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The solution of the Schrödinger equation for Makarov potential and homogeneous manifold $SL(2,\mathbb{C})/GL(1,\mathbb{C})$

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Abstract. In this study, we are going to obtain the energy spectrum and the corresponding solution of the non-central Makarov potential. In this case, we consider the arbitrary angular momentum with quantum number l. In order to calculate the energy spectrum and eigenfunction we use the factorisation method. The factorisation method leads us to discuss the shape-invariance condition with respect to any index as n and m. Here, we also achieve the shape invariance with respect to the main quantum number n. It leads to the quantum-solvable models on real forms of the homogeneous manifold $SL(2,\mathbb{C})/GL(1,\mathbb{C})$ with infinite-fold degeneracy for $\gamma_v = 0$ and $\gamma_v \neq 0$. These processes also help us to obtain raising and lowering operators of states on the above-mentioned homogeneous manifold.

Keywords. Non-central potential; shape-invariance condition; homogeneous manifold $SL(2,\mathbb{C})/GL(1,\mathbb{C})$; hypergeometric and Laguerre equation.

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1. Introduction

In recent years, there has been a growing interest in the field of differential equations [1–4]. As we know, the Schrödinger equation for Makarov potential can be solved by several methods. Here, this potential in the spherical coordinates is defined as [5]

$$V(r,\theta) = \frac{\alpha_v}{r} + \frac{\beta_v}{r^2\sin^2\theta} + \frac{\gamma_v\cos\theta}{r^2\sin^2\theta}, \qquad (1)$$

where the first term represents the Coulomb potential, the second and the third terms are the short-range ring-shaped terms. This potential can be used to describe ring-shaped molecules such as benzene and interactions between deformed pairs of nuclei where $\alpha_v > 0$. The potential has been investigated by using asymptotic iteration method (AIM) [6] and Nikiforov–Uvarov method [7–9]. AIM also offers an efficient and accurate investigation in non-relativistic and relativistic quantum mechanics. Also, the energy eigenvalues of the Makarov potential using supersymmetry and shape invariance have been investigated using several techniques [10,11]. Furthermore, three types of ring-shaped potentials have been studied by using the factorisation method and ladder operators [12–16]

All the above information gives us motivation to arrange our paper as follows: In §2, we introduce the Schrödinger equation for the Makarov potential. We separate this equation in terms of radial and angular momentum parts respectively and we compare the radial part of the equation with the associated Laguerre equation and obtain the radial part wave function. This comparison helps us to calculate the energy spectrum and also arrange α and β with the corresponding case. In §3, we introduce the angular part of the equation corresponding to θ and by using different variables for the angular part of the equation, we conclude with two cases: $\gamma_v \neq 0$ and $\gamma_v = 0$. Then, we compare these equations with the corresponding associated hypergeometric equation and obtain the angular-part wave function. Finally, we took advantage of the hypergeometric equation and achieved the first-order operators and the quantum-solvable models on $SL(2,\mathbb{C})/GL(1,\mathbb{C})$.

2. Separating variables in the Schrödinger equation

The Schrödinger equation for a particle in general non-central potential V with the mass μ can be written as [6]:

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$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{2\mu}{\hbar^2}\left(E-V(r,\theta)\right)\right]=\psi(r,\theta,\varphi), \tag{2}$$

where the potential $V(r,\theta)$ takes the Makarov form as in eq. (1). The wave function can be written as

$$\psi(r,\theta,\varphi)=\frac{1}{r}R(r)Y(\theta,\varphi).$$

Then by selecting

$$Y(\theta,\varphi)=\frac{1}{\sin^{1/2}\theta}H(\theta)\Phi(\varphi),$$

eq. (2) can be separated into variables and the following equations are obtained:

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+\left(\frac{2\mu}{\hbar^2}(E-V(r))-\frac{l(l+1)}{r^2}\right)R(r)=0, \tag{3}$$

$$\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2}-\left(\frac{2\mu}{\hbar^2}V(\theta)+\frac{m'^2}{\sin^2\theta}-\frac{1}{2}-\frac{1}{4}\frac{\cos^2\theta}{\sin^2\theta}-l(l+1)\right)H(\theta)=0, \tag{4}$$

$$\frac{\mathrm{d}^2\Phi(\varphi)}{\mathrm{d}\varphi^2}=-m'^2\Phi(\varphi). \tag{5}$$

