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New closed form solutions of the new coupled Konno–Oono equation using the new extended direct algebraic method

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Abstract. In this paper, we apply the new extended direct algebraic method (NEDAM) to solve new exact solutions of the new coupled Konno–Oono (CKO) equation, and construct exact solution expressed in terms of hyperbolic functions and trigonometric functions with arbitrary parameters. A comparison between our established results and the results obtained by the existing ones is also presented. As a newly developed mathematical tool, the proposed method is an effective and straightforward technique to work out new solutions of various types of nonlinear partial differential equations (NLPDEs) in applied sciences and engineering.

Keywords. New extended direct algebraic method; coupled Konno–Oono equation; nonlinear partial differential equation.

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1. Introduction

Nonlinear partial differential equations (NLPDEs) have been the subject of study in various branches of mathematical-physical sciences such as biology, chemistry, physics, quantum mechanics, ecology, etc.

The nonlinear Konno–Oono equation system was introduced by Konno and Oono [1] as the coupled integrable dispersionless system

$$\begin{cases} q_{xt} - 2\alpha q r_x - 2\beta q s_x + \gamma (rs)_x = 0, \\ r_{xt} - 2\alpha r r_x - 2\beta (2qq_x + r_x s) - 2\gamma q_x r = 0, \\ s_{xt} - 2\beta s s_x - 2\alpha (2qq_x + r_x s) - 2\gamma q_x s = 0, \end{cases}$$

where α, β and γ are constants. The Konno–Oono equation system has been investigated as applications for current-fed string interacting with an external magnetic field [1–3], and the parallel transport of each point of the curve along the direction of time where the connection is magnetic-valued [4]. Kocak *et al* [5] obtained trawling wave solution for the coupled Konno-Oono (CKO) equation using the modified exponential function method. A special case of a system, considered to be transformed into new Konno–Oono equation system which is a coupled integrable dispersionless equations, is given in the form of [6–10]

$$\begin{cases} u_{xt} - 2uv = 0, \\ v_t + 2uu_x = 0. \end{cases} \tag{1}$$

This model attracted the attention of various scholars from all over the world. Many scientists have analysed this equation in terms of new and different properties by using several mathematical methods. In [6], the tanh-function method and extended tanh-function method have been obtained to construct the exact solutions for this equation. In [7], the modified simple equation (MSE) method has been employed for eq. (1) and kink solutions, bell-shaped solutions were obtained by using MSE method. Also, some solitons and singular periodic solutions of this equation are found in [8–10].

A special type of exact solutions, called travelling wave solutions for NLPDEs, has a lot of importance, because most of the phenomena that arise in mathematics, physics and engineering fields can be described by NLPDEs. However, from the past few decades, a

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variety of effective exact solution methods, such as the auxiliary equation method [11], the improved (G'/G)-expansion method [12], the exp-function method [13], the generalised Kudryashov method [14], the first integral method [15,16], the extended Jacobi elliptic function expansion method [17], the improved $\tan(\phi/2)$-expansion method [18], the Bernoulli subequation function method [19], the sine-Gordon expansion method [20,21], the Khater method [22], the subequation method [23–25] and many more [27–35] have been developed for solving NLPDEs.

In this study, a new and effective generalised algebra method is used to produce new exact travelling wave solutions of the new CKO equation. In this paper, our goal is to apply new extended direct algebraic method (NEDAM) for extracting exact solutions of the new CKO equation [26]. This method is considered among those general ones from which, under certain cases, various methods such as the extended tanh-function method [6], sine-Gordon expansion method [8] etc. can be deduced. It has been established that the executed method is powerful, skilled to examine NLPDEs, compatible to computer algebra and provides further general wave solutions. Thus, the investigation of exact solutions to other NLEES through the NEDAM deserves further research.

This paper is organised as follows: In §2, we introduce the description of the NEDAM. In §3, solutions to the new CKO equation are given. In §4, we present the results of the study comprehensively. In §5, the physical explanation is given and finally in §6, we provide the conclusions of the study.

2. The NEDAM

In this section, we shall outline the main steps of the NEDAM.

Step 1. Let us consider the NLPDEs

$$F(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0. \tag{2}$$

Combine the travelling wave transformation by

$$u(x,t) = U(\xi), \qquad \xi = k(x - \lambda t)$$

which can be converted to an ordinary differential equation (ODE) as follows:

$$G(u, u', u'', \ldots) = 0. \tag{3}$$

Step 2. Suppose that the solution of ODE (3) can be expressed by a polynomial in $Q(\xi)$ as follows:

$$U(\xi) = \sum_{j=0}^{N} b_j Q^j(\xi), \quad b_N \neq 0, \tag{4}$$

where $b_k (0 \leq j \leq N)$ are constant coefficients to be determined later and $Q(\xi)$ satisfies the ODE in the form

$$Q'(\xi) = Ln(A)(\alpha + \beta Q(\xi) + \sigma Q^2(\xi)), \quad A \neq 0, 1, \tag{5}$$

and the solutions of ODE (5) are

(1) When $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$,

$$Q_1(\xi) = -\frac{\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma} \times \tan_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\xi\right),$$

$$Q_2(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma} \times \cot_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2}\xi\right),$$

$$Q_3(\xi) = -\frac{\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma} \times \left(\tan_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\xi\right) \pm \sqrt{pq}\sec_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\xi\right)\right),$$

$$Q_4(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{2\sigma} \times \left(\cot_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\xi\right) \pm \sqrt{pq}\csc_A\left(\sqrt{-(\beta^2 - 4\alpha\sigma)}\xi\right)\right),$$

$$Q_5(\xi) = -\frac{\beta}{2\sigma} + \frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{4\sigma} \times \left(\tan_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha\sigma)}}{4}\xi\right) - \cot_A\left(\frac{\sqrt{-(\beta^2 - 4\alpha)\sigma}}{4}\xi\right)\right).$$

(2) When $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$,

$$Q_6(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma} \times \tanh_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}\xi\right),$$

$$Q_7(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma} \times \coth_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2}\xi\right),$$

$$Q_8(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma} \times \left(\tanh_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\xi\right) \pm i\sqrt{pq}\,\mathrm{sech}_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\xi\right)\right),$$

$$Q_9(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{2\sigma} \times \left(\coth_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\xi\right) \pm\sqrt{pq}\,\mathrm{csch}_A\left(\sqrt{\beta^2 - 4\alpha\sigma}\xi\right)\right),$$

$$Q_{10}(\xi) = -\frac{\beta}{2\sigma} - \frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4\sigma} \times \left(\tanh_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4}\xi\right) + \coth_A\left(\frac{\sqrt{\beta^2 - 4\alpha\sigma}}{4}\xi\right)\right).$$

