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Numerical and perturbation solutions of third-grade fluid in a porous channel: Boundary and thermal slip effects

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Abstract. The steady flow of a third-grade fluid due to pressure gradient is considered between parallel plane walls which are kept at different temperatures. The space between the plane walls is assumed to be a porous medium of constant permeability. The viscosity of the fluid is taken as constant as well as a function of temperature. It is further assumed that the fluid may slip at the wall surfaces. The consequence of this assumption results in non-linear boundary conditions at the plane walls. The temperature field is also supposed to satisfy thermal slip condition at the walls. The governing equations are modelled under these assumptions and the approximate solution is obtained using the perturbation theory. The skin friction coefficient is a decreasing function of slip parameters in the case of temperature-dependent viscosity models while no variation is noted for the case of constant viscosity via boundary slip parameter. The heat transfer rate increases with the boundary slip parameter and decreases with the thermal slip parameter. The validity of the approximated solution is checked by calculating the numerical solution as well. The absolute error is calculated and listed in tabular form in the case of constant and temperature-dependent viscosity via boundary and thermal slip parameters. The influence of various emerging parameters on flow velocity and temperature profile is discussed through graphs.

Keywords. Third-grade fluid; perturbation solution; slip effects; channel flow; porous medium.

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1. Introduction

Research on non-Newtonian fluids has gained momentum in the past few decades because of the importance of these fluids in technological and industrial processes. Various constitutive equations have been proposed in literature to describe the behaviour of non-Newtonian fluids. Amongst these, differential constitutive equations have received much attention. Differential constitutive models which are widely used to describe polymer behaviour are: second-order model, third-order model, Maxwell model, Oldroyd-B model, Giesekus model etc. Second- and third-order fluid models are special cases of the models proposed by Rivlin and Ericksen [1]. A thermodynamic study of second- and third-order fluids was performed by Fosdick and Rajagopal [2] and Dunn and Rajagopal [3]. In these studies, some constraints have been put on the model parameters for thermodynamics compatibility. Some recent studies using these models in different scenarios were conducted in refs [4–17].

Tripathi [18] designed a mathematical model to investigate variable viscosity on swallowing of food bolus in a finite esophagus. He took food bolus as a viscous fluid with a variable viscosity and concluded that the effort needed to swallow a low viscous fluid is less compared to that when the fluid has higher viscosity. The electric, magnetic and thermal radiation effects on the peristaltic flow of Jaffery nanofluid was analysed by Prakash *et al* [19]. They transformed the highly nonlinear system of differential equations into a simple form by using dimensionless analysis and lubrication theory and presented the closed form solution. Sharma *et al* [20] used the Mathematica software to find the computational results of electroosmotic-driven flow altered by peristaltic wave propulsion under the consideration of wavy velocity. Their results can be helpful in clinical implications such as: drug delivery, cell therapeutics

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etc. Jayavel *et al* [21] used the perturbation method to find analytical solution of electro-osmotic flow of biorheological fluids in a vertical wavy channel. They observed that the skin friction and local Nusselt number vs. Biot number have opposite behaviour. Akber *et al* [22] used the symbolic software (Mathematica) to obtain the closed form solution of MHD peristaltic flow of copper–water nanofluid in a vertical curve geometry. They considered three types of nanoshaped particles namely, brick, platelet and cylindrical. They concluded that the temperature of the nanofluid is greater for the platelet nanoparticles and lower for brick nanoparticles. The mathematical model of peristaltic-driven flow of the nanofluid was presented by Prakash *et al* [23] to account for the magnetic and radiation effects. They performed homotopy perturbation method to obtain the analytical expression of velocity and temperature. They validated their homotopy perturbation solution with the help of Bvp4c. Akbar *et al* [24] picked various shapes of nanoparticles to discuss the MHD peristaltic transport flow in a horizontal curve channel. They used the symbolic software (Mathematica) and obtained exact solution of velocity and temperature. Tripathi *et al* [25] analysed the peristaltic flow of Jaffrey fluid in a circular pipe under uniform magnetic field. They used numerical approach to obtain numerical solution of the problem. A theoretical analysis of the magneto-biomimetic nanofluid in a vertical sinusoidal wavy channel under variable viscosity and thermal radiation was done by Prakash *et al* [26]. They used Maple and Mathematica software to evaluate the solution of the given problem. An analytical study of blood flow in a trapped porous channel under the effects of radiation was investigated by Prakash *et al* [27]. They used convective and slip boundary conditions to capture the performance of motion of the particles that may be beneficial for cardiac surgery. Akbar *et al* [28] studied the creeping Newtonian fluid in a ciliated porous tube. The developed nonlinear system of differential equations was solved with the help of Mathematica and exact solution is obtained. Prakash *et al* [29] presented a theoretical analysis of the electro-osmotic peristaltic flow under thermal radiation, electric current and magnetic field in a microchannel. The nonlinear system was solved with the help of Matlab software and the numerical solution was presented. Bhatti *et al* [30] analysed the Hall effects on non-Newtonian flow of hyperbolic tangent fluid over a porous stretching surface. They used the shooting method to find the solution of nonlinear system of equations and presented variation of heat transfer rate and Sherwood number vs. the involved physical parameter in tabular form. Exact solution of the peristaltic flow of Jefrey fluid within a porous duct under partial slip boundary conditions was analysed by Ellahi *et al* [31]. Ghosh and Mukhopadhyay [32] reported the boundary layer flow of nanofluid over a stretching surface under the combined boundary and slip effects.

Motivated by the wide usage and applicability of third-grade model, in this study, we shall focus on slip effects and heat transfer analysis of channel flow of this model. In fact, work presented in this dissertation is an extension of the result due to Aksoy and Pakdemirli [4]. But, we have used wall slip and thermal slip conditions in our analysis unlike the analysis performed by Aksoy and Pakdemirli [4]. The problem is formulated in the form of differential equation and boundary conditions. An approximate as well as numerical solution of the problem is obtained for the case of constant and temperature-dependent viscosity. A discussion about the obtained results is presented at the end.

2. Problem formulation

2.1 *Flow geometry*

We consider a subclass of Rivlin–Ericken fluids, i.e. a third-grade fluid filling the porous media between two parallel plates at different temperature (figure 1).

$$\bar{u} = -\frac{l}{\mu}\left(\mu\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right) + 2\beta_3\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right)^3\right)_{\bar{y}=h},$$

$$\bar{\theta} = \theta_{\mathrm{w}} - k_1\left(\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\bar{y}}\right)_{\bar{y}=h},$$

where $\bar{u}$ is the dimensional form of velocity, μ is the viscosity, β_3 is the third-grade parameter, $\bar{\theta}$ is the dimensional temperature, $\bar{y}$ is the dimension of space coordinate, h is the distance between parts, θ_{w} is the temperature of the wall and k is the permeability of the porous medium.

2.2 *Governing equations*

The assumptions that flow is in steady state and unidirectional imply that $\bar{V}$ and $\bar{\theta}$ must be of the following form:

$$\mathbf{V} = (\bar{u}(\bar{y}), 0, 0), \quad \bar{\theta} = \bar{\theta}(\bar{y}). \tag{1}$$

The dimensional form of momentum and energy is given by [4]

$$\frac{\mathrm{d}\bar{\mu}}{\mathrm{d}\bar{y}}\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}} + \bar{\mu}\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}\bar{y}^2} + 6\beta_3\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right)^2\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}\bar{y}^2} - \frac{\varphi\bar{u}}{k}\left[\bar{\mu} + 2\beta_3\left(\frac{\mathrm{d}\bar{u}}{\mathrm{d}\bar{y}}\right)^2\right] = p_0, \tag{2}$$

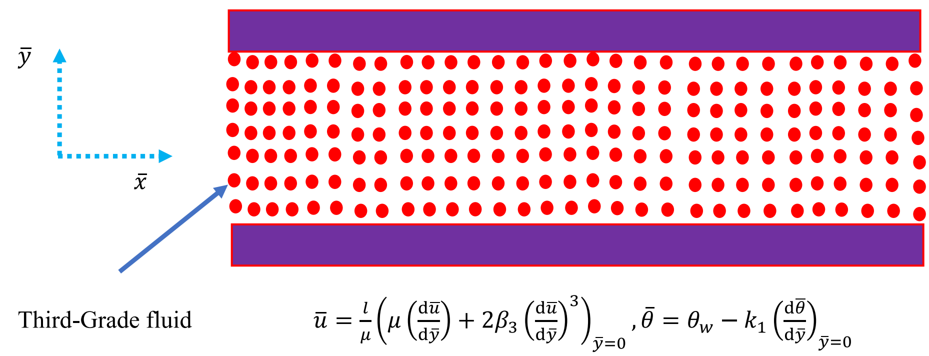

Figure 1. Geometry of the problem.