In order to have definite bound states, we have to apply some boundary conditions. For example, we may have $R(0)=0$ and $R(\infty)=0$ in (3). Also $H(0)$ and $H(\pi)$ must be finite and $\Phi(\varphi)=\Phi(\varphi+2\pi)$ in (5). The solution of eq. (5) is well-known as

$$\Phi_{m'}(\varphi)=\frac{1}{\sqrt{2\pi}}\mathrm{e}^{im'\varphi},\quad m'=0,\pm1,\pm2,\ldots. \tag{6}$$

Only eqs (3) and (4) should be solved by factorisation method. The radial Schrödinger equation for the non-central Coulomb potential is rearranged as

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+\left(-\epsilon^2+\frac{\tilde{\alpha}}{r}-\frac{l(l+1)}{r^2}\right)R(r)=0, \tag{7}$$

where

$$-\epsilon^2=\frac{2\mu E}{\hbar^2},\quad \tilde{\alpha}=-\frac{2\mu\alpha_v}{\hbar^2}, \tag{8}$$

and by defining $l=\Lambda-\frac{1}{2}$ and $l(l+1)=\Lambda^2-\frac{1}{4}$, eq. (5) can be written as

$$\frac{\mathrm{d}^2H(\theta)}{\mathrm{d}\theta^2}+\left(\Lambda^2-\frac{\kappa^2+\eta^2-\frac{1}{4}}{\sin^2\theta}-\frac{2\kappa\eta\cos\theta}{\sin^2\theta}\right)\times H(\theta)=0, \tag{9}$$

where

$$\kappa^2+\eta^2=\frac{2\mu\beta_v}{\hbar^2}+m'^2,$$
$$2\kappa\eta=\frac{2\mu\gamma_v}{\hbar^2}. \tag{10}$$

Here, we note that the energy spectrum can be obtained by the radial part of the equation. In order to solve eq. (7), we take $R(r)=U(r)L(r)$. In such an approach, $L(r)$ is the associated Laguerre differential equation [17–20].

So, we compare eq. (7) as a form of $R(r)=U(r)L(r)$ with the following associated Laguerre differential equation:

$$rL''^{(\alpha,\beta)}_{n,m}+(1+\alpha-\beta r)L'^{(\alpha,\beta)}_{n,m}+\left[\left(n-\frac{m}{2}\right)\beta-\frac{m}{2}\left(\alpha+\frac{m}{2}\right)\right]L^{(\alpha,\beta)}_{n,m}=0. \tag{11}$$

One can obtain the energy spectrum as

$$E=-\frac{\beta_R^2\hbar^2}{8\mu}. \tag{12}$$

Also, the wave function for the radial part will be

$$U(r)=N_0\mathrm{e}^{-\beta_R r/2}r^{(1+\alpha_R)/2}, \tag{13}$$

where

$$\alpha_R=-m+\sqrt{1+4l(l+1)}, \tag{14}$$

$$\beta_R=\frac{2\tilde{\alpha}}{2n-m+\alpha_R+1}. \tag{15}$$

Finally, for cases $\gamma_v=0$ and $\gamma_v\neq0$ we shall obtain the corresponding α_R and β_R. Therefore, from eq. (13) the corresponding eigenfunction will be

$$R(r)=N_0\mathrm{e}^{-\beta_R r/2}r^{(1+\alpha_R)/2}L^{(\alpha_R,\beta_R)}_{n,m}(r),$$

where $L^{(\alpha,\beta)}_{n,m}(r)$ is the associated Laguerre functions that has the following Rodrigues representation:

$$L^{(\alpha,\beta)}_{n,m}=\frac{a_{n,m}(\alpha,\beta)}{r^{\alpha+(m/2)}\mathrm{e}^{-\beta r}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}(r^{\alpha+n}\mathrm{e}^{-\beta r}),$$

where $a_{n,m}(\alpha,\beta)$ is the normalisation coefficient. Now we are going to investigate the angular part of the equation. Here we consider two cases, the first one is $\gamma_v=0$ and the second one is $\gamma_v\neq0$.