(3) When $\alpha\sigma > 0$ and $\beta = 0$,

$$Q_{11}(\xi) = \sqrt{\frac{\alpha}{\sigma}}\tan_A\left(\sqrt{\alpha\sigma}\xi\right),$$

$$Q_{12}(\xi) = -\sqrt{\frac{\alpha}{\sigma}}\cot_A\left(\sqrt{\alpha\sigma}\xi\right),$$

$$Q_{13}(\xi) = \sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(2\sqrt{\alpha\sigma}\xi\right) \pm\sqrt{pq}\sec_A\left(2\sqrt{\alpha\sigma}\xi\right)\right),$$

$$Q_{14}(\xi) = -\left(\sqrt{\frac{\alpha}{\sigma}}\cot_A\left(2\sqrt{\alpha\sigma}\xi\right) \pm\sqrt{pq}\csc_A\left(2\sqrt{\alpha\sigma}\xi\right)\right),$$

$$Q_{15}(\xi) = \frac{1}{2}\sqrt{\frac{\alpha}{\sigma}}\left(\tan_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right) - \cot_A\left(\frac{\sqrt{\alpha\sigma}}{2}\xi\right)\right).$$

(4) When $\alpha\sigma < 0$ and $\beta = 0$,

$$Q_{16}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\tanh_A\left(\sqrt{-\alpha\sigma}\xi\right),$$

$$Q_{17}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\coth_A\left(\sqrt{-\alpha\sigma}\xi\right),$$

$$Q_{18}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\left(\tanh_A\left(2\sqrt{-\alpha\sigma}\xi\right) \pm i\sqrt{pq}\,\mathrm{sech}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right),$$

$$Q_{19}(\xi) = -\sqrt{-\frac{\alpha}{\sigma}}\left(\coth_A\left(2\sqrt{-\alpha\sigma}\xi\right) \pm\sqrt{pq}\,\mathrm{csch}_A\left(2\sqrt{-\alpha\sigma}\xi\right)\right),$$

$$Q_{20}(\xi) = -\frac{1}{2}\sqrt{-\frac{\alpha}{\sigma}}\left(\tanh_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right) + \coth_A\left(\frac{\sqrt{-\alpha\sigma}}{2}\xi\right)\right).$$

(5) When $\beta = 0$ and $\sigma = \alpha$,

$$Q_{21}(\xi) = \tan_A(\alpha\xi),$$

$$Q_{22}(\xi) = -\cot_A(\alpha\xi),$$

$$Q_{23}(\xi) = \tan_A(2\alpha\xi) \pm \sqrt{pq}\sec_A(2\alpha\xi),$$

$$Q_{24}(\xi) = -\cot_A(2\alpha\xi) \pm \sqrt{pq}\csc_A(2\alpha\xi),$$

$$Q_{25}(\xi) = \frac{1}{2}\left(\tan_A\left(\frac{\alpha}{2}\xi\right) - \cot_A\left(\frac{\alpha}{2}\xi\right)\right).$$

(6) When $\beta = 0$ and $\sigma = -\alpha$,

$$Q_{26}(\xi) = -\tanh_A(\alpha\xi),$$

$$Q_{27}(\xi) = -\coth_A(\alpha\xi),$$

$$Q_{28}(\xi) = -\tanh_A(2\alpha\xi) \pm i\sqrt{pq}\,\mathrm{sech}_A(2\alpha\xi),$$

$$Q_{29}(\xi) = -\coth_A(2\alpha\xi) \pm \sqrt{pq}\,\mathrm{csch}_A(2\alpha\xi),$$

$$Q_{30}(\xi) = -\frac{1}{2}\left(\tanh_A\left(\frac{\alpha}{2}\xi\right) + \coth_A\left(\frac{\alpha}{2}\xi\right)\right).$$

(7) When $\beta^2 = 4\alpha\sigma$,

$$Q_{31}(\xi) = \frac{-2\alpha(\beta\xi LnA + 2)}{\beta^2\xi LnA}.$$

(8) When $\beta = \lambda, \alpha = m\lambda (m \neq 0)$ and $\sigma = 0$,

$$Q_{32}(\xi) = A^{\lambda\xi} - m.$$

(9) When $\beta = \sigma = 0$,

$$Q_{33}(\xi) = \alpha\xi LnA.$$

(10) When $\beta = \alpha = 0$,

$$Q_{34}(\xi) = \frac{-1}{\sigma\xi LnA}.$$

(11) When $\alpha = 0$ and $\beta \neq 0$,

$$Q_{35}(\xi) = -\frac{p\beta}{\sigma(\cosh_A(\beta\xi) - \sinh_A(\beta\xi) + p)},$$

$$Q_{36}(\xi) = -\frac{\beta(\sinh_A(\beta\xi) + \cosh_A(\beta\xi))}{\sigma(\sinh_A(\beta\xi) + \cosh_A(\beta\xi) + q)}.$$

(12) When $\beta = \lambda$, $\sigma = m\lambda (m \neq 0)$ and $\alpha = 0$,

$$Q_{37}(\xi) = \frac{pA^{\lambda\xi}}{p - mqA^{\lambda\xi}}.$$

Here the generalised hyperbolic and trigonometric functions are defined as [22]

$$\sinh_A(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{2},$$

$$\cosh_A(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{2},$$

$$\tanh_A(\xi) = \frac{pA^{\xi} - qA^{-\xi}}{pA^{\xi} + qA^{-\xi}},$$

$$\coth_A(\xi) = \frac{pA^{\xi} + qA^{-\xi}}{pA^{\xi} - qA^{-\xi}},$$

$$\mathrm{sech}_A(\xi) = \frac{2}{pA^{\xi} + qA^{-\xi}},$$

$$\mathrm{csch}_A(\xi) = \frac{2}{pA^{\xi} - qA^{-\xi}},$$

$$\sin_A(\xi) = \frac{pA^{i\xi} - qA^{-i\xi}}{2i},$$

$$\cos_A(\xi) = \frac{pA^{i\xi} + qA^{-i\xi}}{2},$$

$$\tan_A(\xi) = -i\frac{pA^{i\xi} - qA^{-i\xi}}{pA^{i\xi} + qA^{-i\xi}},$$

$$\cot_A(\xi) = i\frac{pA^{i\xi} + qA^{-i\xi}}{pA^{i\xi} - qA^{-i\xi}},$$

$$\sec_A(\xi) = \frac{2}{pA^{i\xi} + qA^{-i\xi}},$$

$$\csc_A(\xi) = \frac{2i}{pA^{i\xi} - qA^{-i\xi}}.$$

Here ξ is an independent variable, p and q are arbitrary constants greater than zero and called deformation parameters.

