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Optimised wave perturbation for the linear instability of magnetohydrodynamics in plane Poiseuille flow

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Abstract. In this work, linear stability of an electrically conductive fluid experiencing Poiseuille flow for minimum Reynolds value under a normal magnetic field is analysed using the Chebyshev collocation method. The neutral curves of linear instability are derived by utilising Qualitat and Zuverlassigkeit (QZ) method. Instability of the magnetohydrodynamics for plane Poiseuille flow is introduced by solving the generalised Orr–Sommerfeld equation to determine the growth rates, wave number and spatial shapes of the eigenmodes. To solve linear problems, we use numerical methods which help us at each time step of the simulation, uncoupled by physical processes, which can improve the computational performance. This article provides the stability and error analysis, presents a concise study of the Poiseuille flow, and produces computational tests to support the given theory.

Keywords. Computational fluid dynamics; instability; magnetohydrodynamics; Chebyshev collocation method; electrically conducting fluid.

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1. Introduction

Many surveys have been made on the effect of transverse magnetic field on the stability of plane Poiseuille flow and Couette flow between two parallel flat plates is investigated using the Chebyshev collocation method with the Qualitat and Zuverlassigkeit (QZ) algorithm for Oldroyd-B fluid [1,2]. The critical values of Reynolds number (Re), wave numbers (k) and wave speed are computed in [1,2]. The stability of the magnetohydrodynamic (MHD) flow in a duct with conducting walls is introduced in [3] for a homogeneous and constant static magnetic field. The temporal growth and spatial distribution are obtained using the direct and adjoin problems by adding some perturbations where weak jetliners appear near the sidewalls as a centrist magnetic field and the velocity of the jetliner increases by increasing the magnetic field intensity. The response of the transverse magnetic field and the proportion of the duct flow are considered. The flow in the duct is stable at the strong or weak magnetic field, but appears unstable at a moderate magnetic field, and the stability remains unchanged with the aspect ratio of the duct. The instability of the duct flow is related to the exponential growth of perturbations derived by the generated jetliners. The transient growth of perturbations is analysed on the optimal perturbation in the form of streamwise vortices and localised within the side-wall layers. In comparison, the Hartmann layer normal direction of the magnetic field is irrelevant to the stability of the flow. The MHD flow effects are analysed by Sultan *et al* [4]. The MHD flows occur at large Re, but the liquid metals, the case considered in this work, has smaller Re. Two decoupled methods based on the artificial compression method and the partitioned method for the MHD flows at low magnetic Re are constructed. Yao *et al* [5] studied the stability, error analysis and provided computational tests to support the theory. The plasma flow has been generated and derived in the plasma dynamic experiment with magnetic Re using 10 thermally emissive lanthanum hexaboride cathodes which generate torque in helium and argon plasmas [6]. Measurements of the plasma velocity are introduced: (i) edge-localised flow drive using multicusp boundary field and (ii) volumetric flow drive using an axial-Helmholtz field. Radial velocity profiles show that the edge-driven flow is established

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through ion viscosity, but is limited by the drag force. The volumetric drive is shown to produce larger shear velocity and has the correct flow profile for studying the magnetorotational instability. Chuanfei [7] studied the role of plasmoid instability in the MHD turbulence in evolving current sheets, due to its effects on the resultant fast magnetic reconnection. The temporal linear stability analysis to trace the time evolution of an infinitesimal and two-dimensional (2D) disturbance has been imposed on the flow [8]. The electrically conducting fluid under the influence of the applied magnetic field is considered to investigate the instability of 2D flows [9]. Furthermore, the stability of the fully developed flow of an electrically conducting fluid driven by pressure when a uniform external magnetic field is perpendicular to the plates are applied theoretically [10,11]. The linear instability for a small value of Re has been introduced for the case of plane Couette flow [12]. Pekeris [13] has presented Couette and plane Poiseuille flow with some details. Ho and Denn [14] have found the stability of low Re using a numerical method based on the shooting method. Moreover, they have observed that at low Re no instabilities occur, but the numerical method led to an artificial instability. Recently, many researchers published their works on different aspects of MHD in cross fluid flow [15–22]. Hussain [23] has investigated the MHD instability between parallel plates of the conducting fluid where increase in the Hartmann number (Ha) leads to increases in the flow stability.

