & 0 \\ 0 & e^{at_1} \end{bmatrix} \begin{bmatrix} \cos(bt_1) & \sin(bt_1) \\ -\sin(bt_1) & \cos(bt_1) \end{bmatrix} \\ &= U \cdot V, \end{aligned}$$

where

$$U = \begin{bmatrix} e^{at_1} & 0 \\ 0 & e^{at_1} \end{bmatrix}$$

is the scaling matrix and

$$V = \begin{bmatrix} \cos(bt_1) & \sin(bt_1) \\ -\sin(bt_1) & \cos(bt_1) \end{bmatrix}$$

is the rotation matrix. □

Comment: Scaling factor depends on real part of the eigenvalue whereas imaginary part of the eigenvalue represents angle of rotation. The curves $X(t)$ and $U^{-1}Y(t)$ are congruent.

Example 1. Consider the two classical systems

$$\dot{X}(t) = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} X(t) \tag{16}$$

and

$$\dot{X}(t) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} X(t). \tag{17}$$

In figures 7a and 7b we sketch the solutions of systems (16) and (17) respectively with initial conditions $X(0) = [1, 1]^T$ (blue curve) and $Y(0) = X(1)$ (red curve). It can be checked that both the trajectories follow the same path.

Lemma 2. *Consider a planar system* ${}_0^C\mathrm{D}_t^\alpha X(t) = AX(t)$, $0 < \alpha < 1$. *New trajectory $Y(t)$ starting at some point $X(t_1)$ on the original trajectory $X(t)$ is given by the linear transformation*

$$Y(t) = TX(t), \tag{18}$$

where $T = E_\alpha(At_1^\alpha)$.

(i) *If A has real distinct eigenvalues, then T represents scaling (only).*
(ii) *If A has complex conjugate eigenvalues $a \pm ib$, then T represents both scaling and rotation.*

Proof. Solution of a system of FDEs

$${}_0^C\mathrm{D}_t^\alpha X(t) = AX(t), \; 0 < \alpha < 1, \; X(0) = X_0 \tag{19}$$

is given by

$$X(t) = E_\alpha(At^\alpha)X_0.$$

Now, let us consider the system

$${}_0^C\mathrm{D}_t^\alpha Y(t) = AY(t), \; 0 < \alpha < 1, \; Y(0) = X_1, \tag{20}$$

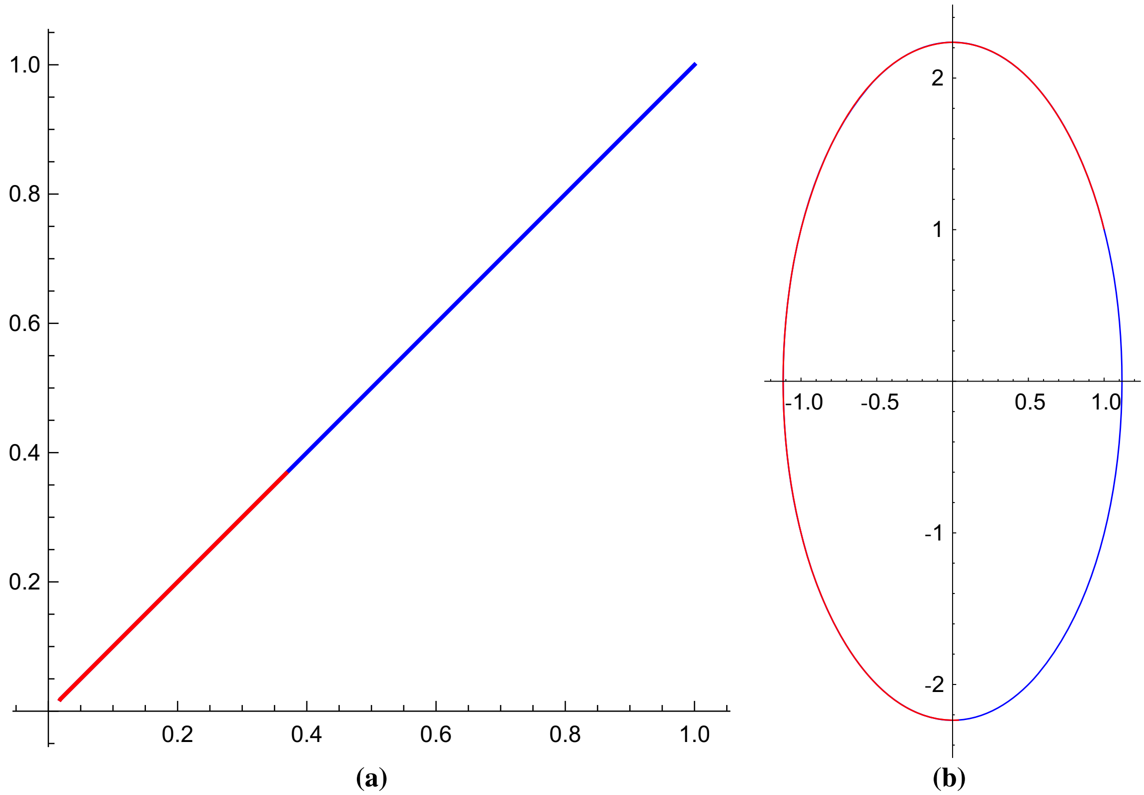


Figure 7. Solution trajectories of (**a**) system (16) and (**b**) system (17).

where $X_1 = X(t_1) = E_\alpha(At_1^\alpha)X_0$.