3. Angular part of the Schrödinger equation

3.1 *Case* 1. $\gamma_v=0$

We investigate the angular part of the equation by introducing a new variable $x=-\cot\theta$ and rewriting eq. (9) as follows:

$$(1+x^2)\frac{d^2H}{dx^2}+2x\frac{dH}{dx}+\left(\frac{\Lambda^2}{1+x^2}-\left(\kappa^2+\eta^2-\frac{1}{4}\right)-2\kappa\eta\sqrt{\frac{x^2}{1+x^2}}\right)H=0. \quad (16)$$

With the choice of the variable $H(x)=U(x)J(x)$, we rewrite eq. (16) as follows:

$$(1+x^2)J''(x)+\left(2(1+x^2)\frac{U'(x)}{U(x)}+2x\right)J'(x)+\left((1+x^2)\frac{U''(x)}{U(x)}+2x\frac{U'(x)}{U(x)}+\frac{\Lambda^2}{1+x^2}-\left(\kappa^2+\eta^2-\frac{1}{4}\right)-2\kappa\eta\sqrt{\frac{x^2}{1+x^2}}\right)J(x)=0. \quad (17)$$

In order to obtain the function U, we compare eq. (17) with the following associated hypergeometric differential equation:

$$(1+x^2)J''^{\alpha,\beta}_{n,m}(x)+[\beta+2(\alpha+l)x]\,J'^{\alpha,\beta}_{n,m}(x)-\left[n(n+2\alpha+1)-m(m+2\alpha)+m(\beta+(m+2\alpha)x)\frac{x}{1+x^2}\right]J^{\alpha,\beta}_{n,m}(x)=0, \quad (18)$$

to obtain the eigenfunction $U(x)$,

$$U(x)=N_0(1+x^2)^{\alpha_{1,H}/2}e^{\beta_{1,H}/2\arctan x},$$

where

$$\beta_{1,H}=0,\quad \alpha_{1,H}=\frac{-1-2m+\sqrt{1+4m}}{2} \quad (19)$$

and

$$l(l+1)=\frac{1}{4}-\alpha_{1,H}. \quad (20)$$

As $2\kappa\eta=(2\mu\gamma_v/\hbar^2)=0$, we conclude γ_v will be zero in eq. (10). Also, we can obtain the corresponding eigenfunction and the raising and lowering operators, respectively, as

$$H(\theta)=\frac{N_0}{(\sin\theta)^{\alpha_{1,H}}}J^{\alpha_{1,H},0}_{n,m}(\theta),$$
$$A_+(n)=\frac{d}{d\theta}-(n+1)\alpha_{1,H}\cot\theta,$$
$$A_-(n)=-\frac{d}{d\theta}-(n+1)\alpha_{1,H}\cot\theta. \quad (21)$$

Therefore, the associated hypergeometric differential equation with respect to the parameter n can be factorised as follows:

$$A_+(n)A_-(n)J^{\alpha_{1,H},0}_{n,m}(\theta)=E_1(n,m)J^{\alpha_{1,H},0}_{n,m}(\theta),$$
$$A_-(n)A_+(n)J^{\alpha_{1,H},0}_{n-1,m}(\theta)=E_1(n,m)J^{\alpha_{1,H},0}_{n-1,m}(\theta), \quad (22)$$

where

$$J^{\alpha_{1,H},0}_{n,m}(\theta)=\left[\frac{a_{n,m}}{(1+x^2)^{(m/2)+\alpha_{1,H}}}\left(\frac{d}{dx}\right)^{n-m}\times\left((1+x^2)^{n+\alpha_{1,H}}\right)\right]_{x=-\cot\theta} \quad (23)$$

and

$$E_1(n,m)=-(n-m)(n+m+2\alpha_{1,H}) \quad (24)$$

in which $a_{n,m}$ is the normalisation coefficient. In the next step, we shall try to show the corresponding model as a form of some quantum-solvable model. Here, we want to show that the above results will be in the form of some quantum model.