The objective of this work is to analyses the linear instability of Poiseuille flow at low Re of electrically conductive fluid under the influence of transverse magnetic field. The Orr–Sommerfeld equation of fourth-order linear stability is investigated numerically using the collocation technique by Chebyshev's polynomials. The numerical computations are implemented using MATLAB 2014. We study the case of 2D MHD turbulence using high-resolution direct numerical simulations. At sufficiently large Re, the combined effects of dynamic alignment and turbulent intermittency lead to a copious formation of plasmoids.

The remaining work is organised as follows: In §2, the 2D governing Navier–Stokes (NS) equation with appropriate boundary conditions are introduced. In §3 the Chebyshev polynomial expansion is investigated and the eigenvalue problem is solved numerically. In §4, results and discussions are presented. Furthermore, the Chebyshev method for the non-linear dynamical equilibrium of electrical fluid is considered and different cases are analysed with external forces. Moreover, the numerical results refer to the effect on the evolving stability solutions. Finally, the conclusion is related to the practical application of the results obtained in §5.

2. Mathematical formulation

To analyse the MHD instability, consider the electrically conducting, incompressible, viscous and fully developed fluid in parallel planes driven by pressure p for the transversal magnetic field. Distance between parallel plates is $2L$. The flow is along the y-axis, and the applied magnetic field is normal to the direction of flow (the magnetic field is along the x-axis). The magnetic field is supposed to be constant as shown in figure 1. Dimensionless NS equations are described as follows:

$$\nabla \cdot \vec{V} = 0, \tag{1}$$

$$\frac{D\vec{V}}{Dt} = -\nabla P - \frac{1}{\mathrm{Re}}(g_x - \Delta\vec{V}) + N(\vec{J} \times \vec{B}), \tag{2}$$

$$\nabla \cdot \vec{J} = 0 \tag{3}$$

and

$$\vec{J} = -\nabla\phi + \vec{V} \times \vec{B}. \tag{4}$$

The boundary conditions for the above system is described by the following equation:

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial \phi}{\partial x} = u = \text{ at } \; x = \pm L. \tag{5}$$

Here, $V = (u, v, w)$ is a three-dimensional velocity vector, $j = (j_x, j_y, j_z), \phi$, p and $B = (1, 0, 0)$ represent the current density, electric potential, pressure and external magnetic field along the x-axis, respectively. We assume that all variables are independent of coordinate z and are only functions of x and y. The operators are represented as $\nabla = (\partial_x, \partial_x)$ and $\Delta = (\partial_x^2, \partial_x^2)$. B is the magnitude of the magnetic field along the x-direction. The problem is defined by Reynolds and interaction parameter $N = \mathrm{Ha}^2/\mathrm{Re}$ where Ha represents Hartmann number ($\mathrm{Ha} = BL\sqrt{\sigma/\rho v_0}$). The parameters appearing in eq. (2) are: Re $(= v_0 L/v)$ which is the ratio of inertial forces and viscous forces and interaction parameter $N (= \sigma L B^2/\rho\vartheta_0)$ which is the ratio of electromagnetic forces and inertial forces. Here ϑ_0 is the kinematic viscosity.