Step 3. We can find the value N in eq. (4) by balancing the highest-order derivative term and the highest-order nonlinear term in (3).

Step 4. Substitute eq. (4) along with its required derivatives into eq. (3) and compare the coefficients of powers of $Q(\xi)$ in the resultant equation for obtaining the set of algebraic equations.

3. Solutions of the new CKO equation

Now, in order to solve eq. (1) by the NEDAM, we kick off with the following wave transformation:

$$u(x,t) = U(\xi), \quad \xi = \quad k(x - \lambda t),$$

$$v(x,t) = V(\xi), \quad \xi = \quad k(x - \lambda t), \tag{6}$$

where k is the wave number and λ is the wave velocity. Equation (6) reduces eq. (1) into the following ODEs:

$$-\lambda k^2 U'' - 2UV = 0, \tag{7}$$

$$-\lambda k V' + 2kUU' = 0. \tag{8}$$

By integrating eq. (8) with respect to ξ, we obtain

$$V = \frac{1}{\lambda}(U^2 + \theta), \tag{9}$$

where θ is the constant of integration. When we put eq. (9) into eq. (7), we obtain

$$\lambda^2 k^2 U'' + 2U^3 + 2\theta U = 0. \tag{10}$$

Now balancing the highest-order derivative U'' and the nonlinear term U^3 in eq. (10), we obtain $N = 1$.

According to $N = 1$, the finite expansion (4) of the generalised algebra method admits

$$U(\xi) = b_0 + b_1 Q(\xi). \tag{11}$$

Substituting eq. (11) into (10) and equating all the coefficients of $Q(\xi)$ to zero, we yield an extremely complicated system of algebraic equations.

Solving this system of equations for b_0, b_1 and λ with the help of symbolic computation, we obtain the following values:

$$b_0 = \pm\beta\sqrt{\frac{\theta}{4\alpha\sigma - \beta^2}}, \quad b_1 = \pm 2\sigma\sqrt{\frac{\theta}{4\alpha\sigma - \beta^2}},$$

$$\lambda = \pm\frac{2}{kLnA}\sqrt{-\frac{\theta}{4\alpha\sigma - \beta^2}}. \tag{12}$$

The solutions of (1) corresponding to (2), (4), (5) and (12) are

(1) When $\beta^2 - 4\alpha\sigma < 0$ and $\sigma \neq 0$,

$$u_1(x,t) = \pm\beta\sqrt{\theta}\tan_A\left(\frac{k\sqrt{4\alpha\sigma - \beta^2}}{2}x \pm \frac{\sqrt{-\theta}}{LnA}t\right),$$

$$v_1(x,t) = \frac{\theta}{\lambda}\left(\beta^2\tan_A^2\left(\frac{k\sqrt{4\alpha\sigma - \beta^2}}{2}x \pm \frac{\sqrt{-\theta}}{LnA}t\right) + 1\right),$$

$$u_2(x,t) = \pm\beta\sqrt{\theta}\cot_A\left(\frac{k\sqrt{4\alpha\sigma - \beta^2}}{2}x \pm \frac{\sqrt{-\theta}}{LnA}t\right),$$

$$v_2(x,t) = \frac{\theta}{\lambda}\left(\beta^2\cot_A^2\left(\frac{k\sqrt{4\alpha\sigma - \beta^2}}{2}x \pm \frac{\sqrt{-\theta}}{LnA}t\right) + 1\right),$$

$$u_3(x,t) = \pm\beta\sqrt{\theta}\left(\tan_A\left(k\sqrt{4\alpha\sigma - \beta^2}x \pm \frac{2\sqrt{-\theta}}{LnA}t\right) \pm\sqrt{pq}\sec_A\left(k\sqrt{4\alpha\sigma - \beta^2}x \pm \frac{2\sqrt{-\theta}}{LnA}t\right)\right),$$

$$v_3(x,t) = \frac{\theta}{\lambda}\left(\beta^2\left(\tan_A\left(k\sqrt{4\alpha\sigma - \beta^2}x \pm \frac{2\sqrt{-\theta}}{LnA}t\right) \pm\sqrt{pq}\sec_A\left(k\sqrt{4\alpha\sigma - \beta^2}x \pm \frac{2\sqrt{-\theta}}{LnA}t\right)\right)^2 + 1\right),$$

$$u_4(x,t) = \pm\beta\sqrt{\theta}\left(\cot_A\left(k\sqrt{4\alpha\sigma - \beta^2}x \pm \frac{2\sqrt{-\theta}}{LnA}t\right) \pm\sqrt{pq}\csc_A\left(k\sqrt{4\alpha\sigma - \beta^2}x \pm \frac{2\sqrt{-\theta}}{LnA}t\right)\right),$$

$$v_4(x,t) = \frac{\theta}{\lambda}\left(\beta^2\left(\cot_A\left(k\sqrt{4\alpha\sigma - \beta^2}x \pm \frac{2\sqrt{-\theta}}{LnA}t\right) \pm\sqrt{pq}\csc_A\left(k\sqrt{4\alpha\sigma - \beta^2}x \pm \frac{2\sqrt{-\theta}}{LnA}t\right)\right)^2 + 1\right),$$

$$u_5(x,t) = \pm\frac{\beta\sqrt{\theta}}{2}\left(\tan_A\left(\frac{k\sqrt{4\alpha\sigma - \beta^2}}{4}x \pm \frac{\sqrt{-\theta}}{2LnA}t\right) - \cot_A\left(\frac{k\sqrt{4\alpha\sigma - \beta^2}}{4}x \pm \frac{\sqrt{-\theta}}{2LnA}t\right)\right),$$

$$v_5(x,t) = \frac{\theta}{\lambda}\left(\frac{\beta^2}{4}\left(\tan_A\left(\frac{k\sqrt{4\alpha\sigma - \beta^2}}{4}x \pm \frac{\sqrt{-\theta}}{2LnA}t\right) - \cot_A\left(\frac{k\sqrt{4\alpha\sigma - \beta^2}}{4}x \pm \frac{\sqrt{-\theta}}{2LnA}t\right)\right)^2 + 1\right).$$

