Pramana – J. Phys. (2020) 94:55
https://doi.org/10.1007/s12043-020-1920-2

Analysis of the evolution equation of a hyperbolic curve flow via Lie symmetry method

BEN GAO* and ZHANG SHI

College of Mathematics, Taiyuan University of Technology, Taiyuan 030024, People's Republic of China
*Corresponding author. E-mail: benzi0116@163.com

MS received 25 August 2019; revised 20 November 2019; accepted 12 December 2019

Abstract. In this paper, based on the classical symmetry method, the group-invariant solutions of the evolution equation of a hyperbolic curve flow are investigated. The optimal system of the obtained symmetries is found, and the reduced equations and exact solutions of the evolution equation are discussed. Then explicit solutions are obtained by the power series method. In addition, the convergence of the power series solutions is proved. The objective shapes of the solutions of the evolution equation are performed.

Keywords. Evolution equation of a hyperbolic curve flow; Lie group of symmetry; optimal system; power series solutions.

PACS Nos **02.30.Jr; 11.30.−j; 05.45.Yv; 04.20.Jb**

1. Introduction

In this research article, we consider the following fully nonlinear, hyperbolic equation which was introduced as the evolution equation of a hyperbolic curve flow [1–3]:

$$u_{tt}u_{xx} - u_{xt}^2 = \frac{u_{xx}^2}{1+u_x^2}, \tag{1}$$

where $u = u(x, t)$ is the unknown real function and the subscripts denote the partial derivatives. Equation (1) has significant applications in physics, image enhancement and computer vision [4,5]. It has arisen worldwide concerns, the finite time blow-up was provided in [3], the local solvability under certain conditions and asymptotic behaviour of the solutions were obtained in [1].

As we know, Lie symmetry method has played a significant role in the research on the properties of solutions of the curve flow [6–8]. The basic view of the symmetry method is to look for group invariance and do some reductions to differential equations [9–14]. Once symmetry group of differential equations has been obtained, one can construct large classes of the corresponding special solutions. In order to classify all the reduced equations, we need an optimal system of one-dimensional subalgebra of Lie algebra established by Lie group analysis [9]. Using symmetry method, we shall construct the optimal system of (1), from which interesting exact solutions are obtained.

The outline of this paper is as follows. In §2, symmetries and an optimal system of one-dimensional subalgebra of eq. (1) are discussed. Section 3 concentrates on the reduced equations and some exact solutions. In §4, the explicit solutions of the reduced equations are established by means of the power series method, the convergence of the power series solutions is proved, and some objective shapes of the solutions are performed. The last section gives the conclusion.

2. Lie point symmetry

In this section, we get infinitesimal generators of eq. (1) by using Lie's theory of symmetries, then we obtain the commutator of Lie algebra.

First, let us consider a Lie algebra of infinitesimal symmetries of eq. (1) of the form

$$X = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \eta(x,t,u)\partial_u. \tag{2}$$

According to the invariance conditions for eq. (1) with respect to the transformation (2), we have [9,15]

$$\mathrm{pr}^{(2)} X(\Delta)|_{\Delta=0} = 0,$$

where $\mathrm{pr}^{(2)} X$ is the second-order prolongation of X [9,15] and $\Delta = (u_{tt}u_{xx} - u_{xt}^2)(1 + u_x^2) - u_{xx}^2$, on this condition,

Published online: 12 March 2020

Table 1. Table of Lie brackets.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	$-X_2$	0	0	$-X_5$	0	$-X_7$
X_2	X_2	0	X_5	0	0	X_7	0
X_3	0	$-X_5$	0	$-X_6$	0	0	0
X_4	0	0	X_6	0	X_7	X_3	$-X_5$
X_5	X_5	0	0	$-X_7$	0	0	0
X_6	0	$-X_7$	0	$-X_3$	0	0	0
X_7	X_7	0	0	X_5	0	0	0

$$\mathrm{pr}^{(2)}X = X + \eta_x^{(1)}\frac{\partial}{\partial u_x} + \eta_{xx}^{(2)}\frac{\partial}{\partial u_{xx}} + \eta_{xt}^{(2)}\frac{\partial}{\partial u_{xt}} + \eta_{tt}^{(2)}\frac{\partial}{\partial u_{tt}},$$

where

$$\eta_x^{(1)} = D_x(\eta) - u_x D_x(\xi) - u_t D_x(\tau),$$

$$\eta_{xt}^{(2)} = D_x D_t(\eta - \xi u_x - \tau u_t) + \xi u_{xxt} + \tau u_{xtt},$$

$$\eta_{xx}^{(2)} = D_x^2(\eta - \xi u_x - \tau u_t) + \xi u_{xxx} + \tau u_{xxt},$$

$$\eta_{tt}^{(2)} = D_t^2(\eta - \xi u_x - \tau u_t) + \xi u_{xtt} + \tau u_{ttt},$$

and D_x, D_t stand for the operators of the total differentiation. For instance,

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{tt}\frac{\partial}{\partial u_t} + \cdots .$$