The plane Poiseuille flow perturbation for two dimensions can be written as

$$u = u', \quad v = v_0 + v', \quad p = p^* - g_x/\mathrm{Re} + p'. \tag{6}$$

By substituting these values in eqs (1)–(4) we get

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{7}$$

such that x and y components of eq. (2) are described as

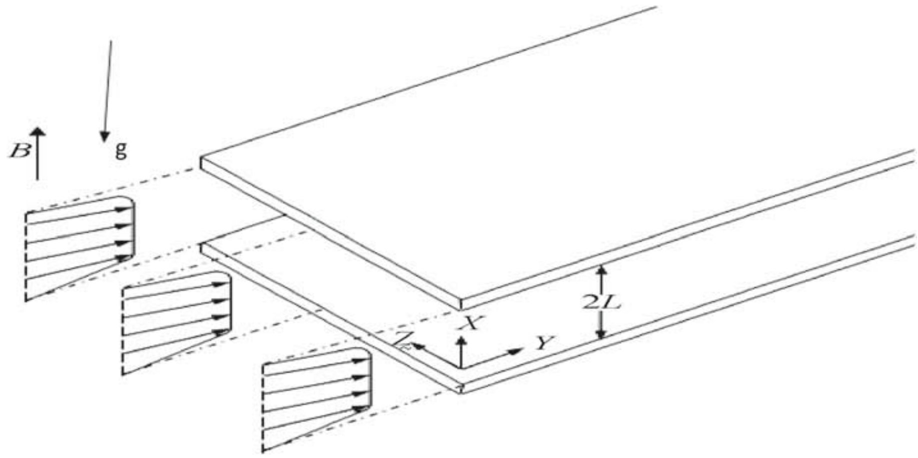

Figure 1. The geometry of the problem.

$$\frac{\partial u'}{\partial t}+v_0\frac{\partial u'}{\partial y}=-\frac{\partial p'}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u'}{\partial x^2}+\frac{\partial^2 u'}{\partial y^2}\right), \tag{8}$$

$$\frac{\partial v'}{\partial t}+u'\frac{\partial v_0}{\partial x}+v_0\frac{\partial v'}{\partial y} =-\frac{\partial p'}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v'}{\partial x^2}+\frac{\partial^2 v'}{\partial y^2}\right)-Nv', \tag{9}$$

$$\frac{\partial^2\phi'}{\partial x^2}+\frac{\partial^2\phi'}{\partial y^2}=0. \tag{10}$$

The characteristics of basic velocity for this problem is described as follows:

$$v_0=\frac{g}{\mathrm{Ha}^2}\left(\frac{\cosh(\mathrm{Ha}x)}{\cosh(\mathrm{Ha})}+1\right). \tag{11}$$

Small perturbations for normal modes u', v', p', ϕ' can be described by the following relation:

$$\left(u',v',p',\phi'\right) =\left[\hat{u}(x),\hat{v}(x),\hat{p}(x),\hat{\phi}(x)\right]\mathrm{e}^{i(kx-\omega t)}. \tag{12}$$

Here k is an arbitrary real wave number and $\omega=\omega_{\mathrm{r}}+\omega_{\mathrm{i}}$ is the phase velocity. The real part ω_{r} represents the amplification velocity and the imaginary part ω_{i} is known as the frequency in oscillation form. All variations with parallel coordinate y and time t are assumed to be contained in $\mathrm{e}^{i(kx-\omega t)}$. Substituting eq. (12) into (7)–(10), we obtain

$$D\hat{u}+ik\hat{v}=0, \tag{13}$$

$$-i\omega\hat{u}+ikv_0\hat{u}=-D\hat{p}+\frac{1}{\mathrm{Re}}\left(D^2-k^2\right)\hat{u}, \tag{14}$$

$$-i\omega\hat{v}+ikv_0\hat{v}+\hat{u}Dv_0 =-ik\hat{p}+\frac{1}{\mathrm{Re}}\left(D^2-k^2\right)\hat{v}-N\hat{v}, \tag{15}$$

where $D=\mathrm{d}/\mathrm{d}x$. The linear stability equations (13)–(15) are reduced to

$$D^4\hat{u}+(-\mathrm{Re}N-2k^2+ik\,\mathrm{Re}\,v_0)D^2\hat{u} +(k^4+ik\,\mathrm{Re}\,D^2v_0+ik^3\mathrm{Re}v_0)\hat{u} =-\omega(i\mathrm{Re}D^2\hat{u}-ik^2\mathrm{Re}\hat{u}), \tag{16}$$

$$(D^2-k^2)\hat{\phi}=0. \tag{17}$$

The boundary conditions for the above system is reduced to

$$\hat{u}=0 \quad\text{and}\quad D\hat{u}=D\hat{\phi}=0 \quad\text{at}\quad x=\pm L. \tag{18}$$

The following section is set to solve the eigenvalue problem numerically.