$$k_{\rm th}\frac{d^2\bar{\theta}}{d\bar{y}^2} + \bar{\mu}\left(\frac{d\bar{u}}{d\bar{y}}\right)^2 + 2\beta_3\left(\frac{d\bar{u}}{d\bar{y}}\right)^4 = 0, \tag{3}$$

$$u = \frac{l}{\mu}S_{xy},$$

where

$$S_{xy} = 2\beta_3\left(\frac{d\bar{u}}{d\bar{y}}\right)^3 + \bar{\mu}\left(\frac{d\bar{u}}{d\bar{y}}\right), \tag{4a}$$

$$\bar{\theta}(0) = \theta_l + k_1\left(\frac{d\bar{\theta}}{d\bar{y}}\right)_{\bar{y}=0},$$

$$\bar{\theta}(h) = \theta_u - k_1\left(\frac{d\bar{\theta}}{d\bar{y}}\right)_{\bar{y}=h}, \tag{4b}$$

where φ is the porosity of porous space, p_0 is the pressure, $k_{\rm th}$ is the thermal conductivity, u is the dimensional form of the velocity component, θ_l is the temperature of the lower wall, θ_u is the temperature of the upper wall. The non-dimensional parameters are given by

$$a_1 = \frac{\beta_3 U^2}{\mu_0 h^2}, \quad a_2 = \frac{\varphi h^2}{k}, \; a_3 = \frac{p_0 h^2}{\mu_0 U}, \quad u = \frac{\bar{u}}{U},$$

$$y = \frac{\bar{y}}{h}, \quad \Gamma = \frac{\mu_0 U^2}{k_{\rm th}(\theta_u - \theta_l)}, \quad \theta = \frac{\bar{\theta} - \theta_l}{(\theta_u - \theta_l)}, \tag{5}$$

where a_1, a_2 and a_3 are respectively the dimensionless parameter of non-Newtonian behaviour, dimensionless parameter related to the porous media, pressure gradient parameter, μ_0 is the constant viscosity, U is the arbitrary reference velocity and Γ is the viscous heating parameter. In view of above normalised quantities, eqs (1)–(4) take the following form:

$$\frac{d\mu}{dy}\frac{du}{dy} + \mu\frac{d^2u}{dy^2} + 6a_1\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} - a_2u\left[\mu + 2a_1\left(\frac{du}{dy}\right)^2\right] = a_3 \tag{6}$$

$$\frac{d^2\theta}{dy^2} + \mu\Gamma\left(\frac{du}{dy}\right)^2 + 2\Gamma a_1\left(\frac{du}{dy}\right)^4 = 0. \tag{7}$$

The boundary conditions read as

$$u(0) = \gamma\left[\frac{2a_1}{\mu}\left(\frac{du}{dy}\right)^3 + \left(\frac{du}{dy}\right)\right]_{y=0},$$

$$u(1) = -\gamma\left[\frac{2a_1}{\mu}\left(\frac{du}{dy}\right)^3 + \left(\frac{du}{dy}\right)\right]_{y=1}, \tag{8a}$$

$$\theta(0) = \alpha\left(\frac{d\theta}{dy}\right)_{y=0}, \quad \theta(1) = 1 - \alpha\left(\frac{d\theta}{dy}\right)_{y=1}, \tag{8b}$$

where $\gamma = l/h$ and $\alpha = k_1/h$ are the thermal and momentum slip parameters, respectively, y is the dimensionless form of the space coordinate and θ is the dimensionless temperature.

As a first case, we first assume that the viscosity is constant. The variable viscosity case will be handled independently.

(i) *The constant viscosity model*

In this case, we shall use two distinct approximations and find the solution of each case separately. First, we use the model when the effects of non-Newtonian and porous medium are small and later we use the model when only non-Newtonian effects is small.

(a) *When non-Newtonian and porous media effects are small*

When $\mu = 1$, eqs (6) and (7) with boundary conditions (8) are defined by

$$\frac{d^2u}{dy^2} + 6a_1\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2} - a_2u\left(1 + 2a_1\left(\frac{du}{dy}\right)^2\right) = a_3 \tag{9a}$$

$$u(0) = \gamma\left(2a_1\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3 + \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)\right)_{y=0},$$

$$u(1) = -\gamma\left(2a_1\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3 + \left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)\right)_{y=1}, \tag{9b}$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}y^2} + \Gamma\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^2 + 2\Gamma a_1\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^4 = 0, \tag{10a}$$

$$\theta(0) = \alpha\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)_{y=0}, \quad \theta(1) = 1 - \alpha\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right)_{y=1}. \tag{10b}$$

2.3 *Perturbation solution*

To find the solution of eqs (2) to (10b), the perturbation method [33–35] is used here. For this, we assume

$$a_1 = \epsilon\lambda_1, \qquad a_2 = \epsilon\lambda_2, \tag{11}$$

where ϵ is the perturbation parameter, and the approximate velocity and temperature profiles can be written as

$$u = u_0 + \epsilon u_1 \quad \text{and} \quad \theta = \theta_0 + \epsilon\theta_1. \tag{12}$$

Substituting (11) and (12) into eqs (9) and (10) and comparing the coefficients of like powers of ϵ, we get the following system of equations:

ϵ^0:

$$\frac{\mathrm{d}^2u_0}{\mathrm{d}y^2} = a_3, \tag{13a}$$

$$u_0(0) = \gamma\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)_{y=0}, \quad u_0(1) = -\gamma\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)_{y=1}. \tag{13b}$$

$$\frac{\mathrm{d}^2\theta_0}{\mathrm{d}y^2} + \Gamma\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)^2 = 0, \tag{14a}$$

$$\theta_0(0) = \alpha\left(\frac{\mathrm{d}\theta_0}{\mathrm{d}y}\right)_{y=0}, \quad \theta_0(1) = 1 - \alpha\left(\frac{\mathrm{d}\theta_0}{\mathrm{d}y}\right)_{y=1}. \tag{14b}$$

ϵ^1:

$$\frac{\mathrm{d}^2u_1}{\mathrm{d}y^2} + 6a_3\lambda_1\left(\frac{\mathrm{d}u_0}{\mathrm{d}x}\right)^2 - \lambda_2 u_0 = 0, \tag{15a}$$

$$u_1(0) = \gamma\left(2\lambda_1\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)^3 + \left(\frac{\mathrm{d}u_1}{\mathrm{d}y}\right)\right)_{y=0},$$

$$u_1(1) = -\gamma\left(2\lambda_1\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)^3 + \left(\frac{\mathrm{d}u_1}{\mathrm{d}y}\right)\right)_{y=1}, \tag{15b}$$

$$\frac{\mathrm{d}^2\theta_1}{\mathrm{d}y^2} = -2\Gamma\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)\left(\frac{\mathrm{d}u_1}{\mathrm{d}y}\right) - 2\lambda_1\Gamma\left(\frac{\mathrm{d}u_0}{\mathrm{d}y}\right)^2, \tag{16a}$$

$$\theta_1(0) = \alpha\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}y}\right)_{y=0}, \quad \theta_1(1) = -\alpha\left(\frac{\mathrm{d}\theta_1}{\mathrm{d}y}\right)_{y=1}. \tag{16b}$$

The solution of the first-order system is given as follows:

$$u_0 = \frac{a_3}{2}(y^2 - y - \gamma) \tag{17a}$$

$$\theta_0 = \frac{(\alpha + y)}{(1 + 2\alpha)} + \frac{\Gamma a_3^2}{24(1 + 2\alpha)}(-2y^4 + 4y^3 - 3y^2 + y - 4\alpha y^4 + 8\alpha y^3 - 6\alpha y^2 + 2\alpha y + \alpha + 2\alpha^2). \tag{17b}$$

The solution of the first-order system is given by

$$u_1 = \frac{\lambda_2 a_3}{24}(y^4 - 2y^3 + y - 6\gamma y^2 + 6\gamma y + 6\gamma^2 + \gamma) + \frac{\lambda_1 a_3^3}{4}(-2y^4 + 4y^3 - 3y^2 + y) \tag{18a}$$

$$\theta_1 = \frac{\Gamma\lambda_2 a_3^2}{720}(-8y^6 + 24y^5 - 15y^4 - 10y^3 + 15y^2 - 6y - 6\alpha + 60\gamma y^4 - 120\gamma y^3 + 90\gamma y^2 - 30\gamma y). \tag{18b}$$

The final approximation of the velocity and temperature profiles are

$$u = \frac{a_3}{2}(y^3 - y - \gamma) + \frac{a_2 a_3}{24}(y^4 - 2y^3 + y - 6\gamma y^2 + 6\gamma y + 6\gamma^2 + \gamma) + \frac{a_1 a_3^3}{4}(-2y^4 + 4y^3 - 3y^2 + y) \tag{19a}$$

$$\theta = \frac{(\alpha + y)}{(1 + 2\alpha)} + \frac{\Gamma a_3^2}{24(1 + 2\alpha)}(-2y^4 + 4y^3 - 3y^2 + y - 4\alpha y^4 + 8\alpha y^3 - 6\alpha y^2 + 2\alpha y + \alpha + 2\alpha^2) + \frac{\Gamma a_2 a_3^2}{720}(-8y^6 + 24y^5 - 15y^4 - 10y^3 + 15y^2 - 6y - 6\alpha + 60\gamma y^4 - 120\gamma y^3 + 90\gamma y^2$$

$$-30\gamma y-30\alpha\gamma-6\alpha\big)+\frac{\Gamma a_1 a_3^4}{240}\big(16y^6-48y^5+60y^4-40y^3+40y^3 15y^2-3y-3\alpha\big). \quad (19b)$$