Next, we get a system of over-determined linear equations of ξ, τ and η, as

$$\xi_x = \tau_t, \quad \xi_{tt} = \xi_{tu} = \xi_{uu} = 0,$$

$$\tau_u = \tau_x = \tau_{tt} = 0,$$

$$\eta_u = \tau_t, \quad \eta_x = -\xi_u, \ \eta_{tt} = 0.$$

Solving these equations, one can get

$$\xi = c_1 x + c_3 t + c_4 u + c_5,$$

$$\tau = c_1 t + c_2, \quad \eta = c_1 u - c_4 x + c_6 t + c_7,$$

where $c_1, c_2, c_3, c_4, c_5, c_6$ and c_7 are real constants. Therefore, Lie algebra L_7 of the transformations of eq. (1) is spanned by the following generators:

$$X_1 = x\partial_x + t\partial_t + u\partial_u, \quad X_2 = \partial_t,$$

$$X_3 = t\partial_x, \quad X_4 = u\partial_x - x\partial_u, \quad X_5 = \partial_x,$$

$$X_6 = t\partial_u, \ X_7 = \partial_u.$$

In order to classify all the group-invariant solutions, we need an optimal system of one-dimensional subalgebras. In this section, the optimal system of subgroups for eq. (1) is constructed by only using the commutator table [16]. First, using the commutator $[X_m, X_n] = X_m X_n - X_n X_m$, we attained the commutation relations of X_1, X_2, X_3, X_4, X_5, X_6 and X_7 listed in table 1.

An arbitrary operator $X \in L_7$ is given as

$$X = l_1 X_1 + l_2 X_2 + l_3 X_3 + l_4 X_4 + l_5 X_5 + l_6 X_6 + l_7 X_7.$$

To establish the linear transformations of the vector $l = (l_1, l_2, l_3, l_4, l_5, l_6, l_7)$, we denote

$$E_i = c_{ij}^k l_j \partial_{l_k}, \quad i = 1, 2, 3, 4, 5, 6, 7, \tag{3}$$

where c_{ij}^k is constructed by the formula $[X_i, X_j] = c_{ij}^k X_k$. Based on eq. (3) and table 1, E_1, E_2, E_3, E_4, E_5, E_6 and E_7 can be written as

$$E_1 = -l_2\partial_{l_2} - l_5\partial_{l_5} - l_7\partial_{l_7},$$

$$E_2 = l_1\partial_{l_2} + l_3\partial_{l_5} + l_6\partial_{l_7},$$

$$E_3 = -l_2\partial_{l_5} - l_4\partial_{l_6},$$

$$E_4 = l_3\partial_{l_6} + l_5\partial_{l_7} + l_6\partial_{l_3} - l_7\partial_{l_5},$$

$$E_5 = l_1\partial_{l_5} - l_4\partial_{l_7},$$

$$E_6 = -l_2\partial_{l_7} - l_4\partial_{l_3},$$

$$E_7 = l_1\partial_{l_7} + l_4\partial_{l_5}.$$

For E_1, E_2, E_3, E_4, E_5, E_6, E_7, the Lie equations with parameters $a_1, a_2, a_3, a_4, a_5, a_6$ and a_7 and the initial condition $\tilde{l}|_{a_i=0} = l, \ i = 1, 2, 3, 4, 5, 6, 7$ are given as

$$\frac{d\tilde{l}_1}{da_1} = 0, \quad \frac{d\tilde{l}_2}{da_1} = -\tilde{l}_2, \quad \frac{d\tilde{l}_3}{da_1} = 0,$$

$$\frac{d\tilde{l}_4}{da_1} = 0, \quad \frac{d\tilde{l}_5}{da_1} = -\tilde{l}_5, \quad \frac{d\tilde{l}_6}{da_1} = 0, \quad \frac{d\tilde{l}_7}{da_1} = -\tilde{l}_7,$$

$$\frac{d\tilde{l}_1}{da_2} = 0, \quad \frac{d\tilde{l}_2}{da_2} = \tilde{l}_1, \quad \frac{d\tilde{l}_3}{da_2} = 0, \quad \frac{d\tilde{l}_4}{da_2} = 0,$$

$$\frac{d\tilde{l}_5}{da_2} = \tilde{l}_3, \quad \frac{d\tilde{l}_6}{da_2} = 0, \quad \frac{d\tilde{l}_7}{da_2} = \tilde{l}_6, \quad \frac{d\tilde{l}_1}{da_3} = 0,$$

$$\frac{d\tilde{l}_2}{da_3} = 0, \quad \frac{d\tilde{l}_3}{da_3} = 0, \quad \frac{d\tilde{l}_4}{da_3} = 0, \quad \frac{d\tilde{l}_5}{da_3} = -\tilde{l}_2,$$

$$\frac{d\tilde{l}_6}{da_3} = -\tilde{l}_4, \quad \frac{d\tilde{l}_7}{da_3} = 0, \quad \frac{d\tilde{l}_1}{da_4} = 0,$$

$$\frac{d\tilde{l}_2}{da_4} = 0, \quad \frac{d\tilde{l}_3}{da_4} = \tilde{l}_6,$$

