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Lump solutions with higher-order rational dispersion relations

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Abstract. This paper aims to explore a kind of lump solutions in nonlinear dispersive waves with higher-order rational dispersion relations. We show that the second member in the commuting Kadomtsev–Petviashvili hierarchy is such an example, and construct its lump solutions, based on a Hirota trilinear form. The presented lump solutions have one peak and two valleys, where the global maximum and minimum values are achieved. A few three-dimensional plots and contour plots are made for a specific example of the lumps.

Keywords. Symbolic computation; lump solution; dispersion relation.

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1. Introduction

The main question in the theory of differential equations is to explore the existence of solutions to given differential equations, for example, dispersive wave equations describing real-world problems. Theoretically, it is also interesting to determine what kind of differential equations can possess particularly interesting given solutions such as solitons and lumps. Initial-value problems are about the existence, uniqueness and stability of solutions which satisfy the given initial data. Laplace's method and the Fourier transform method are powerful in solving initial-value problems for linear ordinary and partial differential equations, respectively. Soliton theory encompasses the systematic study of initial-value problems of soliton equations and the involved technique is called the inverse scattering transform method [1,2].

It is well-known that it is extremely difficult to construct exact solutions even for integrable equations. The Hirota bilinear method provides an efficient technique to determine soliton solutions [3,4], historically found for the Korteweg–de Vries equation. Soliton solutions are analytic functions usually exponentially localised, which are used to describe various wave phenomena in science and technology. Lump solutions are another kind of exact analytical solutions, which are rational and localised in all directions in space (see, e.g., [5–7] in (2 + 1) dimensions). The motivation is the long wave limits of N-soliton solutions [8]. The richness of lump solutions can be seen in many integrable equations in (2 + 1) dimensions (see, e.g., [5,6]). Illustrative examples of such integrable equations include the Kadomtsev–Petviashvili-I (KPI) equation [9], the B-Kadomtsev–Petviashvili (BKP) equation [10,11], the Davey–Stewartson equation II [8], the three-wave resonant interaction [12] and the Ishimori-I equation [13]. Special lump solutions of the KPI equation have been computed from N-soliton solutions [14]. Recent studies also show that there exist lumps in the Kadomtsev–Petviashvili (KP) equation with a self-consistent source [15]. A crucial step in computing lump solution is to look for positive quadratic function solutions to bilinear equations [5]. Then taking the logarithmic transformations of the obtained positive solutions, one can generate lump solutions to nonlinear

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differential equations (see, e.g. [5] and [6] in the cases of Hirota and generalised bilinear equations, respectively).

In this paper, we would like to explore a kind of lump solutions in nonlinear dispersive waves with higher-order rational dispersion relations. We shall show that the second member in the commuting KP hierarchy possesses such lump solutions. Instead of Hirota bilinear forms, we shall use a trilinear form to compute lump solutions, and unlike the cases of using bilinear forms (see, e.g., [5,6,16–18]), the dispersion relations in the presented lump solutions are higher-order rational functions of wave numbers. The peak and two valleys of the lump solutions and their corresponding extreme values will be determined through symbolic computation with Maple. A few three-dimensional plots and contour plots will be made in a specific case via the Maple plot tool, to shed light on the dynamical properties of the lump solutions. Concluding remarks will be given in the final section.

2. Lump solutions

2.1 *Second KPI equation*

It is obvious that under a reduction of $\alpha = (\sqrt{6}/6)i$, the second member of the commuting KP hierarchy (see [19]) reads as

$$P(u,v) = u_t + 2u_x v_x + 4uu_y - \frac{1}{6}v_{yy} + u_{xxy} = 0, \tag{2.1}$$

where $v_{xx} = u_y$. The equation has a trilinear form

$$\begin{aligned} B(f) = {} & f^2 f_{yyy} - 3ff_y f_{yy} - 6f^2 f_{xxt} \\ & +6ff_t f_{xx} - 12 f_t f_x^2 + 12 ff_x f_{tx} \\ & -6f^2 f_{xxxxy} + 6ff_y f_{xxxx} + 24 ff_x f_{xxxy} \\ & -12 ff_{xx} f_{xxy} + 2f_y^3 - 24 f_x f_y f_{xxx} + 12 f_y f_x^2 \\ & -24 f_x^2 f_{xxy} + 24 f_x f_{xx} f_{xy} = 0 \end{aligned} \tag{2.2}$$