According to Nayfeh [36], for perturbation solution to be uniformly valid the correction terms should be much smaller than the leading term. Thus, for expressions (9) and (10) to be uniformly valid, the following conditions must be satisfied:

$$\frac{a_2}{6}\ll 1, \quad 2a_1a_3^2\ll 1 \quad (20a)$$

$$\frac{a_2}{5}\ll 1, \quad \frac{3a_1a_3^2}{2}\ll 1. \quad (20b)$$

Special case:

Solutions in the absence of boundary and thermal-slip ($\alpha=\gamma=0$), are given by

$$u=\frac{a_3}{2}\big(y^2-y\big)+\frac{a_2a_3}{24}\big(y^4-2y^3+y\big)+\frac{a_1a_3^3}{4}\big(-2y^4+4y^3-3y^2+y\big) \quad (21a)$$

$$\theta=y+\frac{\Gamma a_3^2}{24}\big(-2y^4+4y^3-3y^2+y\big)+\frac{\Gamma a_2a_3^2}{720}\times\big(-8y^6+24y^5-15y^4-10y^3+15y^2-6y\big)+\frac{\Omega a_1a_3^4}{240}\big(16y^6-48y^5+60y^4-40y^3+15y^2-3y\big). \quad (21b)$$

(b) *When only the non-Newtonian effects are small*

In this section, the perturbation solution for the velocity and temperature profiles is sought when $a_1\ll 1$. Note that when $a_1\ll 1$ the hydraulic resistance is mainly due to the parallel plates instead of the porous medium.

Assuming $a_1=\epsilon\lambda_1$ and using (9) we get the following systems at various orders of ϵ.

ϵ^0:

$$\frac{d^2u_0}{dy^2}-a_2u_0=a_3, \quad (22a)$$

$$u_0(0)=\gamma\left(\frac{du_0}{dy}\right)_{y=0}, \quad u_0(1)=-\gamma\left(\frac{du_0}{dy}\right)_{y=0}. \quad (22b)$$

ϵ^1:

$$\frac{d^2u_1}{dy^2}-a_2u_1=2a_2u_0\left(\frac{du_0}{dy}\right)^2-6\left(\frac{d^2u_0}{dy^2}\right)\left(\frac{du_0}{dy}\right)^2, \quad (23a)$$

$$u_1(0)=\gamma\left(2\left(\frac{du_0}{dy}\right)^3+\left(\frac{du_1}{dy}\right)\right)_{y=0},$$

$$u_1(1)=-\gamma\left(2\left(\frac{du_0}{dy}\right)^3+\left(\frac{du_1}{dy}\right)\right)_{y=1}. \quad (23b)$$

Solving the above system of equations, we get

$$u_0=-\frac{\left(\cosh\left[\frac{\sqrt{a_2}}{2}\right]-\cosh\left[\frac{1}{2}(1-2y)\sqrt{a_2}\right]+\gamma\sinh\left[\frac{\sqrt{a_2}}{2}\right]\sqrt{a_2}\right)a_3}{\left(\cosh\left[\frac{\sqrt{a_2}}{2}\right]+\gamma\sinh\left[\frac{\sqrt{a_2}}{2}\right]\sqrt{a_2}\right)a_2} \quad (24a)$$

$$\begin{aligned}u_1=&\frac{a_3^3e^{-3y\sqrt{a_2}}}{6(1+e^{\sqrt{a_2}}+(-1+e^{\sqrt{a_2}})\gamma\sqrt{a_2})^4a_2^2}(-(1+e^{\sqrt{a_2}}\\&+(-1+e^{\sqrt{a_2}})\gamma\sqrt{a_2})(3e^{3\sqrt{a_2}}+4e^{5y\sqrt{a_2}}+3e^{6y\sqrt{a_2}}\\&+6e^{2(1+y)\sqrt{a_2}}+4e^{(2+y)\sqrt{a_2}}+4e^{(3+y)\sqrt{a_2}}\\&+24e^{(1+3y)\sqrt{a_2}}+24e^{(2+3y)\sqrt{a_2}}+6e^{(1+4y)\sqrt{a_2}}\\&+4e^{(1+5y)\sqrt{a_2}}+4(3e^{2(1+y)\sqrt{a_2}}y\\&-3e^{(1+4y)\sqrt{a_2}}y-e^{5y\sqrt{a_2}}\gamma-e^{(2+y)\sqrt{a_2}}\gamma\\&+6e^{(2+3y)\sqrt{a_2}}\gamma+e^{(1+5y)\sqrt{a_2}}\gamma)\sqrt{a_2})\\&+\frac{1}{(-1+e^{\sqrt{a_2}}+(1+e^{\sqrt{a_2}})\gamma\sqrt{a_2})e^{(1+2y)\sqrt{a_2}}}\\&\times(-7-27e^{\sqrt{a_2}}+27e^{3\sqrt{a_2}}+7e^{4\sqrt{a_2}}\\&+2(20\gamma+33e^{3\sqrt{a_2}}\gamma+8e^{4\sqrt{a_2}}\gamma+6e^{2\sqrt{a_2}}(1+2\gamma)\\&+e^{\sqrt{a_2}}(-6+9\gamma))\sqrt{a_2}+\gamma(-41\gamma-72e^{2\sqrt{a_2}}\gamma\\&+75e^{3\sqrt{a_2}}\gamma+17e^{4\sqrt{a_2}}\gamma+3e^{\sqrt{a_2}}(8+7\gamma))a_2\\&+4(1+e^{\sqrt{a_2}})\gamma^2(2\gamma-2e^{2\sqrt{a_2}}\gamma+2e^{3\sqrt{a_2}}\gamma\\&-e^{\sqrt{a_2}}(3+2\gamma))a_2^{3/2})\end{aligned}$$

$$+\frac{1}{(-1+e^{\sqrt{a_2}}+(1+e^{\sqrt{a_2}})\gamma\sqrt{a_2})e^{4y\sqrt{a_2}}}$$
$$\times(-7-27e^{\sqrt{a_2}}+27e^{3\sqrt{a_2}}+7e^{4\sqrt{a_2}}+2(8\gamma$$
$$+33e^{\sqrt{a_2}}\gamma+20e^{4\sqrt{a_2}}\gamma+6e^{2\sqrt{a_2}}(1+2\gamma)$$
$$+e^{3\sqrt{a_2}}(-6+9\gamma))\sqrt{a_2}+\gamma(-17\gamma-75e^{\sqrt{a_2}}\gamma$$
$$+72e^{2\sqrt{a_2}}\gamma+41e^{4\sqrt{a_2}}\gamma-3e^{3\sqrt{a_2}}(8+7\gamma))a_2$$
$$+4(1+e^{\sqrt{a_2}})\gamma^2(2\gamma-2e^{\sqrt{a_2}}\gamma$$
$$+2e^{3\sqrt{a_2}}\gamma-e^{2\sqrt{a_2}}(3+2\gamma))a_2^{3/2})). \tag{24b}$$