3. Chebyshev polynomial expansion

To solve Orr–Sommerfeld equation (16) with boundary conditions (18) by 'spectral method' with expansion velocity in Chebyshev polynomial is defined by

$$T_n(\cos\theta)=\cos(n\theta). \tag{19}$$

For all non-negative integers n as in Hamming [24] and Fox [25], the expansion of the above equation in a Chebyshev series can be written as follows:

$$u_j=\sum_{n=0}^{N_1}a_nT_{nj}, \tag{20}$$

$$u'_j=\sum_{n=0}^{N_1}a_nT'_{nj}, \tag{21}$$

$$u''_j=\sum_{n=0}^{N_1}a_nT''_{nj}, \tag{22}$$

$$u_j^{(4)} = \sum_{n=0}^{N_1} a_n T_{nj}^{(4)}, \tag{23}$$

where $j = 1, 2, 3, \ldots, N_1$ and $T_{nj}, T'_{nj}, T''_{nj}, \ldots, T_{nj}^{(4)}$ represent the Chebyshev polynomial and its derivatives at collocation point j and a_n are the coefficients of the series. So, eq. (16) can be written as follows:

$$T_n(\cos\theta)$$

$$= \cos(n\theta) . \sum_{n=0}^{N_1} \left\{ T^{(4)} + \left[-RN - 2k^2 - ikv_0R\right] T'' + \left[k^4 + ik^3Rv_0 + ikRD^2v_0\right] T \right\} a_n$$

$$= \omega \sum_{n=0}^{N_1} \left\{-iR\left[T'' - kT\right]\right\} a_n. \tag{24}$$

Equation (24) is the generalised Orr–Sommerfeld equation which satisfies the expansion coefficient matrix $X = (a_0, a_1, a_2, \ldots, a_n)$ and can be obtained by the following equation:

$$AX = \omega BX. \tag{25}$$

This spectral discretisation process yields a generalised eigenvalue problem with nonsymmetrical $(N_1 + 1)$ by $(N_1 + 1)$ square matrices A and B.

Table 1. The influence of magnetic field (Ha) and wave speed $\omega = \omega_r + \omega_i$.

Ha	ω_r	ω_i
1	0.143888597116726	−0.0350815192418176
5	0.0309187179849157	−0.0295036548515782
10	0.00913243565934038	−0.084498373143767
15	0.00422049838944663	−0.176853602456613
20	0.00241158809706739	−0.305014153746784
25	0.00155639768806923	−0.468836634238067
30	0.00108647370099092	−0.668277873781161
35	0.000801066334470736	−0.903320376674737
40	0.000614900812687241	−1.1739557608002

Table 2. The influence of wave number (k) and wave speed ($\omega = \omega_r + \omega_i$).

k	ω_r	ω_i
1	0.143888597116726	−0.0350815192418176
5	1.6435264493599	−0.0774765281108596
10	3.42430389348372	−0.14710540516576
15	5.19068217260045	−0.277567917760358
20	6.95180363491261	−0.458813415204212
25	8.71067913195485	−0.689578090846745
30	9.61393497736896	−1.22717170408245
35	9.40057432964323	−1.62146637687316
40	9.44130990285073	−2.05408663610492

4. Results and discussions

The eigenvalues with least part are presented in table 1 for unique values of Re = 1000, $k = 1$ and $g = 1$, for various Ha values. Table 1 shows that, the absolute value of ω_i increases gradually, when Ha increases. From our assumption, the magnetic field is stabilised in the given flow field, and so the flow will be stabilising by increasing the value of the magnetic parameter.

For Ha = 1, Re = 1000, $g = 1$ and for various values of k, the eigenvalues are tabulated in table 2. We observed that the imaginary part of ω is negative. We observed that increasing the disturbance in k may have destabilising effect in the flow field. $\mathrm{Re_c}$ increases with increase in magnetic parameter Ha.