Then, its solution is given by

$$Y(t) = E_\alpha(At^\alpha)X_1 = E_\alpha(At_1^\alpha)E_\alpha(At^\alpha)X_0 = TX(t),$$

where $T = E_\alpha(At_1^\alpha)$.

As in Lemma 1, we assume that A is in canonical form.

(i) If

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

then

$$T = E_\alpha(At_1^\alpha) = \begin{bmatrix} E_\alpha(\lambda_1 t_1^\alpha) & 0 \\ 0 & E_\alpha(\lambda_2 t_1^\alpha) \end{bmatrix}.$$

Here T is a scaling matrix.

(ii) If

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

then

$$\begin{aligned} T &= E_\alpha(At_1^\alpha) \\ &= \begin{bmatrix} \mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)] & \mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)] \\ -\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)] & \mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)] \end{bmatrix} \\ &= \begin{bmatrix} |E_\alpha((a+ib)t_1^\alpha)| & 0 \\ 0 & |E_\alpha((a+ib)t_1^\alpha)| \end{bmatrix} \\ &\quad \begin{bmatrix} \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & \frac{\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} \\ -\frac{\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} \end{bmatrix} \\ &= U \cdot V, \end{aligned}$$

where

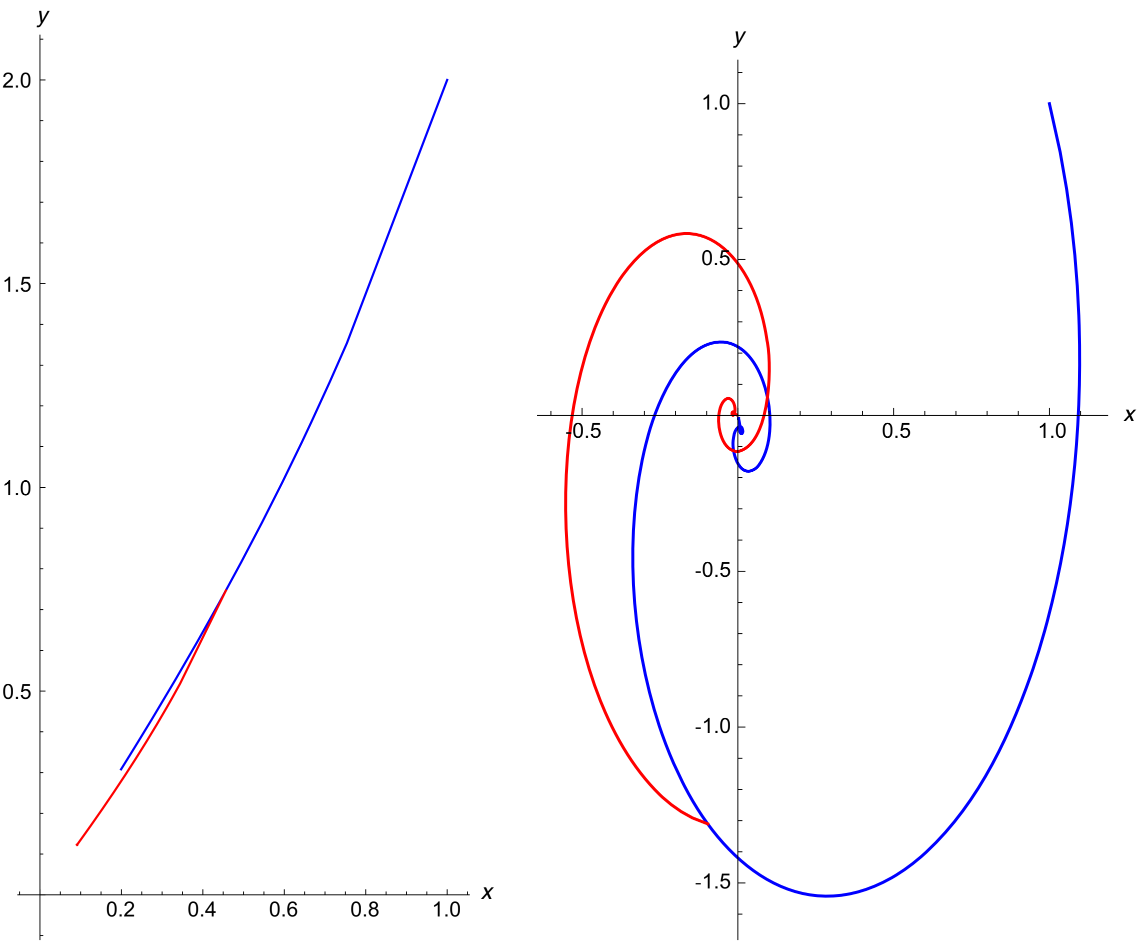
$$U = \begin{bmatrix} |E_\alpha((a+ib)t_1^\alpha)| & 0 \\ 0 & |E_\alpha((a+ib)t_1^\alpha)| \end{bmatrix}$$

is the scaling matrix and

$$V = \begin{bmatrix} \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & \frac{\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} \\ -\frac{\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & \frac{Re[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} \end{bmatrix}$$

is the rotation matrix. □

Comment: Unlike in the integer-order case, the scaling not only depends on a but also on b. The curves $X(t)$ and $U^{-1}Y(t)$ are congruent.