Hence the final form of the velocity profile is given by

$$u=-\frac{\left(\cosh\left[\frac{\sqrt{a_2}}{2}\right]-\cosh\left[\frac{1}{2}(1-2y)\sqrt{a_2}\right]+\gamma\sinh\left[\frac{\sqrt{a_2}}{2}\right]\sqrt{a_2}\right)a_3}{\left(\cosh\left[\frac{\sqrt{a_2}}{2}\right]+\gamma\sinh\left[\frac{\sqrt{a_2}}{2}\right]\sqrt{a_2}\right)a_2}$$
$$+\frac{a_1a_3^3e^{-3y\sqrt{a_2}}}{6(1+e^{\sqrt{a_2}}+(-1+e^{\sqrt{a_2}})\gamma\sqrt{a_2})^4a_2^2}$$
$$\times(-(1+e^{\sqrt{a_2}}+(-1+e^{\sqrt{a_2}})\gamma\sqrt{a_2})(3e^{3\sqrt{a_2}}$$
$$+4e^{5y\sqrt{a_2}}+3e^{6y\sqrt{a_2}}+6e^{2(1+y)\sqrt{a_2}}+4e^{(2+y)\sqrt{a_2}}$$
$$+4e^{(3+y)\sqrt{a_2}}+24e^{(1+3y)\sqrt{a_2}}+24e^{(2+3y)\sqrt{a_2}}$$
$$+6e^{(1+4y)\sqrt{a_2}}+4e^{(1+5y)\sqrt{a_2}}+4(3e^{2(1+y)\sqrt{a_2}}y$$
$$-3e^{(1+4y)\sqrt{a_2}}y-e^{5y\sqrt{a_2}}\gamma-e^{(2+y)\sqrt{a_2}}\gamma$$
$$+e^{(3+y)\sqrt{a_2}}\gamma-6e^{(1+3y)\sqrt{a_2}}\gamma$$
$$+6e^{(2+3y)\sqrt{a_2}}\gamma+e^{(1+5y)\sqrt{a_2}}\gamma)\sqrt{a_2})$$
$$+\frac{1}{(-1+e^{\sqrt{a_2}}+(1+e^{\sqrt{a_2}})\gamma\sqrt{a_2})e^{(1+2y)\sqrt{a_2}}}$$
$$\times(-7-27e^{\sqrt{a_2}}+27e^{3\sqrt{a_2}}+7e^{4\sqrt{a_2}}$$
$$+2(20\gamma+33e^{3\sqrt{a_2}}\gamma+8e^{4\sqrt{a_2}}\gamma$$
$$+6e^{2\sqrt{a_2}}(1+2\gamma)+e^{\sqrt{a_2}}(-6+9\gamma))\sqrt{a_2}$$
$$+\gamma(-41\gamma-72e^{2\sqrt{a_2}}\gamma+75e^{3\sqrt{a_2}}\gamma+17e^{4\sqrt{a_2}}\gamma$$
$$+3e^{\sqrt{a_2}}(8+7\gamma))a_2+4(1+e^{\sqrt{a_2}})\gamma^2$$
$$\times(2\gamma-2e^{2\sqrt{a_2}}\gamma+2e^{3\sqrt{a_2}}\gamma-e^{\sqrt{a_2}}(3+2\gamma))a_2^{3/2})$$
$$+\frac{1}{(-1+e^{\sqrt{a_2}}+(1+e^{\sqrt{a_2}})\gamma\sqrt{a_2})e^{4y\sqrt{a_2}}}$$
$$\times(-7-27e^{\sqrt{a_2}}+27e^{3\sqrt{a_2}}+7e^{4\sqrt{a_2}}$$
$$+2(8\gamma+33e^{\sqrt{a_2}}\gamma+20e^{4\sqrt{a_2}}\gamma+6e^{2\sqrt{a_2}}(1+2\gamma)$$
$$+e^{3\sqrt{a_2}}(-6+9\gamma))\sqrt{a_2}+\gamma(-17\gamma-75e^{\sqrt{a_2}}\gamma$$
$$+72e^{2\sqrt{a_2}}\gamma+41e^{4\sqrt{a_2}}\gamma$$
$$-3e^{3\sqrt{a_2}}(8+7\gamma))a_2$$
$$+4(1+e^{\sqrt{a_2}})\gamma^2(2\gamma-2e^{\sqrt{a_2}}\gamma$$
$$+2e^{3\sqrt{a_2}}\gamma-e^{2\sqrt{a_2}}(3+2\gamma))a_2^{3/2})). \tag{25}$$

Similarly, for the temperature profile we have the following systems:

ϵ^0 :

$$\frac{d^2\theta_0}{dy^2}+\Gamma\left(\frac{du_0}{dy}\right)^2=0, \tag{26a}$$

$$\theta_0(0)=\alpha\left(\frac{d\theta_0}{dy}\right)_{y=0},\quad \theta_0(1)=1-\alpha\left(\frac{d\theta_0}{dy}\right)_{y=1}. \tag{26b}$$

ϵ^1 :

$$\frac{d^2\theta_1}{dy^2}+2\Gamma\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)+2\Gamma\left(\frac{du_0}{dy}\right)^4=0, \tag{27a}$$

$$\theta_1(0)=\alpha\left(\frac{d\theta_1}{dy}\right)_{y=0},\quad \theta_1(1)=-\alpha\left(\frac{d\theta_1}{dy}\right)_{y=1}. \tag{27b}$$

The first-order solution is given by

$$\theta_0=\frac{1}{\left\{\begin{array}{c}(4(1+\cosh[\sqrt{a_2}]+\sinh[\sqrt{a_2}]\\ +\gamma(-1+\cosh[\sqrt{a_2}]\\ +\sinh[\sqrt{a_2}])\sqrt{a_2})^2a_2^2)\end{array}\right\}}$$
$$\times(-\Gamma\cosh[2(-1+y)\sqrt{a_2}]a_3^2-\Gamma\cosh[2y\sqrt{a_2}]a_3^2$$
$$+\Gamma\sinh[2(-1+y)\sqrt{a_2}]a_3^2-\Gamma\sinh[2y\sqrt{a_2}]a_3^2$$
$$+4y^2\Gamma\cosh[\sqrt{a_2}]a_2a_3^2+4y^2\Gamma\sinh[\sqrt{a_2}]a_2a_3^2$$
$$+\frac{1}{1+2\alpha}\Big(4y(\cosh[\sqrt{a_2}]+\sinh[\sqrt{a_2}])a_2$$

$$\times\Big(2\big(1+\cosh[\sqrt{a_2}]\big)a_2+4\gamma\sinh[\sqrt{a_2}]a_2^{3/2}$$

$$+4\gamma^2\sinh\left[\frac{\sqrt{a_2}}{2}\right]^2 a_2^2-(1+2\alpha)\Gamma a_3^2\Big)\Big)$$

$$+\frac{2\big(\cosh[\sqrt{a_2}]+\sinh[\sqrt{a_2}]\big)}{1+2\alpha}$$

$$\times\Big(4\alpha\big(1+\cosh[\sqrt{a_2}]\big)a_2^2+8\alpha\gamma\sinh[\sqrt{a_2}]a_2^{5/2}$$

$$+8\alpha\gamma^2\sinh\left[\frac{\sqrt{a_2}}{2}\right]^2 a_2^3$$

$$+(1+2\alpha)\Gamma\cosh[\sqrt{a_2}]a_3^2+2\alpha(1+2\alpha)$$

$$\times\Gamma\sinh[\sqrt{a_2}]\sqrt{a_2}a_3^2-2\alpha(1+2\alpha)\Gamma a_2 a_3^2\Big)\Big). \quad (28)$$

The expression for θ_1 is not shown here because of its size.

(ii) *The case of temperature-dependent viscosity*

Here, we take that the viscosity of the fluid is temperature-dependent. Two different types of models are discussed. The solutions for the velocity and temperature fields are established by assuming that the effects of porosity are small.

(a) *Reynold's model*

The dimensionless viscosity for the Reynold's model [37–39] is given by

$$\mu=\exp[-M\theta], \quad (29)$$

where M is the Reynolds model parameter. For subsequent analysis, we assume $a_1=\epsilon\lambda_1, a_2=\epsilon\lambda_2, M=\epsilon m$ and

$$u=u_0+\epsilon u_1 \quad \text{and} \quad \theta=\theta_0+\epsilon\theta_1, \quad (30)$$

where ϵ is a small parameter.

With the help of Taylor series, the viscosity and its derivative are given by

$$\mu=1-\epsilon m\theta\,, \quad \frac{d\mu}{dy}=-\epsilon m\frac{d\theta}{dy}, \quad (31)$$

in an approximate form. Making use of the above expressions into (6) and (7) and utilising (30), we get the following systems at various orders of ϵ:

ϵ^0:

$$\frac{d^2u_0}{dy^2}=a_3, \quad (32a)$$

$$\frac{d^2\theta_0}{dy^2}+\Gamma\left(\frac{du_0}{dy}\right)^2=0, \quad (32b)$$

$$u_0(0)=\gamma\left(\frac{du_0}{dy}\right)_{y=0}, \quad u_0(1)=-\gamma\left(\frac{du}{dy}\right)_{y=1}, \quad (32c)$$

$$\theta_0(0)=\alpha\left(\frac{d\theta_0}{dy}\right)_{y=0}, \quad \theta_0(1)=1-\alpha\left(\frac{d\theta_0}{dy}\right)_{y=1}. \quad (32d)$$

ϵ^1:

$$\frac{d^2u_1}{dy^2}-m\left(\frac{d\theta_0}{dy}\right)\left(\frac{du_0}{dy}\right)-ma_3\theta_0$$

$$+6\lambda_1 a_3\left(\frac{du_0}{dy}\right)^2-\lambda_2 u_0=0, \quad (33a)$$

$$\frac{d^2\theta_1}{dy^2}-m\Gamma\theta_0\left(\frac{du_0}{dy}\right)^2+2\Gamma\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right)$$

$$+2\Gamma\lambda_1\left(\frac{du_0}{dy}\right)^4=0, \quad (33b)$$

$$u_1(0)=\gamma\left[2\lambda_1\left(\frac{du_0}{dy}\right)^3+\left(\frac{du_1}{dy}\right)\right]_{y=0},$$

$$u_1(1)=-\gamma\left[2\lambda_1\left(\frac{du_0}{dy}\right)^3+\left(\frac{du_1}{dy}\right)\right]_{y=1}, \quad (33c)$$

$$\theta_1(0)=\alpha\left(\frac{d\theta_1}{dy}\right)_{y=0}, \quad \theta_1(1)=-\alpha\left(\frac{d\theta_1}{dy}\right)_{y=1}. \quad (33d)$$