under the second- and first-order logarithmic transformations

$$u = 2(\ln f)_{xx}, \ v = 2(\ln f)_y. \tag{2.3}$$

Precisely, we have

$$B(f) = -3f^3 P(u,v).$$

We call eq. (2.1) as the second KPI equation, to reflect the second member in the KP hierarchy and the similarity to the KPI equation. In what follows, we would like to determine lump solutions to the second KPI equation (2.1), through symbolic computation with Maple.

2.2 *Higher-order rational dispersion relation*

As usual, we begin with a search for positive quadratic solutions to the corresponding trilinear equation (2.2):

$$\begin{aligned} f = {} & (a_1x + a_2y + a_3t + a_4)^2 \\ & +(a_5x + a_6y + a_7t + a_8)^2 + a_9 \end{aligned} \tag{2.4}$$

in order to generate lump solutions to the second KPI equation (2.1). Substituting this function f into the trilinear equation (2.2) yields a large system of nonlinear algebraic equations on the involved parameters a_i, $1 \le i \le 9$. To get solutions to this nonlinear system, we conduct a direct symbolic computation to obtain a set of solutions for those parameters. The dispersion relations and the translation in the set of solutions read as

$$\begin{cases} a_3 = \dfrac{a_1^2a_2^3 - 3a_1^2a_2a_6^2 + 6a_1a_2^2a_5a_6 - 2a_1a_5a_6^3 - a_2^3a_5^2 + 3a_2a_5^2a_6^2}{6(a_1^2+a_5^2)^2}, \\ a_7 = \dfrac{3a_1^2a_2^2a_6 - a_1^2a_6^3 - 2a_1a_2^3a_5 + 6a_1a_2a_5a_6^2 - 3a_2^2a_5^2a_6 + a_5^2a_6^3}{6(a_1^2+a_5^2)^2}, \\ a_9 = \dfrac{6(a_1^2+a_5^2)^3}{(a_1a_6 - a_2a_5)^2}, \end{cases} \tag{2.5}$$

and all other parameters a_i's are arbitrary.

The first two formulas above exhibit a novel kind of interactive dispersion relations between two dispersive waves. Compared with the case of the KPI equation [9], we find that the dispersion relations in the second KPI equation involve higher-order rational dependences on the wave numbers, a_1, a_2, a_5 and a_6. This is the first example to exhibit such higher-order rational dispersion relations, whose numerators are of degree 5 and whose denominators are of degree 4. All previous examples of lumps in the literature, including higher-order lumps (see, e.g., [8,20,21]), only have the third-order numerators and the second-order denominators of the wave numbers in the wave variables. One reason for the difference should be the differential order of Lax operators of symmetries that are used to formulate soliton equations, which reflect the order of practical perturbations that could be taken for physical systems.

2.3 *Lump characteristic*

The natural condition

$$a_1a_6 - a_2a_5 \neq 0 \tag{2.6}$$

guarantees that the functions

$$u = 2(\ln f)_{xx} = \frac{2(f_{xx}f - f_x^2)}{f^2},$$

$$v = 2(\ln f)_y = \frac{2f_y}{f}, \tag{2.7}$$

decay in all space directions. It also implies that $a_1^2 + a_5^2 > 0$, and so based on (2.5), f is positive and further u and v are analytic. Therefore, under condition (2.6), (2.7) presents lump solutions to the second KPI equation (2.1), together with (2.4) and (2.5).

Condition (2.6) is also necessary for (2.7) to define lump solutions to (2+1)-dimensional soliton equations.