(a) **(b)**

Figure 8. Solution trajectories of (**a**) system (21) and (**b**) system (22).

Example 2. Consider the following systems

$$ {}_0^C\mathrm{D}_t^{0.3} X(t) = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} X(t) \tag{21}$$

and

$$ {}_0^C\mathrm{D}_t^{0.8} X(t) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} X(t). \tag{22}$$

In figures 8a and 8b, we have sketched the solution trajectories of systems (21) and (22) respectively with initial conditions $X(0) = [1, 1]^T$ (blue curve) and $Y(0) = X(1)$ (red curve). It can be seen that both the trajectories cannot follow the same path.

Theorem 3. *Consider a system $\dot{X}(t) = AX(t)$, where A is a 3×3 matrix. New trajectory $Y(t)$ starting at some point $X(t_1)$ on the original trajectory $X(t)$ is given by the linear transformation*

$$Y(t) = TX(t), \tag{23}$$

where $T = \mathrm{e}^{At_1}$.

(i) *If A has real distinct eigenvalues, then T represents scaling (only).*

(ii) *If A has complex conjugate eigenvalues $a \pm ib$ and a real eigenvalue λ, then T represents both scaling and rotation.*

Proof.

(i) If A is in the standard canonical form

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

then

$$T = \mathrm{e}^{At_1} = \begin{bmatrix} \mathrm{e}^{\lambda_1 t_1} & 0 & 0 \\ 0 & \mathrm{e}^{\lambda_2 t_1} & 0 \\ 0 & 0 & \mathrm{e}^{\lambda_3 t_1} \end{bmatrix}.$$

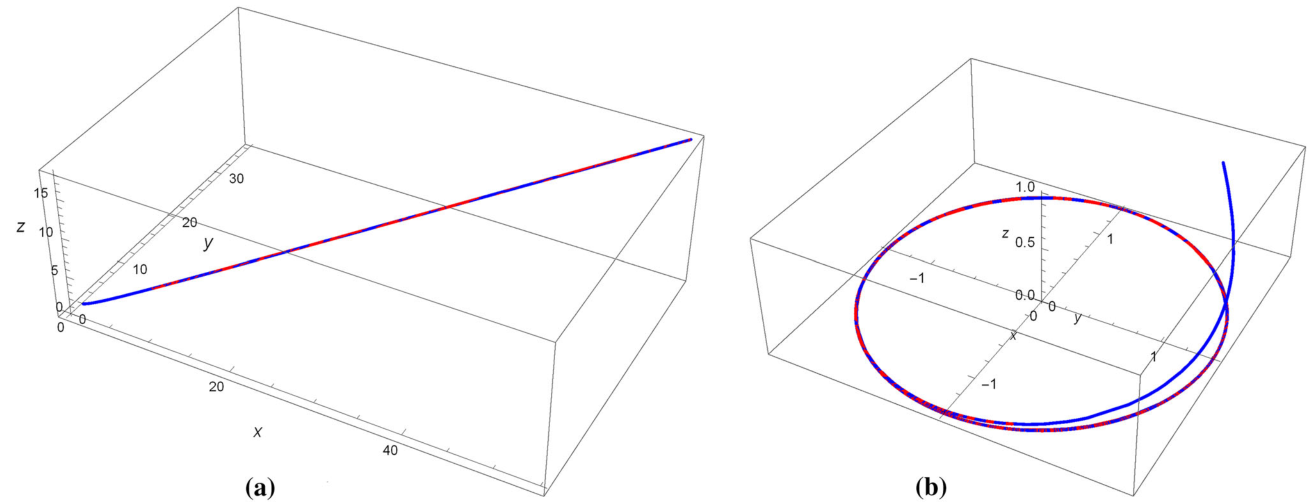


Figure 9. Solution trajectories of (**a**) system (24) and (**b**) system (25).

Here T represents scaling only.

(ii) If A is in the standard canonical form

$$A = \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

then A has eigenvalues $a \pm ib$, λ and

$$\begin{aligned} T = e^{At_1} &= \begin{bmatrix} e^{at_1}\cos(bt_1) & e^{at_1}\sin(bt_1) & 0 \\ -e^{at_1}\sin(bt_1) & e^{at_1}\cos(bt_1) & 0 \\ 0 & 0 & e^{\lambda t_1} \end{bmatrix} \\ &= \begin{bmatrix} e^{at_1} & 0 & 0 \\ 0 & e^{at_1} & 0 \\ 0 & 0 & e^{\lambda t_1} \end{bmatrix} \begin{bmatrix} \cos(bt_1) & \sin(bt_1) & 0 \\ -\sin(bt_1) & \cos(bt_1) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= U \cdot V \end{aligned}$$

where

$$U = \begin{bmatrix} e^{at_1} & 0 & 0 \\ 0 & e^{at_1} & 0 \\ 0 & 0 & e^{\lambda t_1} \end{bmatrix}$$

is the scaling matrix (uniform scaling by factor e^{at_1} of X, Y-coordinates and scaling of Z-coordinate by $e^{\lambda t_1}$) and

$$V = \begin{bmatrix} \cos(bt_1) & \sin(bt_1) & 0 \\ -\sin(bt_1) & \cos(bt_1) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the rotation matrix (rotation about Z-axis; angle of rotation is bt_1). The curves $X(t)$ and $U^{-1}Y(t)$ are congruent. □