Solving the above system of equations subject to the corresponding boundary conditions, we get

$$u_0=\frac{a_3}{2}(y^2-y-\gamma) \quad (34a)$$

$$u_1=\frac{ma_3}{12(1+2\alpha)(1+2\gamma)}(4y^3-3y^2-y+6\alpha y^2$$

$$-6\alpha y+8\gamma y^3-6\gamma y^2-6\gamma y-\gamma+12\alpha\gamma y^2$$

$$-12\alpha\gamma y-6\alpha\gamma-6\alpha^2-12\alpha\gamma^2)$$

$$+\frac{m\Gamma a_3^3}{288}(-4y^6+12y^5-15y^4+10y^3$$

$$-3y^2+6\alpha y^2-6\alpha y-6\alpha\gamma)$$

$$-\frac{\lambda_1 a_3^3}{4}(2y^4-4y^3+3y^2-y)$$

$$+\frac{\lambda_2 a_3^3}{24}(y^4-2y^3+y$$

$$- 6\gamma y^2 + 6\gamma y + 6\gamma^2 + 6\gamma\big) \tag{34b}$$

$$\theta_0 = \left(\frac{y+\alpha}{1+2\alpha}\right) + \frac{\Gamma a_2^2}{24(1+2\alpha)}\big(- 2y^4 + 4y^3 - 3y^2 + y - 4\alpha y^4 + 8\alpha y^3 - 6\alpha y^2 + 2\alpha y + \alpha + 2\alpha^2\big) \tag{34c}$$

$$\theta_1 = \frac{m\Gamma a_3^2}{360(1+2\alpha)^2}\big(8y - 15y^2 - 5y^3 + 30y^4 - 18y^5 + 8\alpha + 30\alpha y - 75\alpha y^2 + 50\alpha y^3 + 30\alpha y^4 - 36\alpha y^5 + 30\alpha^2 + 30\alpha^2 y - 90\alpha^2 y^2 + 120\alpha^2 y^3 - 60\alpha^2 y^4 + 30\alpha^3 + 30\gamma y - 90\gamma y^2 + 30\gamma y^3 + 60\gamma y^4 - 36\gamma y^5 + 36\alpha\gamma + 60\alpha\gamma y - 270\alpha\gamma y^2 + 180\alpha\gamma y^3 + 60\alpha\gamma y^4 - 72\alpha\gamma y^5 + 60\alpha^2\gamma + 60\alpha^2\gamma y - 180\alpha^2\gamma y^2 + 240\alpha^2\gamma y^3 - 120\alpha^2\gamma y^4 + 60\alpha^3\gamma\big) + \frac{\Gamma\lambda_2 a_3^2}{720(1+2\alpha)^2}(-6y + 15y^2 - 10y^3 - 15y^4 + 24y^5 - 8y^6 - 6\alpha - 30\gamma y + 90\gamma y^2 - 120\gamma y^3 + 60\gamma y^4 - 30\alpha\gamma\big) + \frac{m\Gamma^2 a_3^4}{8064(1+2\alpha)^2}\big(y - 14y^3 + 49y^4 - 84y^5 + 84y^6 - 48y^7 + 12y^8 + \alpha + 14\alpha y - 42\alpha y^2 + 56\alpha y^3 - 28\alpha y^4 + 14\alpha^2\big) + \frac{\Gamma\lambda_1 a_3^4}{240(1+2\alpha)^2}\big(- 3y + 15y^2 - 40y^3 + 60y^4 - 48y^5 + 16y^6 - 3\alpha\big). \tag{34d}$$

The final expressions of velocity and temperature are

$$u = \frac{a_3}{2}\big(y^2 - y - \gamma\big) + \frac{Ma_3}{12(1+2\alpha)(1+2\gamma)}\big(4y^3 - 3y^2 - y + 6\alpha y^2 - 6\alpha y + 8\gamma y^3 - 6\gamma y^2 - 6\gamma y - \gamma + 12\alpha\gamma y^2 - 12\alpha\gamma y - 6\alpha\gamma - 6\alpha^2 - 12\alpha\gamma^2\big) + \frac{M\Gamma a_3^3}{288}\big(- 4y^6 + 12y^5 - 15y^4 + 10y^3 - 3y^2 + 6\alpha y^2 - 6\alpha y - 6\alpha\gamma\big) - \frac{a_1 a_3^3}{4}\big(2y^4 - 4y^3 + 3y^2 - y\big) + \frac{a_2 a_3^3}{24}\big(y^4 - 2y^3 + y - 6\gamma y^2 + 6\gamma y + 6\gamma^2 + 6\gamma\big) \tag{35a}$$

$$\theta = \left(\frac{y+\alpha}{1+2\alpha}\right) + \frac{Ma_3^2}{24(1+2\alpha)}\big(- 2y^4 + 4y^3 - 3y^2 + y - 4\alpha y^4 + 8\alpha y^3 - 6\alpha y^2 + 2\alpha y + \alpha + 2\alpha^2\big) + \frac{M\Gamma a_3^2}{360(1+2\alpha)^2}\big(8y - 15y^2 - 5y^3 + 30y^4 - 18y^5 + 8\alpha + 30\alpha y - 75\alpha y^2 + 50\alpha y^3 + 30\alpha y^4 - 36\alpha y^5 + 30\alpha^2 + 30\alpha^2 y - 90\alpha^2 y^2 + 120\alpha^2 y^3 - 60\alpha^2 y^4 + 30\alpha^3 + 30\gamma y - 90\gamma y^2 + 30\gamma y^3 + 60\gamma y^4 - 36\gamma y^5 + 36\alpha\gamma + 60\alpha\gamma y - 270\alpha\gamma y^2 + 180\alpha\gamma y^3 + 60\alpha\gamma y^4 - 72\alpha\gamma y^5 + 60\alpha^2\gamma + 60\alpha^2\gamma y - 180\alpha^2\gamma y^2 + 240\alpha^2\gamma y^3 - 120\alpha^2\gamma y^4 + 60\alpha^3\gamma\big) + \frac{\Gamma a_2 a_3^2}{720(1+2\alpha)^2}\big(-6y + 15y^2 - 10y^3 - 15y^4 + 24y^5 - 8y^6 - 6\alpha - 30\gamma y + 90\gamma y^2 - 120\gamma y^3 + 60\gamma y^4 - 30\alpha\gamma\big) + \frac{M\Gamma^2 a_3^4}{8064(1+2\alpha)^2}\big(y - 14y^3 + 49y^4 - 84y^5 + 84y^6 - 48y^7 + 12y^8 + \alpha + 14\alpha y - 42\alpha y^2 + 56\alpha y^3 - 28\alpha y^4 + 14\alpha^2\big) + \frac{\Gamma a_1 a_3^4}{240(1+2\alpha)^2}\big(- 3y + 15y^2 - 40y^3 + 60y^4 - 48y^5 + 16y^6 - 3\alpha\big). \tag{35b}$$

Special case:

Solutions in the absence of boundary and thermal-slip parameters ($\alpha = \gamma = 0$) are given by

$$u = \frac{a_3}{2}\big(y^2 - y\big) + \frac{Ma_3}{12}\big(4y^3 - 3y^2 - y\big) + \frac{a_2 a_3}{24}\big(y^4 - 2y^3 + y\big) + \frac{M\Gamma a_3}{288}\big(-4y^6 + 12y^5 - 15y^4 + 10y^3 - 3y^2\big)$$

$$-\frac{a_1 a_3^3}{4}\left(2y^4-4y^3+3y^2-y\right) \tag{36a}$$

$$\theta = y+\frac{\Gamma a_3^2}{24}\left(-2y^4+4y^3-3y^2+y\right) + \frac{M\Gamma a_3^2}{360}\left(8y-15y^2-5y^3+30y^4-18y^5\right) + \frac{\Gamma a_2 a_3^2}{720}\left(-6y+15y^2-10y^3-15y^4 +24y^5-8y^6\right)+\frac{M\Gamma^2 a_3^4}{8064}\left(y-14y^3+49y^4 -84y^5+84y^6-48y^7+12y^8\right) + \frac{\Gamma a_1 a_3^4}{240}\left(16y^6-48y^5+60y^4-40y^3 +15y^2-3y\right). \tag{36b}$$

(b) *Vogel's model*

In this model, the viscosity function is assumed to be of the form [38–40]

$$\bar{\mu} = \mu_0 \exp\left[\frac{A}{B+\theta}-\theta_w\right]. \tag{37a}$$

Using the Taylor series expansion, the above expression takes the following dimensionless form:

$$\mu = \omega\left[1-\frac{A\theta}{B^2}\right], \tag{37b}$$

where $\omega = \exp[(A/B)-\theta_w]$, A and B are the Vogel's model parameters. The velocity and temperature expansions are

$$u = u_0+\epsilon u_1 \quad \text{and} \quad \theta = \theta_0+\epsilon\theta_1, \tag{38}$$

where we have again assumed that $a_1 = \epsilon\lambda_1, a_2 = \epsilon\lambda_2$ and $A = \epsilon a$ and performing the usual perturbation analysis, we get the following systems:

ϵ^0:

$$\frac{d^2u_0}{dy^2} = \frac{a_3}{\omega}, \tag{39a}$$

$$\frac{d^2\theta_0}{dy^2}+\omega\Gamma\left(\frac{du_0}{dy}\right)^2 = 0, \tag{39b}$$

$$u_0(0) = \gamma\left(\frac{du_0}{dy}\right)_{y=0}, \quad u_0(1) = -\gamma\left(\frac{du_0}{dy}\right)_{y=1}, \tag{39c}$$

$$\theta_0(0) = \alpha\left(\frac{d\theta_0}{dy}\right)_{y=0}, \quad \theta_0(1) = 1-\alpha\left(\frac{d\theta_0}{dy}\right)_{y=1}. \tag{39d}$$

ϵ^0:

$$\omega\frac{d^2u_1}{dy^2}-\frac{\omega a}{B^2}\left(\frac{d\theta_0}{dy}\right)\left(\frac{du_0}{dy}\right)-\frac{aa_3}{B^2}\theta_0 +\frac{6\lambda_1 a_3}{\omega}\left(\frac{du_0}{dy}\right)^2-\omega\lambda_2 u_0 = 0, \tag{40a}$$

$$\frac{d^2\theta_1}{dy^2}-\frac{\omega a\Gamma}{B^2}\theta_0\left(\frac{du_0}{dy}\right)^2+2\omega a\Gamma\left(\frac{du_0}{dy}\right)\left(\frac{du_1}{dy}\right) +2\Gamma\lambda_1\left(\frac{du_0}{dy}\right)^4 = 0, \tag{40b}$$

$$u_1(0) = \gamma\left[\frac{2\lambda_1}{\omega}\left(\frac{du_0}{dy}\right)+\left(\frac{du_1}{dy}\right)\right]_{y=0},$$

$$u_1(1) = -\gamma\left[\frac{2\lambda_1}{\omega}\left(\frac{du_0}{dy}\right)^3+\left(\frac{du_1}{dy}\right)\right]_{y=1}, \tag{40c}$$

$$\theta_1(0) = \alpha\left(\frac{d\theta_1}{dy}\right)_{y=0}, \quad \theta_1(1) = -\alpha\left(\frac{d\theta_1}{dy}\right)_{y=1}. \tag{40d}$$

The solution of the above system is

$$u_0 = \frac{a_3}{2\omega}\left(y^2-y-\gamma\right). \tag{41a}$$

The order of ϵ, solution is given by

$$u_1 = \frac{aa_3}{12B^2\omega(1+2\alpha)(1+2\gamma)} \times\left(4y^3-3y^2-y+6\alpha y^2 -6\alpha y-\gamma-6\gamma y-6\alpha\gamma-12\alpha\gamma y-6\gamma^2 -12\alpha\gamma^2\right)+\frac{\lambda_2 a_3}{24\omega}\left(y^4-2y^3+y-6\gamma y^2+6\gamma y +\gamma+6\gamma^2\right)+\frac{a\Gamma a_3^3}{288B^2\omega^2}\left(-4y^6+12y^5-15y^4 +.10y^3-3y^2-6\alpha y^2-6\alpha y-6\alpha\gamma\right) +\frac{\lambda_1 a_3^3}{4\omega^4}\left(2y^4-4y^3+3y^2-y\right) \tag{41b}$$

$$\theta_0 = \left(\frac{y+\alpha}{1+2\alpha}\right)+\frac{\Gamma a_3^2}{24\omega(1+2\alpha)}\left(-2y^4+4y^3-3y^2\right.$$

$$+ y - 4\alpha y^4 + 8\alpha y^3 - 6\alpha y^2 + 2\alpha y + \alpha + 2\alpha^2\big) \quad (41c)$$

$$\theta_1 = \frac{a\Gamma a_3^2}{360B^2\omega(1+2\alpha)^2(1+2\gamma)}\big(8y - 15y^2 - 5y^3 + 30y^4 - 18y^5 + 8\alpha + 30\alpha y - 45\alpha y^2 + 50\alpha y^3 + 30\alpha y^4 - 36\alpha y^5 + 30\alpha^2 + 30\alpha^2 y - 90\alpha^2 y^2 + 120\alpha^2 y^3 - 60\alpha^2 y^4 + 30\alpha^3 + 30\gamma y - 90\gamma y^2 + 30\gamma y^3 + 60\gamma y^4 - 36\gamma y^5 + 36\alpha\gamma + 60\alpha\gamma y - 270\alpha\gamma y^2 + 180\alpha\gamma y^3 + 60\alpha\gamma y^4 - 72\alpha\gamma y^5 + 60\alpha^2\gamma + 60\alpha^2\gamma y - 180\alpha^2\gamma y^2 + 240\alpha^2\gamma y^3 - 120\alpha^2\gamma y^4 + 60\alpha^3\gamma\big) + \frac{\Gamma\lambda_2 a_3^2}{720\omega(1+2\alpha)^2}\big(-6y + 15y^2 - 10y^3 - 15y^4 + 24y^5 - 8y^6 - 6\alpha - 30\gamma y + 90\gamma y^2 - 120\gamma y^3 + 60\gamma y^4 - 30\alpha\gamma\big). \quad (41d)$$

Finally,

$$u = \frac{a_3}{2\omega}(y^2 - y - \gamma) + \frac{Aa_3}{12B^2\omega(1+2\alpha)(1+2\gamma)}(4y^3 - 3y^2 - y + 6\alpha y^2 - 6\alpha y + 6\alpha y^2 - \gamma - 6\gamma y - 6\alpha\gamma - 12\alpha\gamma y - 6\gamma^2 - 12\alpha\gamma^2) + \frac{a_2 a_3}{24\omega}\big(y^4 - 2y^3 + y - 6\gamma y^2 + 6\gamma y + \gamma + 6\gamma^2\big) + \frac{A\Gamma a_3^3}{288B^2\omega^2}\big(-4y^6 + 12y^5 - 15y^4 + 10y^3 - 3y^2 - 6\alpha y^2 - 6\alpha y - 6\alpha\gamma\big) + \frac{a_1 a_3^3}{4\omega^4}\big(2y^4 - 4y^3 + 3y^2 - y\big) \quad (42a)$$

$$\theta = \left(\frac{y+\alpha}{1+2\alpha}\right) + \frac{\Gamma a_3^2}{24\omega(1+2\alpha)}\big(-2y^4 + 4y^3 - 3y^2 + y - 4\alpha y^4 + 8\alpha y^3 - 6\alpha y^2 + 2\alpha y + \alpha + 2\alpha^2\big) + \frac{A\Gamma a_3^2}{360B^2\omega(1+2\alpha)^2(1+2\gamma)} \times\big(8y - 15y^2 - 5y^3 + 30y^4 - 18y^5 + 8\alpha + 30\alpha y - 45\alpha y^2 + 50\alpha y^3 + 30\alpha y^4 - 36\alpha y^5 + 30\alpha^2 + 30\alpha^2 y - 90\alpha^2 y^2 + 120\alpha^2 y^3 - 60\alpha^2 y^4 + 30\alpha^3 + 30\gamma y - 90\gamma y^2 + 30\gamma y^3 + 60\gamma y^4 - 36\gamma y^5 + 36\alpha\gamma + 60\alpha\gamma y - 270\alpha\gamma y^2 + 180\alpha\gamma y^3 + 60\alpha\gamma y^4 - 72\alpha\gamma y^5 + 60\alpha^2\gamma \times 60\alpha^2\gamma y - 180\alpha^2\gamma y^2 + 240\alpha^2\gamma y^3 - 120\alpha^2\gamma y^4 + 60\alpha^3\gamma\big) + \frac{\Gamma a_2 a_3^2}{720\omega(1+2\alpha)^2}\big(-6y + 15y^2 - 10y^3 - 15y^4 + 24y^5 - 8y^6 - 6\alpha - 30\gamma y + 90\gamma y^2 - 120\gamma y^3 + 60\gamma y^4 - 30\alpha\gamma\big). \quad (42b)$$

Special case:

Solutions in the absence of boundary and thermal-slip parameters ($\alpha = \gamma = 0$) are given by

$$u = \frac{a_3}{2\omega}(y^2 - y) + \frac{Aa_3}{12B^2\omega}(4y^3 - 3y^2 - y) + \frac{a_2 a_3}{24\omega}\big(y^4 - 2y^3 + y\big) + \frac{A\Gamma a_3^3}{288B^2\omega^2}\big(-4y^6 + 12y^5 - 15y^4 + 10y^3 - 3y^2\big) - \frac{a_1 a_3^3}{4\omega^4}\big(2y^4 - 4y^3 + 3y^2 - y\big) \quad (43a)$$

$$\theta = y + \frac{\Gamma a_3^2}{24\omega}\big(-2y^4 + 4y^3 - 3y^2 + y\big) + \frac{A\Gamma a_3^2}{360B^2\omega}\big(8y - 15y^2 - 5y^3 + 30y^4 - 18y^5\big) + \frac{\Gamma a_2 a_3^2}{720\omega}\big(-6y + 15y^2 - 10y^3 - 15y^4 + 24y^5 - 8y^6\big) + \frac{A\Gamma^2 a_3^4}{8064B^2\omega^2}\big(y - 14y^3 + 49y^4 - 84y^5 + 84y^6 - 48y^7 + 12y^8\big) + \frac{\Gamma a_1 a_3^4}{240\omega^4}\big(-3y + 15y^2 - 40y^3 + 60y^4 - 48y^5 + 16y^6\big). \quad (43b)$$

3. Skin-friction coefficient and heat transfer rate

The skin-friction coefficient and heat transfer rate ($\overline{Nu}$) of the given flow are generally defined by

$$\bar{C}_{\mathrm{f}} = \frac{S_{\bar{x}\bar{y}}}{(1/2)\rho U^2} \tag{44a}$$

$$\overline{Nu} = \frac{h}{(\theta_{\mathrm{u}} - \theta_{\mathrm{l}})} \frac{\partial \bar{\theta}}{\partial \bar{y}}. \tag{44b}$$

With the help of dimensionless quantities which are given in eq. (5), the dimensionless form of the skin-friction coefficient and Nu is given by

$$\frac{1}{2}\mathrm{Re}C_{\mathrm{f}} = \mu\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right) + 2a_1\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3$$

or

$$C_{\mathrm{F}} = \mu\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right) + 2a_1\left(\frac{\mathrm{d}u}{\mathrm{d}y}\right)^3 \tag{45a}$$

and

$$Nu = -\left(\frac{\mathrm{d}\theta}{\mathrm{d}y}\right), \tag{45b}$$

where $\mathrm{Re} = \rho/Uh\mu$ is called the Reynolds number.