$$\frac{d\tilde{l}_4}{da_4}=0,\quad \frac{d\tilde{l}_5}{da_4}=-\tilde{l}_7,\quad \frac{d\tilde{l}_6}{da_4}=\tilde{l}_3,\quad \frac{d\tilde{l}_7}{da_4}=\tilde{l}_5,$$

$$\frac{d\tilde{l}_1}{da_5}=0,\quad \frac{d\tilde{l}_2}{da_5}=0,\quad \frac{d\tilde{l}_3}{da_5}=0,\quad \frac{d\tilde{l}_4}{da_5}=0,$$

$$\frac{d\tilde{l}_5}{da_5}=\tilde{l}_1,\quad \frac{d\tilde{l}_6}{da_5}=0,\quad \frac{d\tilde{l}_7}{da_5}=-\tilde{l}_4,$$

$$\frac{d\tilde{l}_1}{da_6}=0,\quad \frac{d\tilde{l}_2}{da_6}=0,\quad \frac{d\tilde{l}_3}{da_6}=-\tilde{l}_4,$$

$$\frac{d\tilde{l}_4}{da_6}=0, \frac{d\tilde{l}_5}{da_6}=0,\quad \frac{d\tilde{l}_6}{da_6}=0,$$

$$\frac{d\tilde{l}_7}{da_6}=-\tilde{l}_2, \frac{d\tilde{l}_1}{da_7}=0,\quad \frac{d\tilde{l}_2}{da_7}=0,$$

$$\frac{d\tilde{l}_3}{da_7}=0,\quad \frac{d\tilde{l}_4}{da_7}=0,\quad \frac{d\tilde{l}_5}{da_7}=\tilde{l}_4,$$

$$\frac{d\tilde{l}_6}{da_7}=0,\quad \frac{d\tilde{l}_7}{da_7}=\tilde{l}_1.$$

The solutions of the above equations construct the following transformations:

$$T_1: \tilde{l}_1=l_1,\quad \tilde{l}_2=e^{-a_1}l_2,\quad \tilde{l}_3=l_3,\quad \tilde{l}_4=l_4,$$
$$\tilde{l}_5=e^{-a_1}l_5,\quad \tilde{l}_6=l_6,\quad \tilde{l}_7=e^{-a_1}l_7,$$
$$T_2: \tilde{l}_1=l_1,\quad \tilde{l}_2=a_2l_1+l_2,\quad \tilde{l}_3=l_3,\quad \tilde{l}_4=l_4,$$
$$\tilde{l}_5=a_2l_3+l_5,\quad \tilde{l}_6=l_6,\quad \tilde{l}_7=a_2l_6+l_7,$$
$$T_3: \tilde{l}_1=l_1,\quad \tilde{l}_2=l_2,\quad \tilde{l}_3=l_3,\quad \tilde{l}_4=l_4,$$
$$\tilde{l}_5=-a_3l_2+l_5,\quad \tilde{l}_6=-a_3l_4+l_6, \tilde{l}_7=l_7,$$
$$T_4: \tilde{l}_1=l_1,\quad \tilde{l}_2=l_2,\quad \tilde{l}_3=l_6\sin a_4+l_3\cos a_4,$$
$$\tilde{l}_4=l_4,\quad \tilde{l}_5=-l_7\sin a_4+l_5\cos a_4,$$
$$\tilde{l}_6=l_6\cos a_4+l_3\sin a_4,\quad \tilde{l}_7=l_7\cos a_4+l_5\sin a_4,$$
$$T_5: \tilde{l}_1=l_1,\quad \tilde{l}_2=l_2,\quad \tilde{l}_3=l_3,\quad \tilde{l}_4=l_4,$$
$$\tilde{l}_5=a_5l_1+l_5,\quad \tilde{l}_6=l_6,\quad \tilde{l}_7=-a_5l_4+l_7,$$
$$T_6: \tilde{l}_1=l_1,\quad \tilde{l}_2=l_2,\quad \tilde{l}_3=-a_6l_4+l_3,\quad \tilde{l}_4=l_4,$$
$$\tilde{l}_5=l_5,\quad \tilde{l}_6=l_6,\quad \tilde{l}_7=-a_6l_2+l_7,$$
$$T_7: \tilde{l}_1=l_1,\quad \tilde{l}_2=l_2,\quad \tilde{l}_3=l_3,\quad \tilde{l}_4=l_4,$$
$$\tilde{l}_5=a_7l_4+l_5,\quad \tilde{l}_6=l_6,\quad \tilde{l}_7=a_7l_1+l_7.$$

The establishment of the optimal system demands a simplification of the vector

$$l=(l_1,l_2,l_3,l_4,l_5,l_6,l_7), \tag{4}$$

by applying the transformations T_1–T_7. Our task is to construct the simplest representative of each class of similar vectors (4). Two cases will be considered separately.