Now, these results will be used to obtain the quantum-solvable models on $SL(2,\mathbb{C})/GL(1,\mathbb{C})$. In this case, if we look at eqs (21) and (22), one can obtain the L_+, L_-, L_3 and I generators on the homogeneous manifold $SL(2,\mathbb{C})/GL(1,\mathbb{C})$. So, by using two functions in eq. (18) and ref. [21] one can obtain the generators as

$$L_+=e^{i\tilde{\phi}}\left[\frac{\partial}{\partial\theta}+\frac{i}{\tan\theta}\frac{\partial}{\partial\tilde{\phi}}-\frac{1}{\tan\theta}-\alpha_{1,H}\cot\theta\right],$$
$$L_-=e^{-i\tilde{\phi}}\left[-\frac{\partial}{\partial\theta}+\frac{i}{\tan\theta}\frac{\partial}{\partial\tilde{\phi}}-\alpha_{1,H}\cot\theta\right],$$
$$L_3=-i\frac{\partial}{\partial\tilde{\phi}},\quad I=1$$

and the quantum states $Z_{n,m}(\theta,\tilde{\phi})$ as

$$Z_{n,m}(\theta,\tilde{\phi})=e^{in\tilde{\phi}}J^{\alpha_{1,H},0}_{n,m}(\theta),\quad 0\le\tilde{\phi}<2\pi. \quad (25)$$

If we look at the generators, we see that they are satisfied by the commutation relations of Lie algebra $\mathbf{gl}(2,\mathbb{C})$. Also we note that the generators can factorise the corresponding quantum states which are given by

$$L_+L_-Z_{n,m}(\theta,\tilde{\phi})=E_1(n,m)Z_{n,m}(\theta,\tilde{\phi}),$$
$$L_-L_+Z_{n-1,m}(\theta,\tilde{\phi})=E_1(n,m)Z_{n-1,m}(\theta,\tilde{\phi}). \quad (26)$$

3.2 *Case* 2. $\gamma_v\neq 0$

In this case, by defining a new variable $x=\frac{1}{2}(1-\cos\theta)$ ($\theta\in[0,2\pi)$, $x\in[0,1]$), and choosing the function $H(x)=U(x)F(x)$, eq. (9) will be as follows:

$$x(1-x)\frac{d^2F}{dx^2}+\left(2x(1-x)\frac{U'}{U}+\left(\frac{1}{2}-x\right)\right)\frac{dF}{dx}+\left(x(1-x)\frac{U''(x)}{U(x)}\right.$$

$$+\left(\frac{1}{2}-x\right)\frac{U'(x)}{U(x)}+\Lambda^2 - \frac{(\kappa+\eta)^2-\frac{1}{4}-4\kappa\eta x}{4x(1-x)}\Bigg)F(x)=0. \quad (27)$$

In order to obtain eigenfunction and eigenvalue for the Makarov potential, eq. (27) with the following associated hypergeometric differential equation are compared:

$$x(1-x)F''^{\alpha,\beta}_{n,m}(x)+\Big[\alpha+1-(\alpha+\beta+2)x\Big] \times F'^{\alpha,\beta}_{n,m}(x)+\Big[n(n+\alpha+\beta+1)-m(m+\alpha+\beta) -\frac{1}{2}m\frac{(\alpha+\frac{1}{2})-(m+\alpha+\beta)x}{x(1-x)}\Big]F^{\alpha,\beta}_{n,m}(x)=0 \quad (28)$$

and the eigenfunction $U(x)$ is obtained as

$$U(x)=\tilde{N}_0 x^{\frac{2\alpha_{2,H}+1}{4}}(1-x)^{\frac{2\beta_{2,H}+1}{4}},$$

where

$$\alpha_{2,H}=-m\pm(\kappa+\eta),$$
$$\beta_{2,H}=-1\pm(\kappa+\eta+2\sqrt{m}) \quad (29)$$

and

$$l(l+1)=\frac{1}{4}+n(n-m\pm2\sqrt{m}) -m(\pm2\sqrt{m}-1). \quad (30)$$

So, finally one can rewrite the corresponding wave function as

$$H(\theta)=\tilde{N}_0\frac{\sqrt{\sin\theta}}{2^{\frac{\alpha_{2,H}+\beta_{2,H}+1}{2}}}(1-\cos\theta)^{\alpha_{2,H}/2} \times(1+\cos\theta)^{\beta_{2,H}/2}F^{\alpha_{2,H},\beta_{2,H}}_{n,m}(\theta). \quad (31)$$