2.4 *Extreme values*

We point out that in what follows, all formulas are obtained under simplification processes with Maple. A direct computation shows that the lump solution u has three critical points:

$$(x_i, y_0), \ i = 1, 2, 3, \tag{2.8}$$

where

$$\begin{cases} x_1 = \dfrac{(a_2^2 + a_6^2)(a_1a_2 + a_5a_6)}{3(a_1^2 + a_5^2)^2}t + \dfrac{a_2a_8 - a_4a_6}{a_1a_6 - a_2a_5}, \\ x_{2,3} = \dfrac{z_\pm}{3(a_1^2 + a_5^2)^2(a_1a_6 - a_2a_5)}, \\ y_0 = -\dfrac{3a_1^2a_2^2 - a_1^2a_6^2 + 8a_1a_2a_5a_6 - a_2^2a_5^2 + 3a_5^2a_6^2}{6(a_1^2+a_5^2)^2}t \\ \qquad - \dfrac{a_1a_8 - a_4a_5}{a_1a_6 - a_2a_5}. \end{cases} \tag{2.9}$$

The values $z_\pm$ are two solutions of the following quadratic equation:

$$z^2 + (p_1t + p_2)z + p_3t^2 + p_4t + p_5 = 0, \tag{2.10}$$

where

$$\begin{cases} p_1 = -2(a_2^2 + a_6^2)(a_1a_2 + a_5a_6)(a_1a_6 - a_2a_5), \\ p_2 = -6(a_1^2 + a_5^2)^2(a_2a_8 - a_4a_6), \\ p_3 = (a_2^2 + a_6^2)^2(a_1a_2 + a_5a_6)^2(a_1a_6 - a_2a_5)^2, \\ p_4 = 6(a_2^2 + a_6^2)(a_1^2 + a_5^2)^2(a_1a_2 + a_5a_6) \\ \qquad \times (a_1a_6 - a_2a_5)(a_2a_8 - a_4a_6), \\ p_5 = -9(a_1^2+a_5^2)^4[18(a_1^2+a_5^2)^2 - (a_2a_8 - a_4a_6)^2]. \end{cases} \tag{2.11}$$

The discriminant of this quadratic polynomial can be worked out as follows:

$$(p_1t + p_2)^2 - 4(p_3t^2 + p_4t + p_5) = 648(a_1^2 + a_5^2)^6,$$

which is always positive when (2.6) holds and so guarantees the existence of the two real solutions $z_\pm$.

By the second derivative test in calculus, the lump solution u has a peak at the first critical point (x_1, y_0), because we have

$$u_{xx} = -\frac{2(a_1a_6 - a_2a_5)^4}{3(a_1^2 + a_5^2)^4} < 0,$$

$$u_{xx}u_{yy} - u_{xy}^2 = \frac{4(a_1a_6 - a_2a_5)^{10}}{27(a_1^2 + a_5^2)^{10}} > 0,$$

at the critical point (x_1, y_0), and two valleys at the second and third critical points $(x_{2,3}, y_0)$, because we have

$$u_{xx} = \frac{(a_1a_6 - a_2a_5)^4}{48(a_1^2 + a_5^2)^4} > 0,$$

$$u_{xx}u_{yy} - u_{xy}^2 = \frac{(a_1a_6 - a_2a_5)^{10}}{1728(a_1^2 + a_5^2)^{10}} > 0,$$

at the critical points $(x_{2,3}, y_0)$. The values of the lump solution u at the peak and the two valleys are

$$u_1 = \frac{2(a_1a_6 - a_2a_5)^2}{3(a_1^2 + a_5^2)^2}, \quad u_{2,3} = -\frac{(a_1a_6 - a_2a_5)^2}{12(a_1^2 + a_5^2)^2}, \tag{2.12}$$

which are also the global maximum and minimum values of the lump solution u, because u decays in all space directions. Based on (2.9) and (2.10), we see that the single peak moves in a straight line and the two valleys move in algebraic curves in space, when time t changes.