The calculations of skin-friction coefficients and heat transfer are not presented here due to lengthy expressions.

4. Comparison between numerical and perturbation solutions

The comparison of perturbation and numerical solutions are presented in tables 2–4. The pseudospectral collocation method is used for the numerical solution. In pseudospectral, we discretised the derivative operators with Chebyshev or Jacobi orthogonal polynomials. The nonlinearity is handled with help of Newton method [40–51] and discrete Jacobian, i.e. finite difference approximation of Jacobian. For comparison, we show the values of skin-friction coefficient and Nu using slip parameters in the case of constant viscosity and temperature-dependent viscosity models. It is noted that both solutions are in good agreement with each other.

5. Graphical results and discussion

In this section, the graphical results are displayed to examine the effect of various emerging parameters on the velocity and temperature profiles. The analytical expressions are obtained by solving dimensionless equations with boundary conditions with the help of perturbation method. The authors depicted eight figures in which figure 1 shows the flow behaviour of the fluid and figures 2–9 show the velocity and temperature profiles. The variation of Nu and C_{F} using slips parameters are given in table 1. The ranges of the emerging parameters are selected based on the previous studies and that ranges are given by: Non-Newtonian parameter $= 0 \leq a_1 \leq 1$, porous media parameter $= 0 \leq a_2 \leq 1$, pressure gradient parameter $= -2 \leq a_1 \leq -0.1$, boundary slip-parameter $= 0 \leq \alpha \leq 0.5$, thermal slip-parameter $= 0 \leq \gamma \leq 0.5$, Reynolds model parameter $= 0.1 \leq M \leq 1$, Vogel's model parameter $= 0.1 \leq A \leq 1$, Vogel's model parameter $= 0.1 \leq B \leq 1.5$, wall's temperature parameter $= 0.1 \leq \theta_{\mathrm{w}} \leq 1$ and viscous heating parameter $= 0 \leq \Gamma \leq 100$.

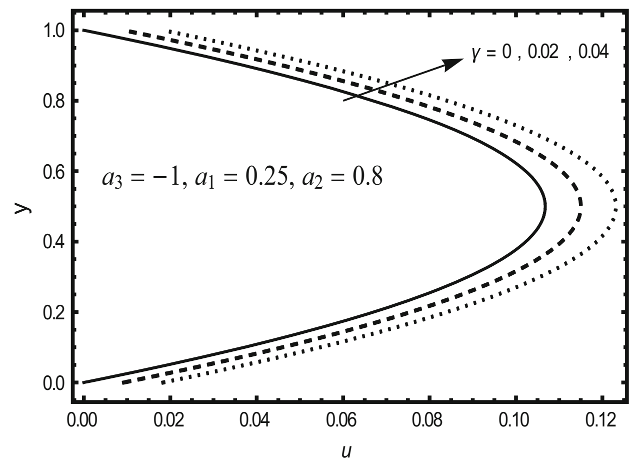

Figure 2. u vs. y for different values of γ.

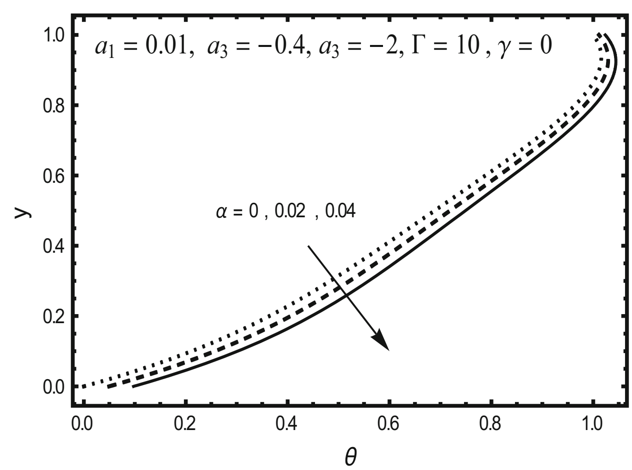

Figure 3. θ vs. y for different values of α.

Figures 2 and 3 display the results when both $a_1 \ll 1$ and $a_2 \ll 1$. From these figures, the following important observations can be made:

1. Velocity increases by increasing γ.
2. Temperature is a decreasing function of γ.

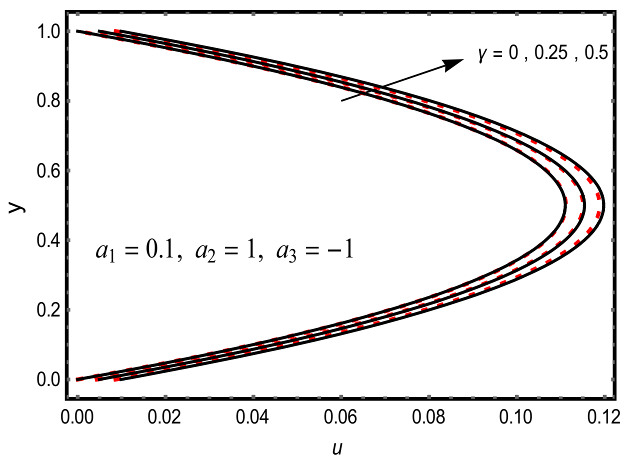

Figure 4. u vs. y for different values of γ.

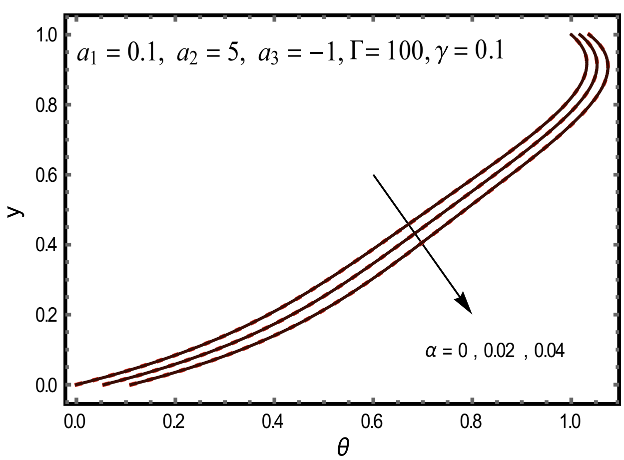

Figure 5. θ vs. y for different values of α.

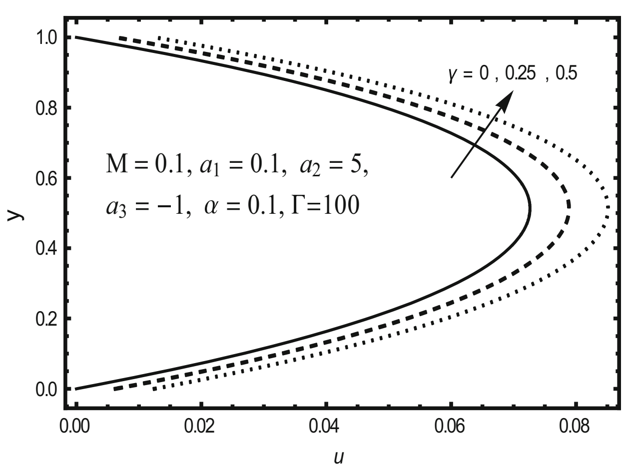

Figure 6. u vs. y for different values of γ.

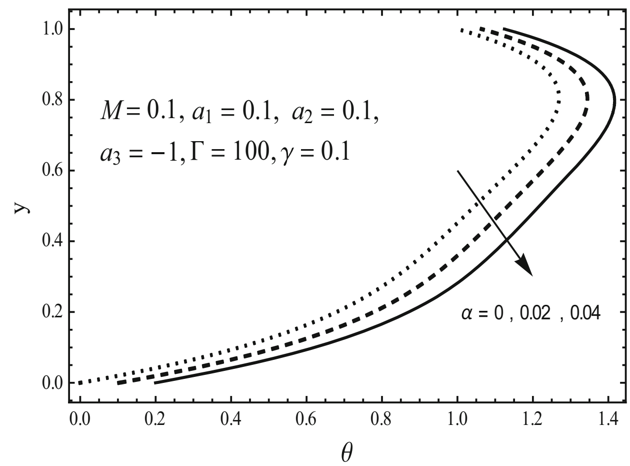

Figure 7. θ vs. y for different values of α.