Figure 2 represents the natural stability curve as parabola for k and Re. When Re becomes smaller than its critical number, the flow is stable against perturbation of each k. The critical value of k corresponds with $\mathrm{Re_c}$ and the unstable area is getting smaller with increasing Ha. The resulting force from the magnetic field applies damping on the fluid flow. The transversal magnetic field stabilises Poiseuille flow of the electrically conductive fluid.

Figure 3 shows the growth rate of ω vs. Re for different values of Ha. Temporal growth rate curves with Re for increasing Ha from 0.9 to 3.3, is shown in figure 3. Maximum growth in this figure is observed at Ha = 3.3 and Re = 4200 at $k = 1.0$. Ha values steadily increase with the growth rate.

Figure 4 shows the growth rate curve of ω vs. Re. Temporal growth velocities with Ha are shown for various ranges of Re = (905–5405). We conclude that Ha correlates with smaller growth rate. The magnetic field suppresses the instability of Couette flow. Each curve in the figure has one global maximum for a distinct magnetic field. As magnetic field decays the disturbances, the weaker magnetic field will give maximum Re.

Figure 5 shows the growth rate curve of ω vs. k, for different Re = 1405–5405 with temporal growth frequencies. In this figure, each curve has two maxima for various values of Re. The longest wave curve is observed for Re = 2905 near zero growth rate. It also shows that higher Re leads to a greater growth rate. By increasing Re, the disturbances will draw more energy from the mean flow and the disturbances will grow swifter. In figure 6 the outside region stands for the stable state while inside the neutral stability curves is the unstable region.

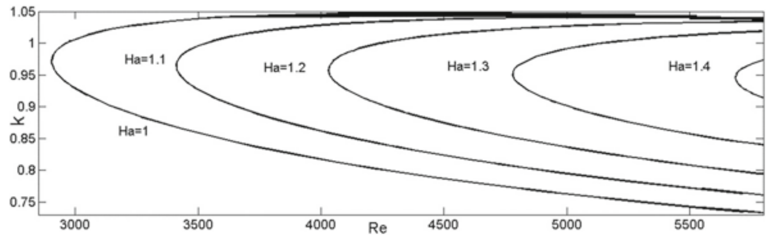

Figure 2. Neutral stability curve for various values of Ha.

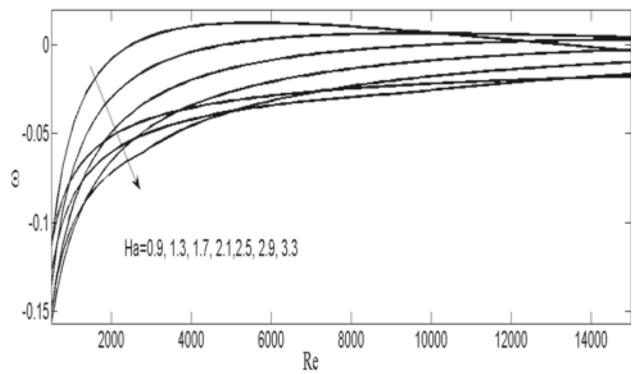

Figure 3. Growth rate curves of ω vs. Re for different values of Ha.

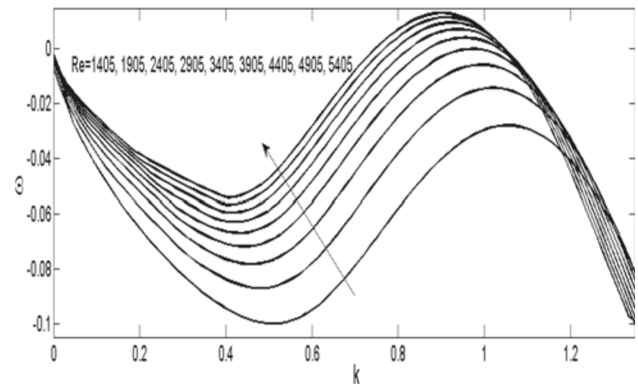

Figure 5. Growth rate curves of ω vs. k for different values of Re.