Case 1. $l_1 \neq 0$
When $a_2=-(l_2/l_1),\ a_5=-(l_5/l_1)$ and $a_7=-(l_7/l_1)$ in T_2, T_5 and T_7, we can enable $\tilde{l}_2, \tilde{l}_5, \tilde{l}_7=0$. Vector (4) becomes

$$(l_1,0,l_3,l_4,0,l_6,0). \tag{5}$$

1.1 $l_4 \neq 0$
When $a_3=(l_6/l_4)$ and $a_6=-(l_3/l_4)$ in T_3 and T_6, we can enable $\tilde{l}_3, \tilde{l}_6=0$. Vector (5) is equivalent to $(l_1,0,0,l_4,0,0,0)$. We get the following representatives:

$$X_1 \pm X_4. \tag{6}$$

1.2 $l_4=0$
Vector (5) is equivalent to

$$(l_1,0,l_3,0,0,l_6,0). \tag{7}$$

1.2.1 $l_3 \neq 0$
When $a_4=-\arctan(l_6/l_3)$ in T_4, we can enable $\tilde{l}_6=0$. Vector (7) is equivalent to

$$(l_1,0,l_3,0,0,0,0). \tag{8}$$

We get the following representatives:

$$X_1 \pm X_3. \tag{9}$$

1.2.2 $l_3=0$
Vector (7) is equivalent to

$$(l_1,0,0,0,0,l_6,0).$$

Making all the possible combinations, we get the following representatives:

$$X_1,\ X_1 \pm X_6. \tag{10}$$

Case 2. $l_1=0$
Vector (4) becomes

$$(0,l_2,l_3,l_4,l_5,l_6,l_7). \tag{11}$$

2.1 $l_4 \neq 0$
When $a_3=(l_6/l_4)$, $a_5=(l_7/l_4)$, $a_6=(l_3/l_4)$ and $a_7=-(l_5/l_4)$ in T_3, T_5, T_6 and T_7, we can enable $\tilde{l}_3, \tilde{l}_5, \tilde{l}_6, \tilde{l}_7=0$. Vector (11) is equivalent to

$$(0,l_2,0,l_4,0,0,0).$$

Making all the possible combinations, we get the following representatives:

$$X_2 \pm X_4,\ X_4. \tag{12}$$

2.2 $l_4 = 0$
Vector (11) becomes

$$(0, l_2, l_3, 0, l_5, l_6, l_7). \tag{13}$$

2.2.1 $l_3 \neq 0$
When $a_2 = -(l_5/l_3)$ and $a_4 = -\arctan(l_6/l_3)$ in T_2 and T_4, we can enable $\tilde{l}_5, \tilde{l}_6 = 0$. Vector (13) is equivalent to

$$(0, l_2, l_3, 0, 0, 0, l_7). \tag{14}$$

2.2.1.1 $l_2 \neq 0$
When $a_6 = (l_7/l_2)$ in T_6, we can enable $\tilde{l}_7 = 0$. Vector (14) is equivalent to

$$(0, l_2, l_3, 0, 0, 0, 0).$$

We get the following representatives:

$$X_2 \pm X_3. \tag{15}$$

2.2.1.2 $l_2 = 0$
Vector (14) is equivalent to

$$(0, 0, l_3, 0, 0, 0, l_7).$$

Making all the possible combinations, we get the following representatives:

$$X_3 \pm X_7,\ X_3. \tag{16}$$

2.2.2 $l_3 = 0$
Vector (13) becomes

$$(0, l_2, 0, 0, l_5, l_6, l_7). \tag{17}$$

2.2.2.1 $l_6 \neq 0$
When $a_2 = -(l_7/l_6)$ in T_2, we can enable $\tilde{l}_7 = 0$. Vector (17) is equivalent to

$$(0, l_2, 0, 0, l_5, l_6, 0). \tag{18}$$

2.2.2.1.1 $l_2 \neq 0$
When $a_3 = (l_5/l_2)$ in T_3, we can enable $\tilde{l}_5 = 0$. Vector (18) is equivalent to

$$(0, l_2, 0, 0, 0, l_6, 0).$$

We get the following representatives:

$$X_2 \pm X_6. \tag{19}$$

2.2.2.1.2 $l_2 = 0$
Vector (18) is equivalent to

$$(0, 0, 0, 0, l_5, l_6, 0).$$

Making all the possible combinations, we get the following representatives:

$$X_5 \pm X_6,\ X_6. \tag{20}$$

2.2.2.2 $l_6 = 0$
Vector (17) is equivalent to

$$(0, l_2, 0, 0, l_5, 0, l_7). \tag{21}$$

2.2.2.2.1 $l_2 \neq 0$
When $a_3 = (l_5/l_2)$ and $a_6 = (l_7/l_2)$ in T_3 and T_6, we can enable $\tilde{l}_5, \tilde{l}_7 = 0$. Vector (21) is equivalent to

$$(0, l_2, 0, 0, 0, 0, 0).$$

We get the following representative:

$$X_2. \tag{22}$$

2.2.2.2.2 $l_2 = 0$
Vector (21) is equivalent to

$$(0, 0, 0, 0, l_5, 0, l_7). \tag{23}$$

2.2.2.2.2.1 $l_5 \neq 0$
When $a_4 = -\arctan(l_7/l_5)$ in T_4, we can enable $\tilde{l}_7 = 0$. Vector (23) is equivalent to

$$(0, 0, 0, 0, l_5, 0, 0).$$

We get the following representative:

$$X_5. \tag{24}$$

2.2.2.2.2.2 $l_5 = 0$
Vector (23) is equivalent to

$$(0, 0, 0, 0, 0, 0, l_7).$$

We get the following representative:

$$X_7. \tag{25}$$

Finally, by gathering the operators (eqs (6), (9), (10), (12), (15), (16), (19), (20), (22), (24) and (25)), we obtain the following theorem.