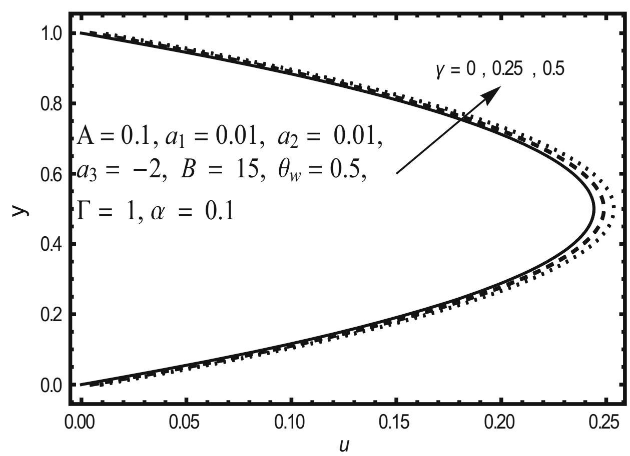

Figure 8. u vs. y for different values of γ.

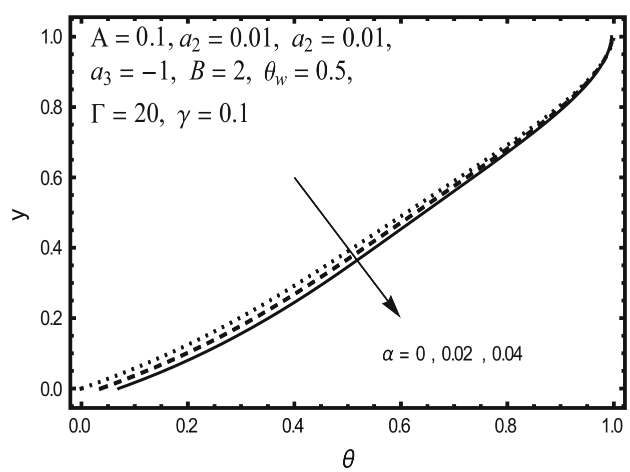

Figure 9. θ vs. y for different values of α.

3. Approximate solution is more sensitive to the slip parameter and less sensitive to the porosity parameter.

Figures 4 and 5 show the comparison between approximate and numerical solution when $a_1 \ll 1$. The red dashed lines show the numerical solution while

Table 1. Variation in C_F and Nu when $\Gamma = 100$, $a_1 = a_2 = 1$, $a_3 = -2$ and $M = 0.25$.

		Constant viscosity model		Reynolds viscosity model		Vogel's viscosity model	
α	γ	C_F	Nu	C_F	Nu	C_F	Nu
0	0.5	0.7287	5.7161	0.8014	6.4628	0.8586	6.4193
0.2	–	0.7287	6.0018	0.7345	7.9350	0.6942	7.5601
0.4	–	0.7287	6.1605	0.6639	8.2205	0.6691	7.7786
–	0	0.9368	9.1335	0.9546	14.4735	0.9469	12.7114
–	0.15	0.8601	7.9268	0.8132	11.1774	0.8279	10.6288
–	0.25	0.8169	7.5327	0.7564	10.0482	0.7693	9.6261

Table 2. Variation in C_F and Nu when $\Gamma = 10$, $a_1 = a_2 = 1$ and $a_3 = -2$.

		Per. solution		Num. solution		Absolute error	
α	γ	C_F	Nu	C_F	Nu	C_F	Nu
0.15	0.2	0.8368	0.0620	0.8378	0.0660	10^{-3}	4×10^{-3}
0.3	–	0.8368	0.2042	0.8378	0.2102	10^{-3}	6×10^{-3}
0.4	–	0.8368	0.2725	0.8378	0.2797	10^{-3}	7.2×10^{-3}
0.45	0.15	0.8601	0.3400	0.8651	0.3433	3×10^{-3}	3.3×10^{-3}
–	0.25	0.8109	0.2712	0.8169	0.2770	6×10^{-3}	5.8×10^{-3}
–	0.35	0.7787	0.2102	0.7787	0.2192	8.6×10^{-3}	9×10^{-3}

Table 3. Variation in C_F and Nu when $\Gamma = 10$, $a_1 = a_2 = 1$, $a_3 = -2$ and $M = 1$.

		Per. solution		Num. solution		Absolute error	
α	γ	C_F	Nu	C_F	Nu	C_F	Nu
0.15	0.2	0.7091	9.6285	0.7136	9.6307	4.5×10^{-3}	2.2×10^{-3}
0.3	–	0.5235	6.3805	0.5289	6.3832	5.4×10^{-3}	2.2×10^{-3}
0.4	–	0.4099	6.3232	0.4159	6.3264	6×10^{-3}	3.2×10^{-3}
0.45	0.15	0.4341	5.9941	0.4381	5.9965	4×10^{-3}	2.4×10^{-3}
–	0.25	0.4045	5.0813	0.4110	5.0863	6.5×10^{-3}	5×10^{-3}
–	0.35	0.3801	4.5935	0.3879	4.6015	7.8×10^{-3}	8×10^{-3}

Table 4. Variation in C_F and Nu when $\Gamma = 10$, $a_1 = a_2 = 1$, $a_3 = -2$ and $A = B = \theta_\mathrm{w} = 1$.

		Per. solution		Num. solution		Absolute error	
α	γ	C_F	Nu	C_F	Nu	C_F	Nu
0.15	0.2	0.8214	9.6252	0.8274	9.6296	6×10^{-3}	4.4×10^{-3}
0.3	–	0.7971	9.9113	0.8069	9.9190	9.8×10^{-3}	7.7×10^{-3}
0.4	–	0.7908	10.0005	0.8008	10.0237	10^{-2}	2.3×10^{-2}
0.45	0.15	0.8253	10.5901	0.8293	10.5914	4×10^{-3}	1.3×10^{-3}
–	0.25	0.7640	9.5850	0.7712	9.5930	7.2×10^{-3}	8×10^{-3}
–	0.35	0.7130	8.7833	0.7236	8.7833	1.06×10^{-2}	10^{-2}

black solid lines indicate the perturbation solution. It is evident from these figures that for small values of non-Newtonian parameter, approximate and numerical solutions of velocity and temperature are in excellent agreement even for large values of the porosity parameter. However, these solutions are sensitive to the slip and thermal parameters respectively, i.e. approximate solution for the velocity diverges from the numerical solution even for small values of γ. Similar is the case with the approximate solution of temperature profile.

Figures 6 and 7 illustrate the effect of various emerging parameters on velocity and temperature profiles when viscosity is temperature-dependent and is given by the Reynold's model. From these figures, it is noted that the effects of α and γ on velocity profile and α on temperature profile are similar as observed in the case of constant viscosity.

The effects of various parameters of interest on temperature and velocity profiles when viscosity is represented by Vogel's model are shown in figures 9. The trend shown in these figures by changing one parameter and fixing the others are similar to the case of Reynold's model.

A few parameters which are additional in Vogel's model are θ_w, A and B. An interesting observation is that velocity and temperature profiles for Vogel's model are less dependent on the thermal-slip parameter (α) in comparison with velocity and temperature profiles for Reynold's model.

The variation of skin-friction coefficient and Nu via boundary slip parameter (α) and thermal slip parameter (γ) in all the three cases are listed in table 1. It is observed that, the skin-friction coefficient does not change with γ for the case of constant viscosity but decreases for variable viscosity model. In the case of temperature-dependent viscosity models, the value of the skin-friction coefficient decreases with increasing slip parameters. Nu is a decreasing function of the slip parameters in all the viscosity models. The validity of the perturbation solution is presented with the numerical solutions and this validity is listed in tables 2–4. The absolute errors between numerical and perturbation solutions are listed in these tables. One- and two-digits of accuracy are obtained in wall shear stress and Nusselt number of the constant and temperature viscosity models respectively.

6. Conclusions

The boundary and thermal slip effects on the flow of third-grade fluid within two parallel porous plates are analysed. The dimensional form of the equations is transformed into dimensionless form. The perturbation method is employed to find the solution of the given boundary value problem. These cases are discussed in the current investigation based on the viscosity of the fluid and they are (a) constant viscosity model, (b) Reynolds viscosity model and (c) Vogel's viscosity model. The variation of skin friction and Nusselt number with slip parameters are presented in tabular form. The important observations of our investigation are:

1. The velocity of the fluid decreases with α and temperature increases with increasing γ.
2. No variation is observed in C_F with α.
3. Nu increases with increasing slip parameters in all the viscosity models.
4. C_F is a decreasing function of slip parameters for the temperature-dependent viscosity models.
5. One- and two-digits of accuracy is obtained between perturbation and numerical solutions in shear stress and heat transfer rate of viscosity models respectively.

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