Example 3. Consider the two classical systems

$$\dot{X}(t) = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{bmatrix} X(t) \tag{24}$$

and

$$\dot{X}(t) = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} X(t). \tag{25}$$

In figures 9a and 9b we sketch the solutions of systems (24) and (25) respectively with initial conditions $X(0) = [1, 1, 1]^T$ (blue curve) and $Y(0) = X(1)$ (red curve). It can be seen that both the trajectories follow the same path.

Theorem 4. *Consider a system* ${}_0^C\mathrm{D}_t^\alpha X(t) = AX(t), 0 < \alpha < 1$ *where A is a 3×3 matrix. New trajectory $Y(t)$ starting at some point $X(t_1)$ on original trajectory $X(t)$ is given by the linear transformation*

$$Y(t) = TX(t), \tag{26}$$

where $T = E_\alpha(At_1^\alpha)$.

(i) *If A has real distinct eigenvalues, then T represents scaling (only).*
(ii) *If A has complex conjugate eigenvalues $a \pm ib$ and a real eigenvalue λ, then T represents both scaling and rotation.*

Proof.

(i) If A is in the standard canonical form

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix},$$

then

$$T = E_\alpha(At_1^\alpha) = \begin{bmatrix} E_\alpha(\lambda_1 t_1^\alpha) & 0 & 0 \\ 0 & E_\alpha(\lambda_2 t_1^\alpha) & 0 \\ 0 & 0 & E_\alpha(\lambda_3 t_1^\alpha) \end{bmatrix}.$$

Here T represents scaling only.
(ii) If A is in the standard canonical form

$$A = \begin{bmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

then

$$T = e^{At_1} = \begin{bmatrix} \mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)] & \mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)] & 0 \\ -\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)] & \mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)] & 0 \\ 0 & 0 & E_\alpha(\lambda t_1^\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} |E_\alpha((a+ib)t_1^\alpha)| & 0 & 0 \\ 0 & |E_\alpha((a+ib)t_1^\alpha)| & 0 \\ 0 & 0 & E_\alpha(\lambda t_1^\alpha) \end{bmatrix} \begin{bmatrix} \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & \frac{\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & 0 \\ -\frac{\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & 0 \\ 0 & 0 & 1 \end{bmatrix} = U \cdot V$$

where

$$U = \begin{bmatrix} |E_\alpha((a+ib)t_1^\alpha)| & 0 & 0 \\ 0 & |E_\alpha((a+ib)t_1^\alpha)| & 0 \\ 0 & 0 & E_\alpha(\lambda t_1^\alpha) \end{bmatrix}$$

is the scaling matrix (uniform scaling by factor $|E_\alpha((a+ib)t_1^\alpha)|$ of X, Y-coordinates and scaling of Z-coordinate by $E_\alpha(\lambda t_1^\alpha)$) and

$$V = \begin{bmatrix} \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & \frac{\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & 0 \\ -\frac{\mathrm{Im}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the rotation matrix (rotation about the Z-axis with angle of rotation is

$$\theta = \cos^{-1}\left[\frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]}{|E_\alpha((a+ib)t_1^\alpha)|}\right].$$

The curves $X(t)$ and $U^{-1}Y(t)$ are congruent). □

Comments: New trajectories are transformed versions of original trajectories. In integer-order case, both trajectories follow the same path because $e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$. This is not the case with fractional-order systems because $E_\alpha(A(t_1+t_2)^\alpha) \neq E_\alpha(At_1^\alpha)E_\alpha(At_2^\alpha)$, in general.

Example 4. Consider the system,

$${}_0^C\mathrm{D}_t^\alpha X(t) = AX(t), \ 0 < \alpha < 1, \tag{27}$$

where

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -2 \\ 0 & 2 & -2 \end{bmatrix} \quad \text{and} \quad X(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Let $Y(t)$ be a solution of ${}_0^C\mathrm{D}_t^\alpha Y(t) = AY(t)$ with $Y(0) = X(t_1)$, $t_1 > 0$. In figure 10a, we sketch the solution trajectories $X(t)$ (blue curve) of system (27) subject to the initial condition $X(0) = [1, 1, 1]^T$ and $Y(t)$ (red curve) with initial condition $Y(0) = X(1)$.

Example 5. Repeat the same exercise as in Example 4, with

$$A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad X(0) = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \tag{28}$$

In figure 10b, we sketch the solution trajectories $X(t)$ (blue curve) of system (28) with initial condition $X(0) = [1, 1, 1]^T$ and $Y(t)$ (red curve) subject to the initial condition $Y(0) = X(1)$.