Theorem 2.1. *An optimal system of* $\{X_1, X_2, X_3, X_4, X_5, X_6, X_7\}$ *is generated by* $X_1,\ X_1 \pm X_3,\ X_1 \pm X_4,\ X_1 \pm X_6,\ X_2,\ X_2 \pm X_3,\ X_2 \pm X_4, X_2 \pm X_6,\ X_3,\ X_3 \pm X_7,\ X_4,\ X_5,\ X_5 \pm X_6,\ X_6,\ X_7$.

3. Reductions and exact solutions

In this section, using Theorem 2.1, we shall find the reduced equations and some exact solutions to eq. (1).

Case $1'$. Reduction by X_1
Integrating the characteristic equation for X_1, we get the invariance

$$z = \frac{t}{x}, \quad p = \frac{u}{x}$$

and the invariant solution takes the form $p = f(z)$, that is $u = xf(t/x)$, eq. (1) can be reduced to

$$f'' = 0, \tag{26}$$

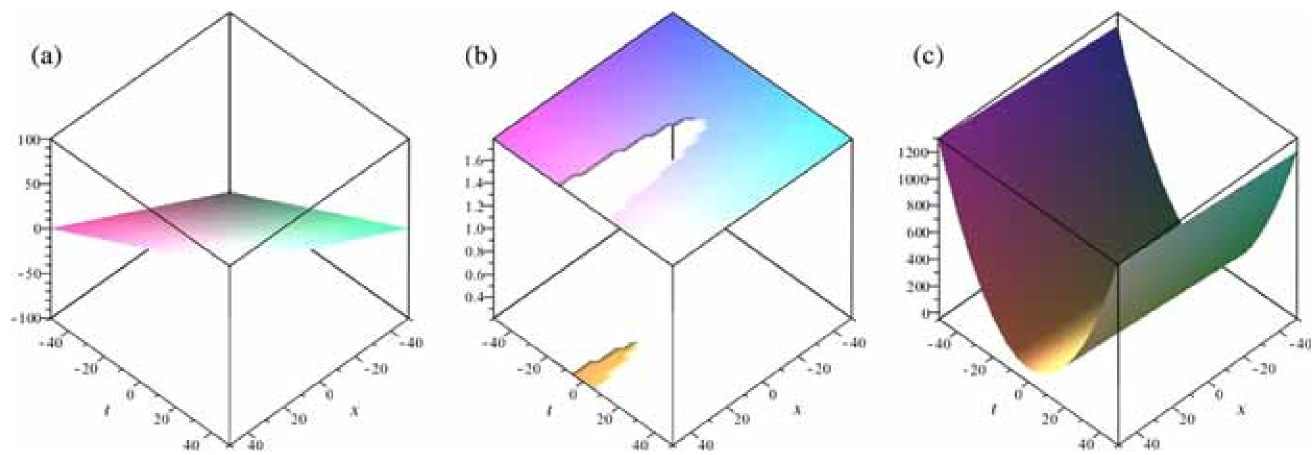

Figure 1. (**a**) $u(x, t)$ of eq. (26) for $c_1 = 1, c_2 = 1$, (**b**) $u(x, t)$ of eq. (29) for $c_1 = 1, c_2 = 1$ and (**c**) $u(x, t)$ of eq. (30) for $c_1 = 1, c_2 = 1$.

where $f' = \mathrm{d}f/\mathrm{d}z$. By solving the above equation, one can get $u(x, t) = c_1 t + c_2 x$, where c_1 and c_2 are arbitrary constants.

When we take $c_1 = 1, c_2 = 1$, the value of u is illustrated in figure 1a.

Case 2′. Reduction by $X_1 + X_6$
Similarly, we have $z = t/x$, $u = t\ln x + xf(z)$. Equation (1) is reduced to

$$\begin{aligned}&-z^4 f''^2 + 3z^3 f'' - 2z^3 f' f'' + z^3 f'^2 f'' \\ &\quad +2z^2 ff'' - 2z^2 ff'f'' + 2z^2 f' \\ &\quad -z^2 f'^2 - 2z^2 + zf^2 f'' + zf'' - 2zf \\ &\quad +2zff' - f^2 - 1 = 0,\end{aligned} \tag{27}$$

where $f' = \mathrm{d}f/\mathrm{d}z$.

Case 3′. For the generator X_4, we have $z = t$, $u = \sqrt{f(z) - x^2}$. Equation (1) is reduced to

$$\frac{1}{4} f'^2 f - \frac{1}{2} f'' f^2 - f^2 = 0, \tag{28}$$

where $f' = \mathrm{d}f/\mathrm{d}z$.