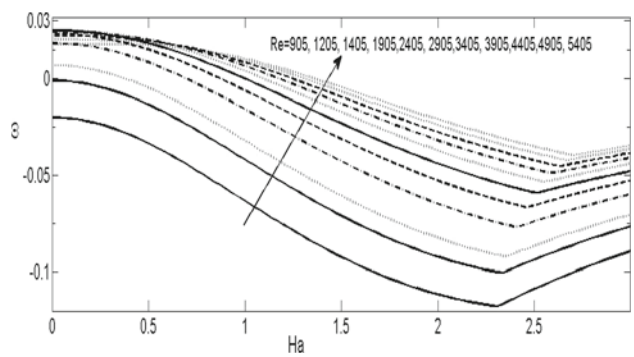

Figure 4. Growth rate curves of ω vs. Ha for different values of Re.

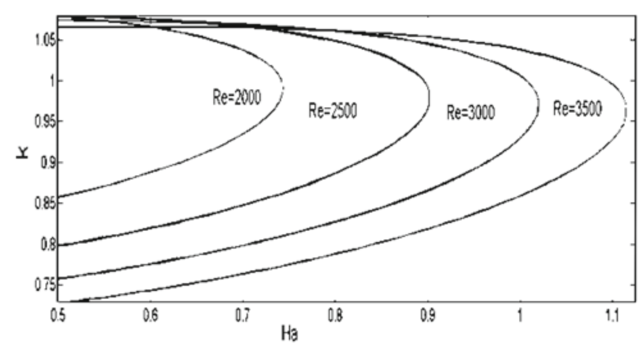

Figure 6. Neutral values of for various values of Re.

The curves in figure 6 show that a specific flow rate in the magnetic field force decreases the growth rate of perturbations when Ha corresponds to the great stable region.

As shown in figure 6 the flow will become unstable as Re increases, but increasing Ha has a great effect of making the flow more stable. As Re increases, the area between the neighbouring curves decrease steadily. Therefore, for a given flow of electrically conductive fluid between parallel plates, it is possible to stabilise the strength of the magnetic field by increasing the normal magnetic field.

In figure 7, the stream-wise disturbance is shown in Re with Ha for $k = 1$. The stable and unstable areas are apart by two neutral instability curves. Area B between two curves is observed as an unstable region, while areas A and C are stable. From the figure we observe that for specific k, perturbations grow in region B between two curves and will decrease in A and C regions. It is observed that there is a coupling relationship between Ha, Re and k of perturbations which will result in instability of flow. There is an important role of the magnetic field in $\mathrm{Re_c}$ and wavenumber.

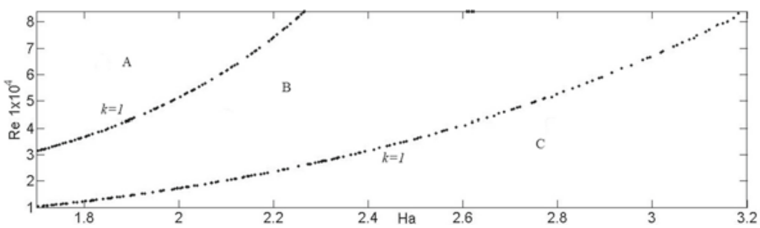

Figure 7. Stream-wise disturbance for Re vs. Ha for $k = 1$.

5. Conclusions

Consider the plane Poiseuille flow with MHD effects. The linear stability analysis in the transverse specific magnetic field is observed with low Reynold values using the Chebyshev collocation technique and the growth of perturbation for various k. The perturbation effect is very little in the span-wise direction, which raises the critical values for different Hartmann values. The pairing influences of Hartmann values, k and Reynold value are investigated. Detailed study of linear stability is presented. It is observed that critical Reynold value rises rapidly as Ha increases, so that the magnetic field stabilises the given flow and the stabilising effect increases by increasing Hartmann values. The magnetic field decreases the unstable region. As Re increases, the growth rate will increase, which makes the flow unstable.

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