4. Differential geometry of trajectories of fractional-order systems

Frenet apparatus is a tool which is very useful to describe the shape of a curve. In this section we find Frenet apparatus for solution trajectories of FDEs

$${}_0^C\mathrm{D}_t^\alpha X(t) = AX(t), \ 0<\alpha<1, \ \text{with } X(0) = X_0 = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{29}$$

where A is in canonical form.
(1) Let

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

where $\lambda_1 \neq \lambda_2$ are real numbers. The Frenet apparatus of solution trajectory of ${}_0^C\mathrm{D}_t^\alpha X(t) = AX(t)$, $0 < \alpha < 1$, $X(0) = (c_1, c_2)^T$ (respectively ${}_0^C\mathrm{D}_t^\alpha Y(t) = AY(t)$, $0 < \alpha < 1$, $Y(0) = X(t_1)$, $t_1 > 0$) is T_1, N_1, κ_1 (respectively T_2, N_2, κ_2). If $\nu_1(t) = \sqrt{(\dot{x})^2 + (\dot{y})^2}$, is the speed of $X = (x, y)^T$ then

$$\nu_1(t) = \sqrt{c_1^2\left(\lambda_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_1 t^\alpha)\right)^2 + c_2^2\left(\lambda_2 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_2 t^\alpha)\right)^2}.$$

Similarly, speed of Y is given by

$$\nu_2(t) = \sqrt{c_1^2(E_\alpha(\lambda_1 t_1^\alpha))^2\left(\lambda_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_1 t^\alpha)\right)^2 + c_2^2(E_\alpha(\lambda_2 t_1^\alpha))^2\left(\lambda_2 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_2 t^\alpha)\right)^2}.$$

If

$$u_1 = \lambda_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_1 t^\alpha)\left[\frac{\mathrm{d}}{\mathrm{d}t}(\lambda_2 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_2 t^\alpha))\right] - \lambda_2 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_2 t^\alpha)\left[\frac{\mathrm{d}}{\mathrm{d}t}(\lambda_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_1 t^\alpha))\right]$$

then we have

$$\kappa_1 = \left|\frac{c_1 c_2 u_1}{\nu_1^3}\right| \quad \text{and} \quad \kappa_2 = \left|\frac{c_1 c_2 E_\alpha(\lambda_1 t_1^\alpha)E_\alpha(\lambda_2 t_1^\alpha)u_1}{\nu_2^3}\right|.$$

Therefore,

$$\kappa_2 = \left|\frac{E_\alpha(\lambda_1 t_1^\alpha)E_\alpha(\lambda_2 t_1^\alpha)\nu_1^3}{\nu_2^3}\right|\kappa_1.$$

The unit tangent vectors

$$T_1 = \frac{\left(c_1\lambda_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_1 t^\alpha), c_2\lambda_2 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_2 t^\alpha)\right)}{\nu_1(t)}$$

and

$$T_2 = \frac{\left(c_1 E_\alpha(\lambda_1 t_1^\alpha)\lambda_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_1 t^\alpha), c_2 E_\alpha(\lambda_2 t_1^\alpha)\lambda_2 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_2 t^\alpha)\right)}{\nu_2(t)}.$$

$$\therefore T_2 = \frac{\nu_1(t)}{\nu_2(t)}\begin{bmatrix} E_\alpha(\lambda_1 t_1^\alpha) & 0 \\ 0 & E_\alpha(\lambda_2 t_1^\alpha)\end{bmatrix} T_1.$$

Similarly, the unit normal vectors

$$N_1 = \frac{\left(-c_2\lambda_2 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_2 t^\alpha), c_1\lambda_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_1 t^\alpha)\right)}{\nu_1(t)}$$

and

$$N_2 = \frac{\left(-c_2 E_\alpha(\lambda_2 t_1^\alpha)\lambda_2 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_2 t^\alpha), c_1 E_\alpha(\lambda_1 t_1^\alpha)\lambda_1 t^{\alpha-1}E_{\alpha,\alpha}(\lambda_1 t^\alpha)\right)}{\nu_2(t)}.$$

$$\therefore N_2 = \frac{\nu_1(t)}{\nu_2(t)}\begin{bmatrix} E_\alpha(\lambda_2 t_1^\alpha) & 0 \\ 0 & E_\alpha(\lambda_1 t_1^\alpha)\end{bmatrix} N_1.$$

Note that, if $\lambda_1 = \lambda_2 = \lambda$ then $\nu_2(t) = E_\alpha(\lambda t_1^\alpha)\nu_1(t),\ T_2 = T_1,\ N_2 = N_1,\ \kappa_2 = \kappa_1 = 0.$