Case 4′. For the generator $X_2 + X_3$, we have $z = t^2 - 2x$, $u = f(z)$. Equation (1) can be reduced to

$$f' + 4f'^3 - 2f'' = 0, \tag{29}$$

where $f' = \mathrm{d}f/\mathrm{d}z$. By solving the above equation, one can get

$$u(x,t) = -\frac{1}{2}\arctan\left(\frac{2c_1 \mathrm{e}^{t^2-2x} - \frac{1}{4}}{\sqrt{c_1 \mathrm{e}^{t^2-2x}(1 - 4c_1 \mathrm{e}^{t^2-2x})}}\right) + c_2,$$

where $0 < c_1 < [1/(4\mathrm{e}^{t^2-2x})]$ and c_2 is an arbitrary constant.

Figure 1b depicts the solution of eq. (29), which is obtained by taking $c_1 = 1, c_2 = 1$.

Case 5′. For the generator $X_2 + X_6$, we have $z = x$, $u = f(z) + \frac{1}{2}t^2$. Equation (1) becomes

$$f'' = 0, \tag{30}$$

where $f' = \mathrm{d}f/\mathrm{d}z$. By solving the above equation, one can get $u(x, t) = c_1 x + \frac{1}{2}t^2 + c_2$, where c_1 and c_2 are arbitrary constants.

Figure 1c portrays the solution of eq. (30), which is obtained by taking $c_1 = 1, c_2 = 1$.

Case 6′. For the generator X_2, we have $z = x$, $u = f(z)$. Equation (1) becomes

$$f'' = 0, \tag{31}$$

where $f' = \mathrm{d}f/\mathrm{d}z$. By solving the above equation, one can get $u(x, t) = c_1 x + c_2$, where c_1 and c_2 are arbitrary constants.

4. The explicit power series solutions

In this section, we use power series method to discuss Case 3′. It is an extremely efficient method for solving differential equations [17].

Now, we construct a power series solution of eq. (28) of the form

$$f(z) = \sum_{n=0}^{\infty} p_n z^n, \tag{32}$$

where the coefficients p_n are all constants to be determined.

Substituting eq. (32) into eq. (28), we have

$$\frac{1}{4}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}(k-l+1)(n-k+1) \times p_l p_{k-l+1} p_{n-k+1} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{l=0}^{k}(n-k+1) \times (n-k+2) p_l p_{k-l} p_{n-k+2} z^n - \sum_{n=0}^{\infty}\sum_{k=0}^{n} p_k p_{n-k} z^n = 0. \quad (33)$$

Comparing coefficients for (33), when $n \geq 0$, we have

$$p_{n+2} = \frac{2}{(n+1)(n+2)p_0^2} \times \Bigg\{\frac{1}{4}(n+1)p_0 p_1 p_{n+1} - p_0 p_n + \sum_{k=1}^{n}\Bigg[\sum_{l=0}^{k}(n-k+1)p_l \times \Bigg(\frac{1}{4}(k-l+1)p_{k-l+1}p_{n-k+1} - \frac{1}{2}(n-k+2)p_{k-l}p_{n-k+2}\Bigg) - p_k p_{n-k}\Bigg]\Bigg\}. \quad (34)$$

Based on eq. (34), we can construct all the coefficients $p_i (i \geq 2)$ of the power series (32), e.g.,

$$p_2 = \frac{p_1^2 - 4p_0}{4p_0},$$

$$p_3 = \frac{p_1^3 - 4p_0 p_1 p_2 - 8p_0 p_1}{12 p_0^2}.$$

Therefore, as $p_0 \neq 0$ and p_1 takes arbitrary constants, the rest of the sequences $\{p_n\}_{n=0}^{\infty}$, according to (34), can be determined. This is to say, there is a power series solution (32) and its coefficients are formed by (34). Moreover, for eq. (34), we prove the convergence of the power series solution (32). Actually, with regard to (34), we have

$$|p_{n+2}| \leq M\Bigg\{|p_{n+1}| + |p_n| + \sum_{k=1}^{n}\Bigg[\sum_{l=0}^{k}|p_l|(|p_{k-l+1}||p_{n-k+1}| + |p_{k-l}||p_{n-k+2}|) + |p_k||p_{n-k}|\Bigg]\Bigg\},$$

where

$$M = \max\left\{\left|\frac{p_1}{2p_0}\right|, \left|\frac{1}{p_0}\right|, \left|\frac{1}{p_0^2}\right|\right\}.$$

Now, we define a power series $R = R(z) = \sum_{n=0}^{\infty} r_n z^n$, by

$$r_i = |p_i|, \quad i = 0, 1$$

and

$$r_{n+2} = M\Bigg\{r_{n+1} + r_n + \sum_{k=1}^{n}\Bigg[\sum_{l=0}^{k} r_l(r_{k-l+1}r_{n-k+1} + r_{k-l}r_{n-k+2}) + r_k r_{n-k}\Bigg]\Bigg\},$$

where $n = 0, 1, \ldots$. Next, it is obviously known that

$$|p_n| \leq r_n, \quad n = 0, 1, 2, \ldots.$$

In consequence, the series $R = R(z) = \sum_{n=0}^{\infty} r_n z^n$ is a majorant series of (32). Then, it is demonstrated that the series $R = R(z)$ has a positive radius of convergence.