2.5 *An illustrative example*

If we take

$$a_1 = 2, \ a_2 = -1, \ a_4 = 1, \ a_5 = -2, \ a_6 = -1, \ a_8 = -1, \tag{2.13}$$

the transformations in (2.7) with (2.4) and (2.5) present the lump solutions u and v for the second KPI

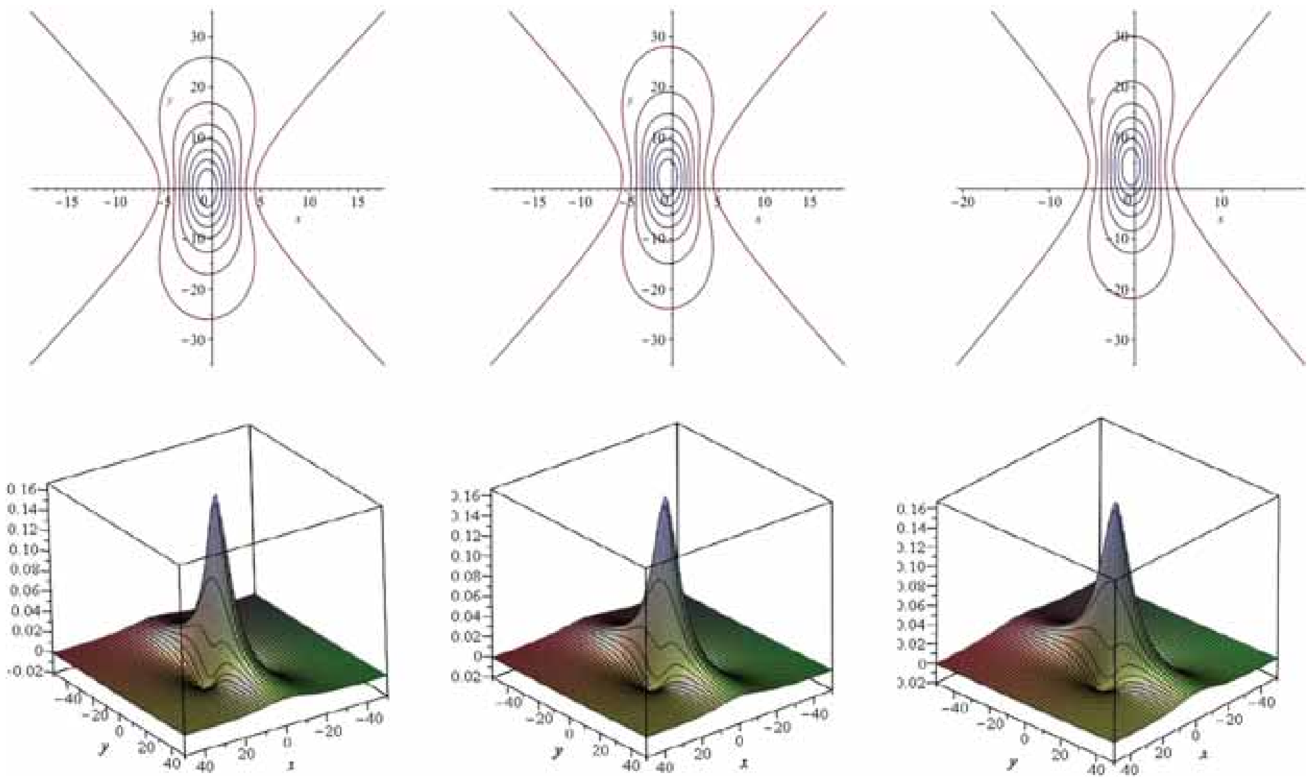

Figure 1. Profiles of u when $t = 0, 50, 100$: contour plots (top) and 3D plots (bottom).