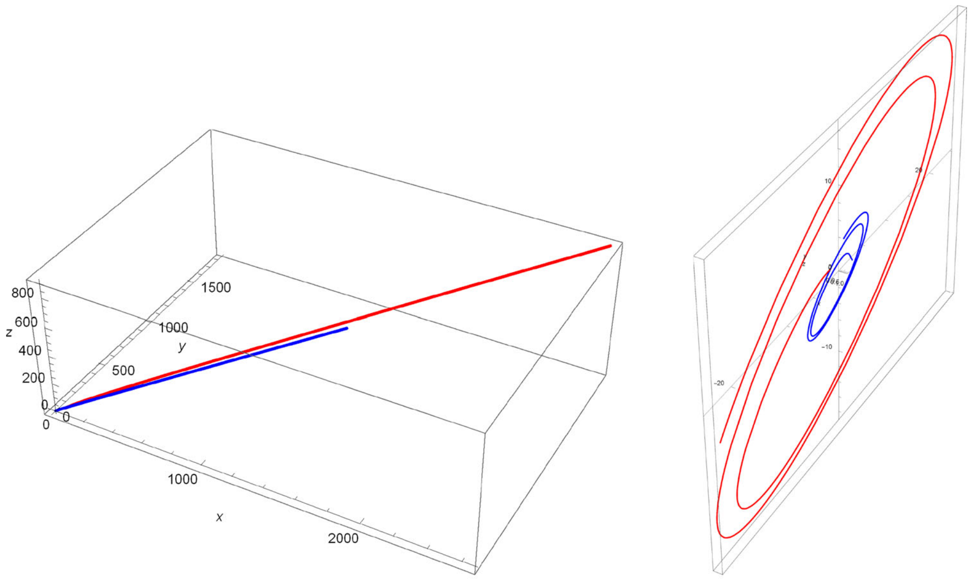
(2) Let

$$A = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$

(i) In this case, the general solution of the system is

$$X(t) = \begin{bmatrix} c_1 E_\alpha(\lambda t^\alpha) + c_2\frac{t^\alpha}{\alpha}E_{\alpha,\alpha}(\lambda t^\alpha) \\ c_2 E_\alpha(\lambda t^\alpha)\end{bmatrix}.$$

$$\nu_1(t) = \left((c_1^2 + c_2^2)\left(\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\right)^2 + \frac{c_2^2}{\alpha^2}\left(\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-1}}{\Gamma(\alpha n+\alpha)}\right)^2 + \frac{2c_1c_2}{\alpha}\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-1}}{\Gamma(\alpha n+\alpha)}\right)^{1/2}.$$



(a) α = 0.6 (b) α = 0.3

Figure 10. Trajectories of fractional-order systems (27) and (28).

If

$$u_2(t)=\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-1}}{\Gamma(\alpha n+\alpha)}\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-2}}{\Gamma(\alpha n-1)} -\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-2}}{\Gamma(\alpha n+\alpha-1)}\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)},$$

then

$$v_2(t)=\left((E_\alpha(\lambda t_1^\alpha))^2\left[(c_1^2+c_2^2)\left(\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\right)^2 +\frac{c_2^2}{\alpha^2}\left(\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-1}}{\Gamma(\alpha n+\alpha)}\right)^2 +\frac{2c_1c_2}{\alpha}\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\right.\right.$$

$$\kappa_1=\left|\frac{\frac{c_2^2}{\alpha}u_2(t)}{(v_1(t))^3}\right|,\ T_1=\frac{\left(c_1\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}+\frac{c_2}{\alpha}\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-1}}{\Gamma(\alpha n+\alpha)},\, c_2\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\right)}{v_1(t)}$$

and

$$N_1=\frac{\left(-c_2\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)},\, c_1\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}+\frac{c_2}{\alpha}\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-1}}{\Gamma(\alpha n+\alpha)}\right)}{v_1(t)}.$$

Similarly,

$$Y(t)=\begin{bmatrix}\left(c_1E_\alpha(\lambda t_1^\alpha)+c_2\frac{t_1^\alpha}{\alpha}E_{\alpha,\alpha}(\lambda t_1^\alpha)\right)E_\alpha(\lambda t^\alpha) +c_2\frac{t^\alpha}{\alpha}E_\alpha(\lambda t_1^\alpha)E_{\alpha,\alpha}(\lambda t^\alpha)\\ c_2E_\alpha(\lambda t_1^\alpha)E_\alpha(\lambda t^\alpha)\times\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-1}}{\Gamma(\alpha n+\alpha)}\end{bmatrix}$$

$$+\frac{t_1^\alpha}{\alpha}E_{\alpha,\alpha}(\lambda t_1^\alpha)E_\alpha(\lambda t_1^\alpha)$$

and

$$\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\left[2c_1c_2\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\right.$$

$$\left.+\frac{2c_2^2}{\alpha}\sum_{n=1}^{\infty}\frac{\lambda^n(\alpha n+\alpha)t^{\alpha n+\alpha-1}}{\Gamma(\alpha n+\alpha)}\right]$$

$$\left.+\frac{c_2^2t_1^{2\alpha}}{\alpha^2}(E_{\alpha,\alpha}(\lambda t_1^{\alpha}))^2\left(\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\right)^2\right)^{1/2}.$$

$$\therefore \kappa_2=\left|\frac{(\nu_1(t))^3}{(\nu_2(t))^3}E_{\alpha}(\lambda t_1^{\alpha})\right|\kappa_1,$$