(2) When $\beta^2 - 4\alpha\sigma > 0$ and $\sigma \neq 0$,

$$u_6(x,t) = \pm\beta\sqrt{-\theta}\tanh_A\left(\frac{k\sqrt{\beta^2 - 4\alpha\sigma}}{2}x \pm \frac{\sqrt{\theta}}{LnA}t\right),$$

$$v_6(x,t) = \frac{\theta}{\lambda}\left(1 - \beta^2\tanh_A^2\left(\frac{k\sqrt{\beta^2 - 4\alpha\sigma}}{2}x \pm \frac{\sqrt{\theta}}{LnA}t\right)\right),$$

$$u_7(x,t) = \pm\beta\sqrt{-\theta}\coth_A\left(\frac{k\sqrt{\beta^2 - 4\alpha\sigma}}{2}x \pm \frac{\sqrt{\theta}}{LnA}t\right),$$

$$v_7(x,t) = \frac{\theta}{\lambda}\left(1 - \beta^2\coth_A^2\left(\frac{k\sqrt{\beta^2 - 4\alpha\sigma}}{2}x \pm \frac{\sqrt{\theta}}{LnA}t\right)\right),$$

$$u_8(x,t) = \pm\beta\sqrt{-\theta}\left(\tanh_A\left(k\sqrt{\beta^2 - 4\alpha\sigma}x \pm \frac{2\sqrt{\theta}}{LnA}t\right) \pm i\sqrt{pq}\,\mathrm{sech}_A\left(k\sqrt{\beta^2 - 4\alpha\sigma}x \pm \frac{2\sqrt{\theta}}{LnA}t\right)\right),$$

$$v_8(x,t) = \frac{\theta}{\lambda}\left(1 - \beta^2\left(\tanh_A\left(k\sqrt{\beta^2 - 4\alpha\sigma}x \pm \frac{2\sqrt{\theta}}{LnA}t\right)\right.\right.$$

$$\pm i\sqrt{pq}\,\mathrm{sech}_A\left(k\sqrt{\beta^2-4\alpha\sigma}\,x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\Bigg)^2\Bigg),$$

$$u_9(x,t)=\pm\beta\sqrt{-\theta}\left(\coth_A\left(k\sqrt{\beta^2-4\alpha\sigma}\,x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right.$$
$$\left.\pm\sqrt{pq}\,\mathrm{csch}_A\left(k\sqrt{\beta^2-4\alpha\sigma}\,x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right),$$

$v_9(x,t)$
$$=\frac{\theta}{\lambda}\left(1-\beta^2\left(\coth_A\left(k\sqrt{\beta^2-4\alpha\sigma}\,x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}\,\mathrm{csch}_A\left(k\sqrt{\beta^2-4\alpha\sigma}\,x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right)^2\right),$$

$u_{10}(x,t)$
$$=\pm\frac{\beta\sqrt{-\theta}}{2}\left(\tanh_A\left(\frac{k\sqrt{\beta^2-4\alpha\sigma}}{4}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right.$$
$$\left.+\coth_A\left(\frac{k\sqrt{\beta^2-4\alpha\sigma}}{4}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right),$$

$v_{10}(x,t)$
$$=\frac{\theta}{\lambda}\left(1-\frac{\beta^2}{4}\left(\tanh_A\left(\frac{k\sqrt{\beta^2-4\alpha\sigma}}{4}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right.\right.$$
$$\left.\left.+\coth_A\left(\frac{k\sqrt{\beta^2-4\alpha\sigma^2}}{4}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right)^2\right).$$

(3) When $\alpha\sigma>0$ and $\beta=0$,

$$u_{11}(x,t)=\pm\sqrt{\theta}\,\tan_A\left(k\sqrt{\alpha\sigma}\,x\pm\frac{\sqrt{-\theta}}{LnA}t\right),$$

$$v_{11}(x,t)=\frac{\theta}{\lambda}\left(\tan_A^2\left(k\sqrt{\alpha\sigma}\,x\pm\frac{\sqrt{-\theta}}{LnA}t\right)+1\right),$$

$$u_{12}(x,t)=\pm\sqrt{\theta}\,\cot_A\left(k\sqrt{\alpha\sigma}\,x\pm\frac{\sqrt{-\theta}}{LnA}t\right),$$

$$v_{12}(x,t)=\frac{\theta}{\lambda}\left(\cot_A^2\left(k\sqrt{\alpha\sigma}\,x\pm\frac{\sqrt{-\theta}}{LnA}t\right)+1\right),$$

$$u_{13}(x,t)=\pm\sqrt{\theta}\left(\tan_A\left(2k\sqrt{\alpha\sigma}\,x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right.$$
$$\left.\pm\sqrt{pq}\,\sec_A\left(2k\sqrt{\alpha\sigma}\,x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right),$$

$$v_{13}(x,t)=\frac{\theta}{\lambda}\left(\left(\tan_A\left(2k\sqrt{\alpha\sigma}\,x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}\,\sec_A\left(2k\sqrt{\alpha\sigma}\,x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right)^2+1\right),$$

$$u_{14}(x,t)=\pm\sqrt{\theta}\left(\cot_A\left(2k\sqrt{\alpha\sigma}\,x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right.$$
$$\left.\pm\sqrt{pq}\,\csc_A\left(2k\sqrt{\alpha\sigma}\,x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right),$$

$$v_{14}(x,t)=\frac{\theta}{\lambda}\left(\left(\cot_A\left(2k\sqrt{\alpha\sigma}\,x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}\,\csc_A\left(2k\sqrt{\alpha\sigma}\,x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right)^2+1\right),$$

$$u_{15}(x,t)=\pm\frac{\sqrt{\theta}}{2}\left(\tan_A\left(\frac{k\sqrt{\alpha\sigma}}{2}x\pm\frac{\sqrt{-\theta}}{2LnA}t\right)\right.$$
$$\left.-\cot_A\left(\frac{k\sqrt{\alpha\sigma}}{2}x\pm\frac{\sqrt{-\theta}}{2LnA}t\right)\right),$$

$$v_{15}(x,t)=\frac{\theta}{\lambda}\left(\frac{1}{4}\left(\tan_A\left(\frac{k\sqrt{\alpha\sigma}}{2}x\pm\frac{\sqrt{-\theta}}{2LnA}t\right)\right.\right.$$
$$\left.\left.-\cot_A\left(\frac{k\sqrt{\alpha\sigma}}{2}x\pm\frac{\sqrt{-\theta}}{2LnA}t\right)\right)^2+1\right).$$