We note that the solution associated with the hypergeometric equation in the Rodrigues representation is

$$F^{\alpha,\beta}_{n,m}(\theta)=\left[\frac{a_{n,m}}{x^{(2m-1+2\alpha)/4}(1-x)^{(2m-1+2\beta)/4}} \times\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}\left(x^{n+\alpha}(1-x)^{n+\beta}\right)\right]_{x=\frac{1}{2}(1-\cos\theta)}, \quad (32)$$

in which $a_{n,m}$ is the normalisation coefficient. Now, we obtain the following shape invariance with respect to the main quantum number m:

$$A_+(m)A_-(m)F^{\alpha_{2,H},\beta_{2,H}}_{n,m}(\theta)=E_2(n,m)F^{\alpha_{2,H},\beta_{2,H}}_{n,m}(\theta),$$
$$A_-(m)A_+(m)F^{\alpha_{2,H},\beta_{2,H}}_{n,m-1}(\theta) = E_2(n,m)F^{\alpha_{2,H},\beta_{2,H}}_{n,m-1}(\theta), \quad (33)$$

where

$$A_+(m)=\frac{\mathrm{d}}{\mathrm{d}\theta}+W_m(\theta),$$
$$A_-(m)=-\frac{\mathrm{d}}{\mathrm{d}\theta}+W_m(\theta) \quad (34)$$

and the superpotential $W_m(\theta)$ and the spectrum $E_2(n,m)$ have been introduced as follows [21]:

$$E_2(n,m)=(n-m+1)\left(n+m+\alpha_{2,H}+\beta_{2,H}\right),$$
$$W_m(\theta)=\frac{1}{2\sin\theta}\Big(\left(2m+\alpha_{2,H}+\beta_{2,H}-1\right)\cos\theta -4m+2-3\alpha_{2,H}-\beta_{2,H}\Big). \quad (35)$$

Also, we can say that the corresponding raising and lowering operators A_+ and A_- are the mathematical operators. In order to obtain the quantum-solvable model, we have to change the corresponding operators as a form of J_+ and J_- which are given by

$$J_+(m)=\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\tilde{\phi}} -\frac{2(m+(\alpha_{2,H}/2))+\beta_{2,H}-1}{2\tan\theta} +\frac{\alpha_{2,H}+\beta_{2,H}}{2\sin\theta},$$
$$J_-(m)=-\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\tilde{\phi}} -\frac{2(m+(\alpha_{2,H}/2))+\beta_{2,H}-1}{2\tan\theta} +\frac{\alpha_{2,H}+\beta_{2,H}}{2\sin\theta}, \quad (36)$$

such that

$$J_+(m)J_-(m)\tilde{Z}_{n,-(\alpha_{2,H}/2),m}(\theta,\tilde{\phi},0) = E_2(n,m)\tilde{Z}_{n,-(\alpha_{2,H}/2),m}(\theta,\tilde{\phi},0),$$
$$J_-(m)J_+(m)\tilde{Z}_{n,-(\alpha_{2,H}/2),m-1}(\theta,\tilde{\phi},0) = E_2(n,m)\tilde{Z}_{n,-(\alpha_{2,H}/2),m-1}(\theta,\tilde{\phi},0). \quad (37)$$

Thus, operators $J_\pm(m)$ describe the shape-invariance symmetry on the homogeneous manifold $SL(2,\mathbb{C})/GL(1,\mathbb{C})$ [21] and

$$\tilde{Z}_{n,-(\alpha_{2,H}/2),m}(\theta,\tilde{\phi},0)=\mathrm{e}^{-i\alpha_{2,H}\tilde{\phi}}F^{\alpha_{2,H},\beta_{2,H}}_{n,m}(\theta). \quad (38)$$

4. Conclusion

Generally, one can say that the non-central Makarov potential can be of the form of the second-order

Schrödinger equation. The corresponding equation factorised in the first-order operators is the raising and lowering operators. These operators and shape-invariance conditions lead us to discuss the algebra point of view for the Makarov system. Also, we have shown that such systems have the same particle moving in homogeneous manifold $SL(2, \mathbb{C})/GL(1, \mathbb{C})$.

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