$$T_2=\frac{\nu_1(t)E_{\alpha}(\lambda t_1^{\alpha})}{\nu_2(t)}T_1+\frac{t_1^{\alpha}E_{\alpha,\alpha}(\lambda t_1^{\alpha})}{\nu_2(t)}V,$$

where

$$V=\left(\frac{c_2}{\alpha}\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)},0\right)$$

and

$$N_2=\frac{\nu_1(t)E_{\alpha}(\lambda t_1^{\alpha})}{\nu_2(t)}N_1+\frac{t_1^{\alpha}E_{\alpha,\alpha}(\lambda t_1^{\alpha})}{\nu_2(t)}W,$$

where

$$W=\left(0,\frac{c_2}{\alpha}\sum_{n=1}^{\infty}\frac{\lambda^n t^{\alpha n-1}}{\Gamma(\alpha n)}\right).$$

(3) Let

$$A=\begin{bmatrix} a & b\\ -b & a\end{bmatrix}.$$

(i) In this case, the general solution of the system is,

$$X(t)=\begin{bmatrix} c_1\ \mathrm{Re}[E_{\alpha}((a+ib)t^{\alpha})]\\ +c_2\ \mathrm{Im}[E_{\alpha}((a+ib)t^{\alpha})]\\ -c_1\ \mathrm{Im}[E_{\alpha}((a+ib)t^{\alpha})]\\ +c_2\ \mathrm{Re}[E_{\alpha}((a+ib)t^{\alpha})]\end{bmatrix}.$$

Let

$$c_{\alpha}(a,b,t)=\sum_{n=1}^{\infty}\frac{(\mathrm{Re}(a+ib)^n)t^{\alpha n-1}}{\Gamma(\alpha n)},$$

$$s_{\alpha}(a,b,t)=\sum_{n=1}^{\infty}\frac{(\mathrm{Im}(a+ib)^n)t^{\alpha n-1}}{\Gamma(\alpha n)}$$

and

$$u_3(t)=(c_1^2+c_2^2)$$

$$\times\left(-c_{\alpha}(a,b,t)\sum_{n=1}^{\infty}\frac{(\mathrm{Im}(a+ib)^n)(\alpha n-1)t^{\alpha n-2}}{\Gamma(\alpha n)}\right.$$

$$\left.+s_{\alpha}(a,b,t)\sum_{n=1}^{\infty}\frac{(\mathrm{Re}(a+ib)^n)(\alpha n-1)t^{\alpha n-2}}{\Gamma(\alpha n)}\right).$$

$$\therefore \nu_1(t)=\sqrt{c_1^2+c_2^2}\left((c_{\alpha}(a,b,t))^2\right.$$

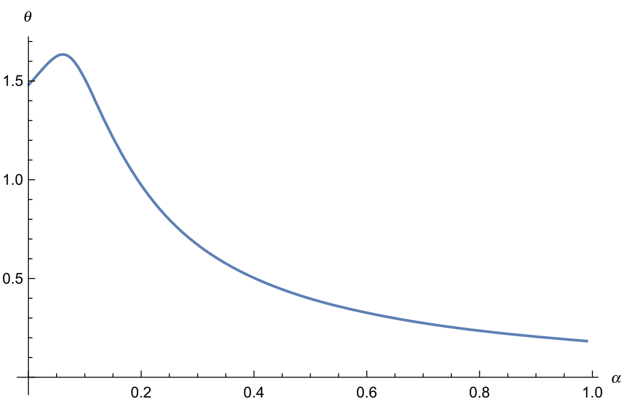


Figure 11. Angle of rotation θ vs. α.

$$\left.+(s_{\alpha}(a,b,t))^2\right)^{1/2},$$

$$\kappa_1=\left|\frac{u_3(t)}{(\nu_1(t))^3}\right|,$$

$$T_1=\frac{1}{\nu_1(t)}\,(c_1c_{\alpha}(a,b,t)+c_2s_{\alpha}(a,b,t),$$
$$-c_1s_{\alpha}(a,b,t)+c_2c_{\alpha}(a,b,t))$$

and

$$N_1=\frac{1}{\nu_1(t)}\,(c_1s_{\alpha}(a,b,t)$$
$$-c_2c_{\alpha}(a,b,t),\ c_1c_{\alpha}(a,b,t)$$
$$+c_2s_{\alpha}(a,b,t)).$$

Similarly,

$$Y(t)=\begin{bmatrix} p\ \ \mathrm{Re}[E_{\alpha}((a+ib)t^{\alpha})]\\ +q\ \mathrm{Im}[E_{\alpha}((a+ib)t^{\alpha})]\\ -p\ \mathrm{Im}[E_{\alpha}((a+ib)t^{\alpha})]\\ +q\ \mathrm{Re}[E_{\alpha}((a+ib)t^{\alpha})]\end{bmatrix}$$

where

$$p=c_1\mathrm{Re}[E_{\alpha}((a+ib)t_1^{\alpha})]+c_2\mathrm{Im}[E_{\alpha}((a+ib)t_1^{\alpha})]$$