(4) When $\alpha\sigma<0$ and $\beta=0$,

$$u_{16}(x,t)=\pm\sqrt{-\theta}\,\tanh_A\left(k\sqrt{-\alpha\sigma}\,x\pm\frac{\sqrt{\theta}}{LnA}t\right),$$

$$v_{16}(x,t)=\frac{\theta}{\lambda}\left(1-\tanh_A^2\left(k\sqrt{-\alpha\sigma}\,x\pm\frac{\sqrt{\theta}}{LnA}t\right)\right),$$

$$u_{17}(x,t)=\pm\sqrt{-\theta}\,\coth_A\left(k\sqrt{-\alpha\sigma}\,x\pm\frac{\sqrt{\theta}}{LnA}t\right),$$

$$v_{17}(x,t)=\frac{\theta}{\lambda}\left(1-\coth_A^2\left(k\sqrt{-\alpha\sigma}\,x\pm\frac{\sqrt{\theta}}{LnA}t\right)\right),$$

$$u_{18}(x,t)=\pm\sqrt{-\theta}\left(\tanh_A\left(2k\sqrt{-\alpha\sigma}x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right.$$
$$\left.\pm i\sqrt{pq}\,\mathrm{sech}_A\left(2k\sqrt{-\alpha\sigma}x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right),$$
$$v_{18}(x,t)=\frac{\theta}{\lambda}\left(1-\left(\tanh_A\left(2k\sqrt{-\alpha\sigma}x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right.\right.$$
$$\left.\left.\pm i\sqrt{pq}\,\mathrm{sech}_A\left(2k\sqrt{-\alpha\sigma}x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right)^2\right),$$
$$u_{19}(x,t)=\pm\sqrt{-\theta}\left(\coth_A\left(2k\sqrt{-\alpha\sigma}x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right.$$
$$\left.\pm\sqrt{pq}\,\mathrm{csch}_A\left(2k\sqrt{-\alpha\sigma}x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right),$$
$$v_{19}(x,t)=\frac{\theta}{\lambda}\left(1-\left(\coth_A\left(2k\sqrt{-\alpha\sigma}x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}\,\mathrm{csch}_A\left(2k\sqrt{-\alpha\sigma}x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right)^2\right),$$
$$u_{20}(x,t)=\pm\frac{\sqrt{-\theta}}{2}\left(\tanh_A\left(\frac{k\sqrt{-\alpha\sigma}}{2}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right.$$
$$\left.+\coth_A\left(\frac{k\sqrt{-\alpha\sigma}}{2}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right),$$
$$v_{20}(x,t)=\frac{\theta}{\lambda}\left(1-\frac{1}{4}\left(\tanh_A\left(\frac{k\sqrt{-\alpha\sigma}}{2}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right.\right.$$
$$\left.\left.+\coth_A\left(\frac{k\sqrt{-\alpha\sigma}}{2}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right)^2\right).$$

(5) When $\beta=0$ and $\sigma=\alpha$,

$$u_{21}(x,t)=\pm\sqrt{\theta}\,\tan_A\left(k\alpha x\pm\frac{\sqrt{-\theta}}{LnA}t\right),$$
$$v_{21}(x,t)=\frac{\theta}{\lambda}\left(\tan_A^2\left(k\alpha x\pm\frac{\sqrt{-\theta}}{LnA}t\right)+1\right),$$
$$u_{22}(x,t)=\pm\sqrt{\theta}\,\cot_A\left(k\alpha x\pm\frac{\sqrt{-\theta}}{LnA}t\right),$$
$$v_{22}(x,t)=\frac{\theta}{\lambda}\left(\cot_A^2\left(k\alpha x\pm\frac{\sqrt{\theta}}{LnA}t\right)+1\right),$$
$$u_{23}(x,t)=\pm\sqrt{\theta}\left(\tan_A\left(2k\alpha x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right.$$
$$\left.\pm\sqrt{pq}\,\sec_A\left(2k\alpha x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right),$$
$$v_{23}(x,t)=\frac{\theta}{\lambda}\left(\left(\tan_A\left(2k\alpha x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}\,\sec_A\left(2k\alpha x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right)^2+1\right),$$
$$u_{24}(x,t)=\pm\sqrt{\theta}\left(\cot_A\left(2k\alpha x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right.$$
$$\left.\pm\sqrt{pq}\,\csc_A\left(2k\alpha x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right),$$
$$v_{24}(x,t)=\frac{\theta}{\lambda}\left(\left(\cot_A\left(2k\alpha x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right.\right.$$
$$\left.\left.\pm\sqrt{pq}\,\csc_A\left(2k\alpha x\pm\frac{2\sqrt{-\theta}}{LnA}t\right)\right)^2+1\right),$$
$$u_{25}(x,t)=\pm\frac{\sqrt{\theta}}{2}\left(\tan_A\left(\frac{k\alpha}{2}x\pm\frac{\sqrt{-\theta}}{2LnA}t\right)\right.$$
$$\left.-\cot_A\left(\frac{k\alpha}{2}x\pm\frac{\sqrt{-\theta}}{2LnA}t\right)\right),$$
$$v_{25}(x,t)=\frac{\theta}{\lambda}\left(\frac{1}{4}\left(\tan_A\left(\frac{k\alpha}{2}x\pm\frac{\sqrt{-\theta}}{2LnA}t\right)\right.\right.$$
$$\left.\left.-\cot_A\left(\frac{k\alpha}{2}x\pm\frac{\sqrt{-\theta}}{2LnA}t\right)\right)^2+1\right).$$