$$\begin{aligned} R(z) &= r_0 + r_1 z + \sum_{n=0}^{\infty} r_{n+2} z^{n+2} = r_0 + r_1 z \\ &+ M\Bigg\{\sum_{n=0}^{\infty} r_{n+1} z^{n+2} + \sum_{n=0}^{\infty} r_n z^{n+2} + \sum_{n=1}^{\infty}\sum_{k=1}^{n}\Bigg[\sum_{l=0}^{k} r_l(r_{k-l+1}r_{n-k+1} + r_{k-l}r_{n-k+2})z^{n+2} + r_k r_{n-k} z^{n+2}\Bigg]\Bigg\} \\ &= r_0 + r_1 z + M[z(R - r_0) + z^2 R + (R - r_0)(R^2 - r_0 R - r_0 r_1 z) + (R - r_0 - r_1 z)(R^2 - r_0^2) + z^2 R(R - r_0)]. \end{aligned}$$

Next, we consider the implicit functional equation about the independent variable z,

$$F(z, R) = R - r_0 - r_1 z - M[z(R - r_0) + z^2 R + (R - r_0)(R^2 - r_0 R - r_0 r_1 z) + (R - r_0 - r_1 z)(R^2 - r_0^2) + z^2 R(R - r_0)].$$

Because F is analytic in the (z, R)-plane and $F(0, r_0) = 0$, $F'_R(0, r_0) = 1 \neq 0$, applying the implicit function theorem [18], we observe that, in a neighbourhood of the point $(0, r_0)$, $R = R(z)$ is analytic and has the positive radius. This means that, in a neighbourhood of the point

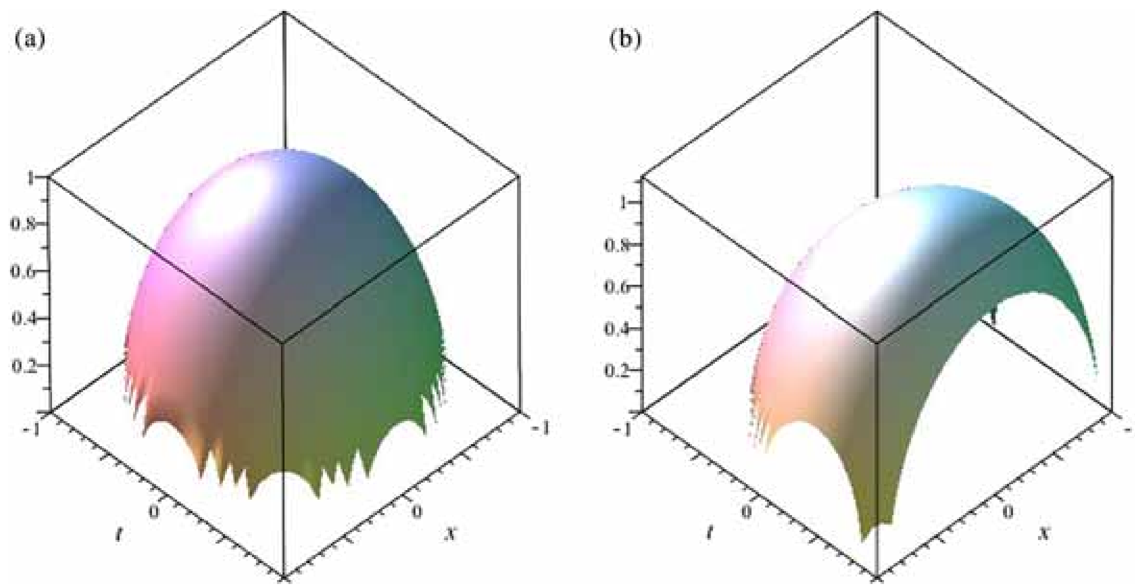

Figure 2. (**a**) $u(x,t)$ of eq. (28) for $p_0 = 1$, $p_1 = 0$ and (**b**) $u(x,t)$ of eq. (32) for $p_0 = 1$, $p_1 = 1$.

$(0, r_0)$ of the plane, the power series (32) converges. The proof is completed.