and

$$q=-c_1\mathrm{Im}[E_{\alpha}((a+ib)t_1^{\alpha})]$$
$$+c_2\mathrm{Re}[E_{\alpha}((a+ib)t_1^{\alpha})].$$

$$\therefore \nu_2(t)=\sqrt{c_1^2+c_2^2}\ |E_{\alpha}((a+ib)t_1^{\alpha})|\ ((c_{\alpha}(a,b,t))^2$$
$$+(s_{\alpha}(a,b,t))^2)^{1/2}$$
$$=|E_{\alpha}((a+ib)t_1^{\alpha})|\,\nu_1(t),$$

$$\kappa_2=\frac{1}{|E_{\alpha}((a+ib)t_1^{\alpha})|}\kappa_1,$$

$$T_2=\frac{\mathrm{Re}[E_{\alpha}((a+ib)t_1^{\alpha})]\nu_1(t)}{\nu_2(t)}T_1$$
$$-\frac{\mathrm{Im}[E_{\alpha}((a+ib)t_1^{\alpha})]\nu_1(t)}{\nu_2(t)}N_1$$

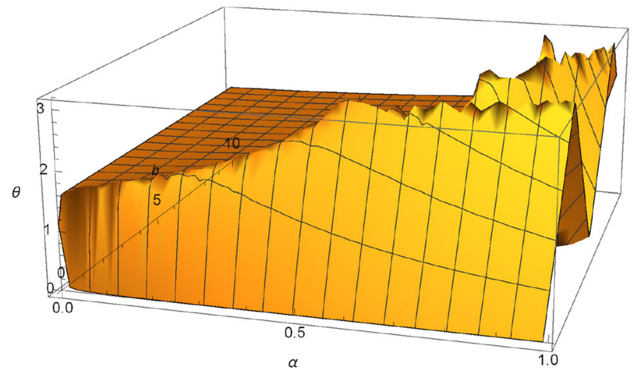


Figure 12. Surface (α, b, θ) for $0 < \alpha < 1$.

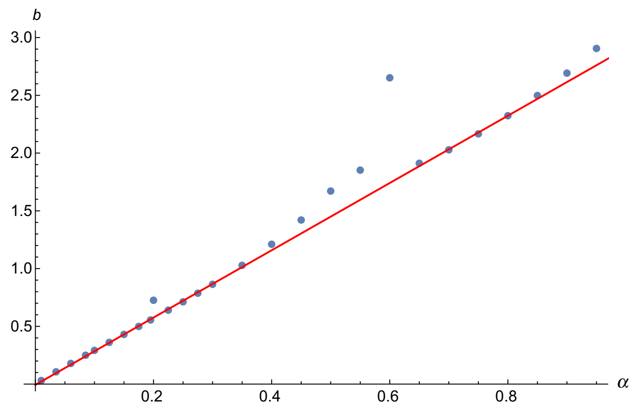


Figure 13. Dotted curve indicates $\max(\theta)$ and it is approximated by a straight line (30) (red curve).

and

$$N_2 = \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]\nu_1(t)}{\nu_2(t)} N_1 + \frac{\mathrm{Re}[E_\alpha((a+ib)t_1^\alpha)]\nu_1(t)}{\nu_2(t)} T_1.$$

Comment: Speed of the new curve is affected by the scaling factor.

5. Bifurcation analysis

Since the new trajectory $Y(t)$ starting at some point $X(t_1)$ on the original trajectory is a transformation of solution $X(t)$ of

$${}_0^C\mathrm{D}_t^\alpha X(t) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} X(t),\ 0 < \alpha < 1,$$

it is worth studying the effect of fractional order on such transformations.

(i) Fix $t_1 = 1$, $a = 0.983469$ and $b = 0.181075$. In the graph of θ, there is local maximum at $\alpha = 0.06144$ as shown in figure 11.

(ii) In figure 12 we fix $a = 1$ and sketch the surface (α, b, θ).

It is observed that angle of rotation θ is having maxima at some values of parameters b and α. In figure 13, we sketch a parametric curve of maximum (θ) for different values of α and b. It can be seen that most of the points of maximum (θ) lie on a straight line

$$b = -0.0066 + 2.9128\alpha. \tag{30}$$

Note: Throughout this manuscript we have chosen the order of fractional derivative $0 < \alpha < 1$. The reason for this choice is given below:

When the system contains a (fractional) derivative of order >1, then it can easily be reduced to an equivalent system containing all the derivatives with order in $(0, 1)$. The proper ways to split such higher-order system to a lower order are discussed in [37].

6. Conclusion

The systems of fractional differential equations are not the dynamical systems in a classical sense. The solution $\phi_t(X_0)$ of fractional-order initial value problem ${}_0^C\mathrm{D}_t^\alpha X = f(X)$, $X(0) = X_0$ does not satisfy the property $\phi_t \circ \phi_s = \phi_{t+s}$ of the flow of classical differential equation. However, the two trajectories $\phi_t(X_0)$ and $\phi_{t+s}(X_0)$ are closely related if we take $f(X) = AX$, a linear function.

In this article, we have shown that the new trajectory is a linear transformation of the original one. Further, we provided analysis of such trajectories with the help of Frenet apparatus.

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