(6) When $\beta=0$ and $\sigma=-\alpha$,

$$u_{26}(x,t)=\pm\sqrt{-\theta}\,\tanh_A\left(k\alpha x\pm\frac{\sqrt{\theta}}{LnA}t\right),$$
$$v_{26}(x,t)=\frac{\theta}{\lambda}\left(1-\tanh_A^2\left(k\alpha x\pm\frac{\sqrt{\theta}}{LnA}t\right)\right),$$

$$u_{27}(x,t)=\pm\sqrt{-\theta}\,\coth_A\left(k\alpha x\pm\frac{\sqrt{\theta}}{LnA}t\right),$$

$$v_{27}(x,t)=\frac{\theta}{\lambda}\left(1-\coth_A^2\left(k\alpha x\pm\frac{\sqrt{\theta}}{LnA}t\right)\right),$$

$$u_{28}(x,t)=\pm\sqrt{-\theta}\left(\tanh_A\left(2k\alpha x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\pm i\sqrt{pq}\,\mathrm{sech}_A\left(2k\alpha x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right),$$

$$v_{28}(x,t)=\frac{\theta}{\lambda}\left(1-\left(\tanh_A\left(2k\alpha x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\pm i\sqrt{pq}\,\mathrm{sech}_A\left(2k\alpha x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right)^2\right),$$

$$u_{29}(x,t)=\pm\sqrt{-\theta}\left(\coth_A\left(2k\alpha x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\pm\sqrt{pq}\,\mathrm{csch}_A\left(2k\alpha x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right),$$

$$v_{29}(x,t)=\frac{\theta}{\lambda}\left(1-\left(\coth_A\left(2k\alpha x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\pm\sqrt{pq}\,\mathrm{csch}_A\left(2k\alpha x\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right)^2\right),$$

$$u_{30}(x,t)=\pm\frac{\sqrt{-\theta}}{2}\left(\tanh_A\left(\frac{k\alpha}{2}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)+\coth_A\left(\frac{k\alpha}{2}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right),$$

$$v_{30}(x,t)=\frac{\theta}{\lambda}\left(1-\frac{1}{4}\left(\tanh_A\left(\frac{k\alpha}{2}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)+\coth_A\left(\frac{k\alpha}{2}x\pm\frac{\sqrt{\theta}}{2LnA}t\right)\right)^2\right).$$

(7) When $\alpha=0$ and $\beta\neq 0$,

$$u_{31}(x,t)=\pm\beta\sqrt{-\theta}\left(1-\frac{2p}{\left(\cosh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)-\sinh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)+p\right)}\right),$$

$$v_{31}(x,t)=\frac{\theta}{\lambda}\left(1-\beta^2\left(1-\frac{2p}{\left(\cosh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)-\sinh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)+p\right)}\right)^2\right),$$

$$u_{32}(x,t)=\pm\beta\sqrt{-\theta}\left(1-\frac{2\left(\sinh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)+\cosh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right)}{\left(\sinh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)+\cosh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)+q\right)}\right),$$

$$v_{32}(x,t)=\frac{\theta}{\lambda}\left(1-\beta^2\left(1-\frac{2\left(\sinh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)+\cosh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)\right)}{\left(\sinh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)+\cosh_A\left(\beta kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)+q\right)}\right)^2\right).$$

(8) When $\beta=\varepsilon$, $\sigma=\mu\varepsilon(\mu\neq 0)$ and $\alpha=0$,

$$u_{33}(x,t)=\pm\varepsilon\sqrt{-\theta}\left(1+\frac{2\mu pA^{\left(\varepsilon kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)}}{p-\mu qA^{\left(\varepsilon kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)}}\right),$$

$$v_{33}(x,t)=\frac{\theta}{\lambda}\left(1-\varepsilon^2\left(1+\frac{2\mu pA^{\left(\varepsilon kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)}}{p-\mu qA^{\left(\varepsilon kx\pm\frac{2\sqrt{\theta}}{LnA}t\right)}}\right)^2\right).$$

It is noteworthy that, all the obtained results have been checked with Maple by putting them back into the original equation and they are found to be correct.

4. Discussion of the results

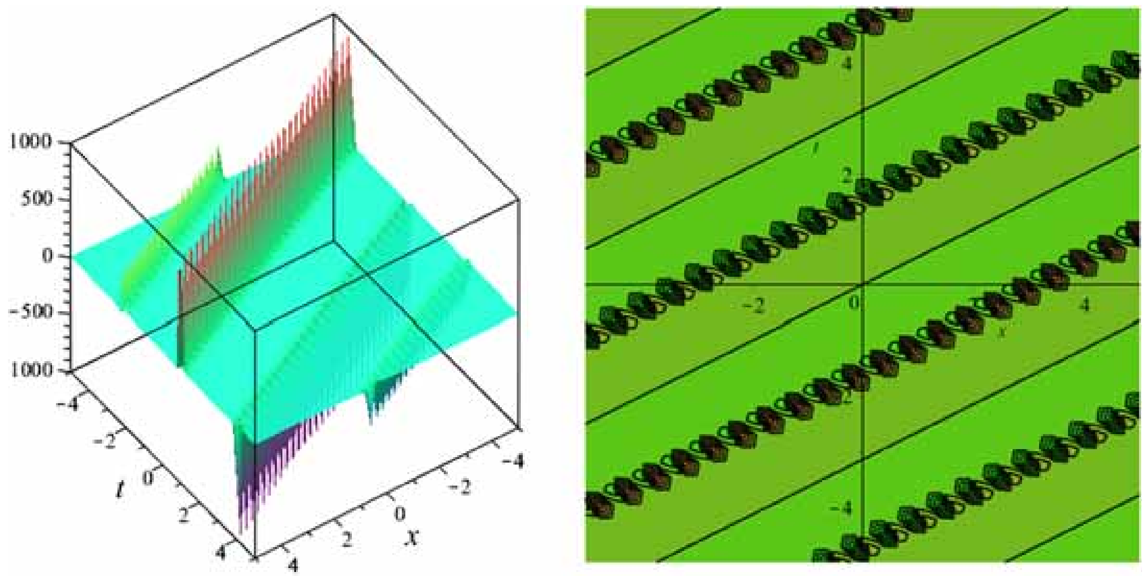
In this investigation, we present a comparison between our results and that obtained by other researchers using various methods. We shortlist our comparison for each pattern as follows:

(1) Solutions of CKO equation (1), namely family (6) are similar to eqs (19)–(26) in Basher *et al* [6] when $p = q = 1, \theta = d, \beta = 1, A = e, k = 2/\sqrt{1-4\alpha\sigma}$ where Bashar *et al* used the travelling wave method (the tanh-function method and the extended tanh-function method).

(2) Solutions of CKO equation (1), namely family (6) are similar to eqs (19)–(26) in Yel *et al* [8] when $p = q = 1, \theta = 1, \beta = 1, A = e, \mu = k\sqrt{1-4\alpha\sigma}$ where Yel *et al* used the travelling wave method (the sine-Gordon expansion method).