Consequently, the power series solution (32) for eq. (28) is analytic and has the following form:

$$\begin{aligned} f(z) &= p_0 + p_1 z + \sum_{n=0}^{\infty} p_{n+2} z^{n+2} \\ &= p_0 + p_1 z + \sum_{n=0}^{\infty} \frac{2}{(n+1)(n+2)p_0^2} \\ &\quad \times \left\{ \frac{1}{4}(n+1)p_0 p_1 p_{n+1} - p_0 p_n \right. \\ &\quad + \sum_{k=1}^{n} \left[\sum_{l=0}^{k} (n-k+1) p_l \right. \\ &\quad \times \left(\frac{1}{4}(k-l+1) p_{k-l+1} p_{n-k+1} \right. \\ &\quad \left. \left. \left. - \frac{1}{2}(n-k+2) p_{k-l} p_{n-k+2} \right) - p_k p_{n-k} \right] \right\} z^{n+2}. \end{aligned}$$

Then, the explicit power series solution of eq. (1) is

$$\begin{aligned} u(x,t) &= \left(p_0 + p_1 t + \sum_{n=0}^{\infty} p_{n+2} t^{n+2} - x^2 \right)^{1/2} \\ &= \left(p_0 + p_1 t + \sum_{n=0}^{\infty} \frac{2}{(n+1)(n+2)p_0^2} \right. \\ &\quad \times \left\{ \frac{1}{4}(n+1)p_0 p_1 p_{n+1} - p_0 p_n \right. \\ &\quad + \sum_{k=1}^{n} \left[\sum_{l=0}^{k} (n-k+1) p_l \right. \\ &\quad \times \left(\frac{1}{4}(k-l+1) p_{k-l+1} p_{n-k+1} \right. \\ &\quad \left. - \frac{1}{2}(n-k+2) p_{k-l} p_{n-k+2} \right) \\ &\quad \left. \left. \left. \times - p_k p_{n-k} \right] \right\} t^{n+2} - x^2 \right)^{1/2}, \end{aligned}$$

where $p_0 \neq 0$ and p_1 are arbitrary constants. The other coefficients $p_n (n \geq 2)$ can be obtained according to eq. (34).

We assign two different values for the dimensionless parameters p_0, p_1, and substitute them into (34). Let the expression

$$\bar{u}(x,t) = \sqrt{p_0 + p_1 t + \sum_{n=0}^{2} p_{n+2} t^{n+2} - x^2} \qquad (35)$$

which denotes the five-term approximation to $u(x,t)$. We get figure 2 of different values of u.

Remark 4.1. The power series method is extremely effective in constructing solutions of eq. (28) and converge rapidly. For Case 2′, unfortunately, this method is invalid. By applying power series method and comparing coefficients to eq. (27), when $n = 0$, we have $1 + p_0^2 = 0$ and when $n = 1$, we have $p_2 = [p_0(1 - 2p_1)/(1 + p_0^2)]$. Obviously, these two equations cannot be established at the same time.

5. Conclusions

In this paper, we apply Lie group analysis method to the evolution equation of a hyperbolic curve flow. Based on this method, the optimal system of the obtained symmetry and reduced equations are derived. Furthermore, exact solutions of the reduced equations are discussed. The results of this paper are of great significance in the study of geometric properties of the hyperbolic curve flow.

Acknowledgements

The authors would like to thank Prof. De-Xing Kong for drawing our attention to hyperbolic geometric flow. The authors are also grateful to the referees for their constructive comments and suggestions, which have greatly improved this paper. The financial help by the Natural Science Foundation of Shanxi (No. 201801D121018) also is acknowledged.

References

[1] K S Chou and W W Wo, *J. Diff. Geom.* **89**, 455 (2011)
[2] W W Wo, F Y Ma and C Z Qu, *Commun. Anal. Geom.* **22**, 219 (2014)
[3] D X Kong and Z G Wang, *J. Diff. Equ.* **247**, 1694 (2009)
[4] L Alvarez, F Guichard, P L Lions and J M Morel, *Arch. Ration. Anal.* **123**, 199 (1993)
[5] G Sapiro, *Geometric partial differential equations in image analysis: Past, present, and future* (Cambridge University Press, New York, 2001)
[6] W W Wo, S X Yang and X L Wang, *Arch. Math.* **108**, 495 (2017)
[7] Z G Wang, *Appl. Math. Comput.* **235**, 560 (2014)
[8] J H Wang, *Sci. China Math.* **56**, 1689 (2013)
[9] P J Olver, Applications of Lie groups to differential equations, in: *Graduate texts in mathematics* (Springer, New York, 1993)
[10] G W Bluman and S Kumei, *Symmetries and differential equations* (Springer-Verlag, Berlin, 1989)
[11] B Gao and Y Zhang, *Physica A* **525**, 1058 (2019)
[12] B Gao and C F He, *Electron. J. Diff. Equ.* **42**, 1 (2017)
[13] A A Afify and M Abdul-Majid, *Pramana – J. Phys.* **88**: 31 (2017)
[14] G W Wang and M S Hashemi, *Pramana – J. Phys.* **88**: 7 (2017)
[15] G W Bluman and S C Anco, *Symmetry and integration methods for differential equations* (Springer, New York, 2004)
[16] Y N Grigoriev, V F Kovalev and S V Meleshko, *Symmetries of integro-differential equations: With applications in mechanics and plasma physics* (Springer, New York, 2010)
[17] N H Asmar, *Partial differential equations with Fourier series and boundary value problems* (China Machine Press, Beijing, 2005)
[18] W Rudin, *Principles of mathematical analysis* (China Machine Press, Beijing, 2004)