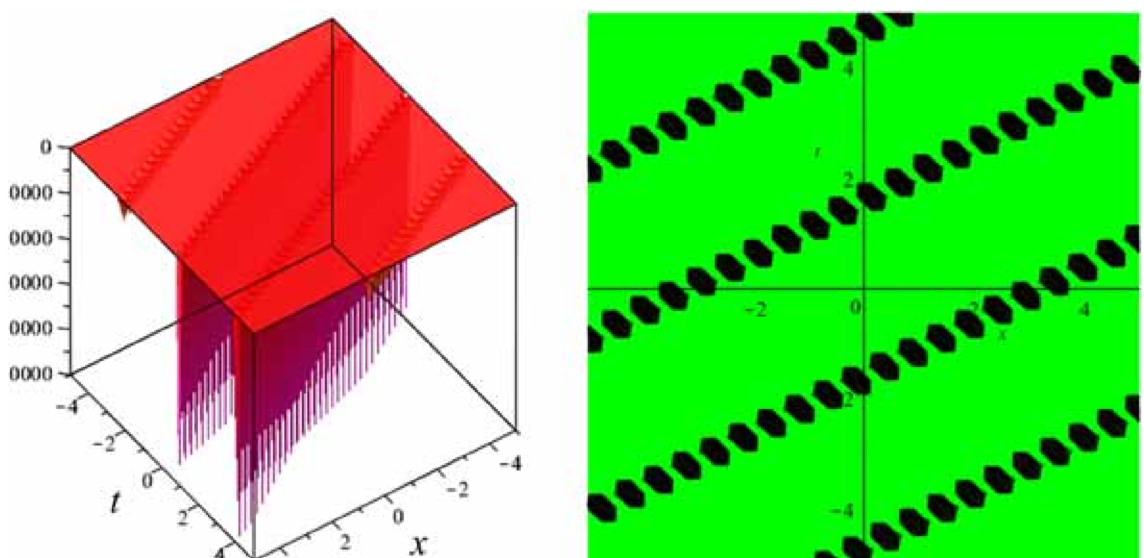
5. Physical explanation

In this section, we shall illustrate the application of the results established above.



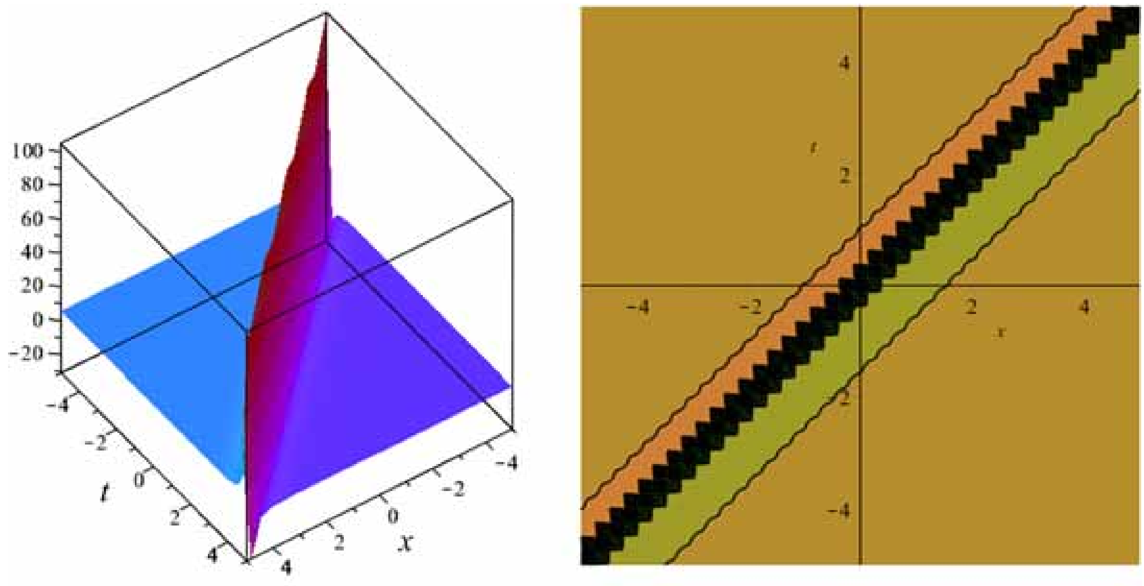
(a) 3D plot (b) contour plot

Figure 1. Shape of u_1 with $\beta = 3, \alpha = 2.5, \sigma = 1, A = e, \theta = -1, k = 1, p = 1, q = 1$ and $-5 \le x, t \le 5$.



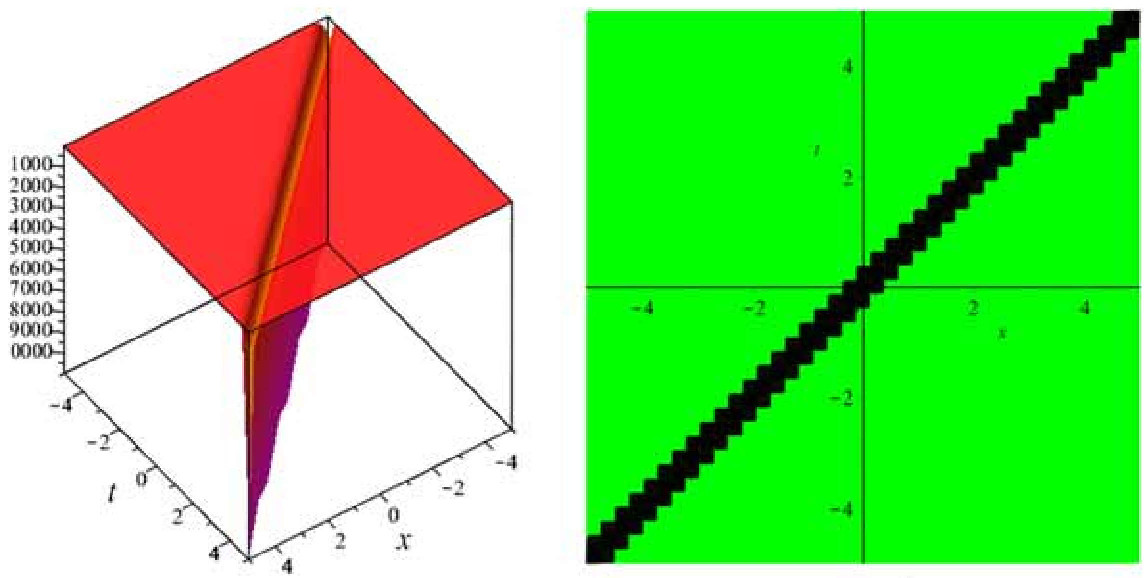
(a) 3D plot (b) contour plot

Figure 2. Shape of v_1 with $\beta = 3, \alpha = 2.5, \sigma = 1, A = e, \theta = -1, k = 1, p = 1, q = 1$ and $-5 \le x, t \le 5$.



(a) 3D plot (b) contour plot

Figure 3. Shape of u_{10} with $\beta = 4, \alpha = 1, \sigma = 2, A = e, \theta = 2, k = 1, p = 0.95, q = 0.9$ and $-5 \leq x, t \leq 5$.



(a) 3D plot (b) contour plot

Figure 4. Shape of v_{10} with $\beta = 4, \alpha = 1, \sigma = 2, A = e, \theta = 2, k = 1, p = 0.95, q = 0.9$ and $-5 \leq x, t \leq 5$.

In this paper, with the aid of the symbolic software program, we applied the NEDAM to investigate the solutions of the new CKO equation. We successfully constructed some new complex hyperbolic function solutions that have not been published elsewhere.

Moreover, we came to the conclusion that the newly obtained hyperbolic function solutions in this study may help to explain some complex physical aspects in the nonlinear physical sciences and are related to such physical properties. To the best of our knowledge, the application of NEDAM has not been published elsewhere.

The graphical illustrations of the solutions are depicted in figures 1–6 with the aid of the commercial software Maple.

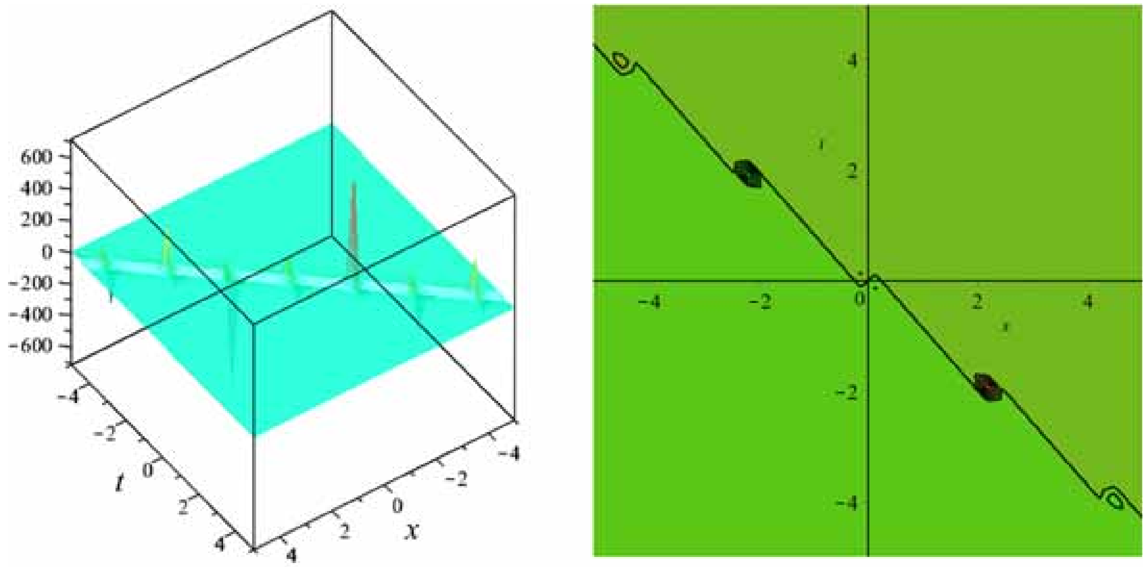


Figure 5. Shape of u_{33} with $\mu = 1, \varepsilon = 1.5, A = 2.9, \theta = 2.5, k = 1.75, p = 1, q = 1$ and $-5 \leq x, t \leq 5$.

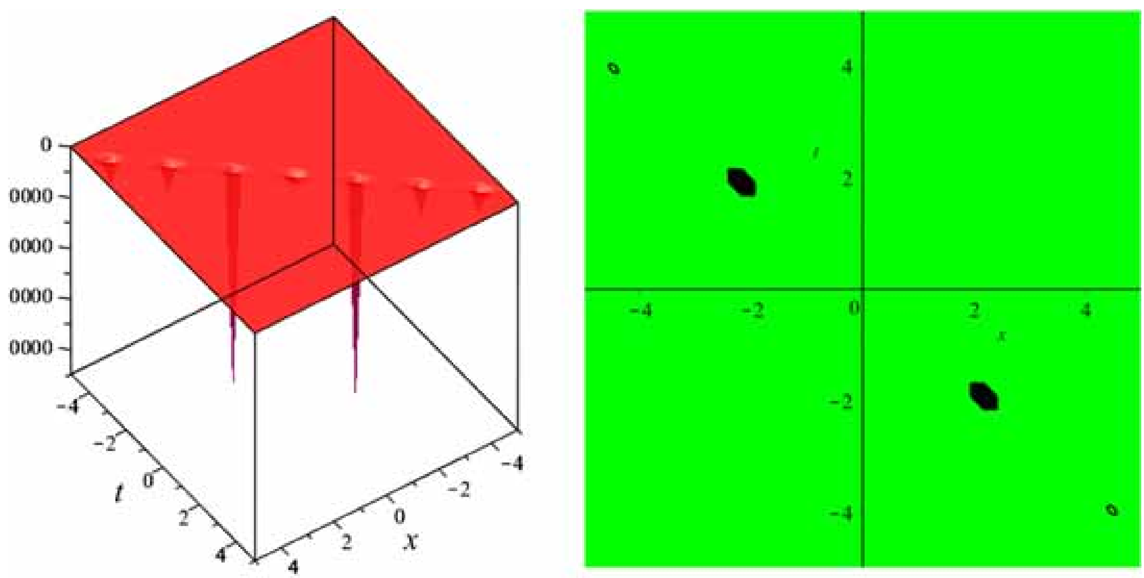


Figure 6. Shape of v_{33} with $\mu = 1, \varepsilon = 1.5, A = 2.9, \theta = 2.5, k = 1.75, \Omega = 2.25, p = 1, q = 1$ and $-5 \leq x, t \leq 5$.

6. Conclusions

In this study, the new approach of the expansion method, namely the NEDAM, has successfully been implemented to investigate the new CKO equation. The equation considered in this study has been discussed using not only this fairly important method but also by different important methods. Many scientists will cite this study and these results will be used in many studies. We have emphasised in this paper that eq. (5) is powerful and can be effectively used to discuss NLPDEs and related models in scientific fields. The results are new and reported for the first time. These new closed form of solutions show the power, capability, reliability and fruitfulness of this method. We can also solve other NLPDEs which are involved in mathematical physics and many other branches of physical sciences by this technique.

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