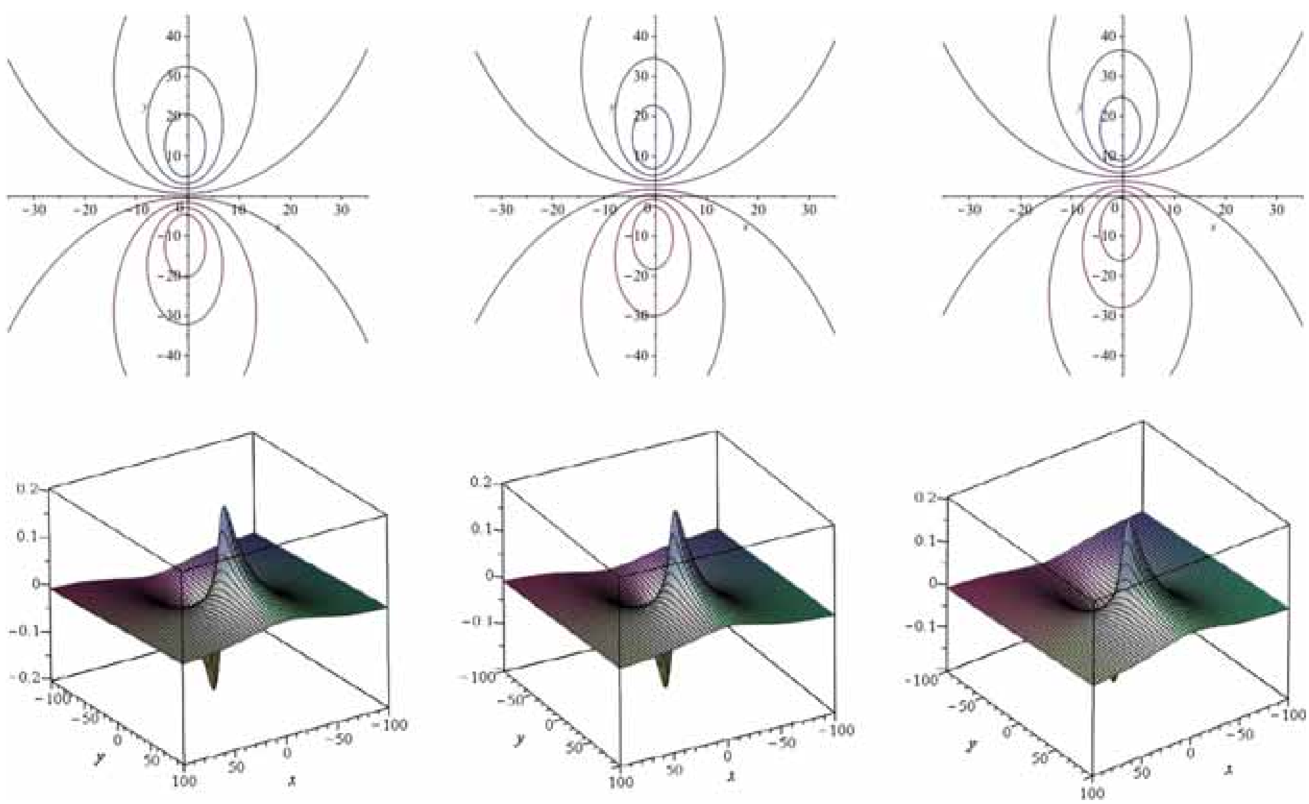

Figure 2. Profiles of v when $t = 0, 50, 100$: contour plots (top) and 3D plots (bottom).

equation (2.1):

$$u=\frac{9216(t^2-48ty-2304x^2+576y^2-2304x+54720)}{(t^2-48ty-2304x^2+576y^2+2304x+55872)^2} \tag{2.14}$$

and

$$v=\frac{2304y-96t}{t^2-48ty-2304x^2+576y^2+2304x+55872}, \tag{2.15}$$

where the denominators have no singularity, as we explicitly have

$$t^2-48ty-2304x^2+576y^2+2304x+55872$$
$$=288\left(\frac{1}{24}t+2x-y+1\right)^2$$
$$+288\left(\frac{1}{24}t-2x-y-1\right)^2+55296>0.$$

Three three-dimensional plots and contour plots of the lump solutions u and v are made via Maple plot tools, to shed light on the characteristic of the lump solutions, in figures 1 and 2, respectively.

All the presented lump solutions above gain invaluable insights into the existing results on exact solutions to nonlinear differential equations, including soliton solutions [1–3] and dromion solutions [22–25]. Lumps, solitons and dromions share the same coherent structures, except lumps decay algebraically but solitons and dromions decay exponentially. Our results also provide some supplements to the existing literature on different effective approaches such as the Hirota bilinear method, the Wronskian approach, the Riemann–Hilbert technique, the algebro-geometric method, and symmetry reductions and constraints (see, e.g., [26–34]).

3. Concluding remarks

We have studied the second member in the KP hierarchy and constructed its lump solutions. The results enrich the discussions in the literature about lumps and solitons, providing the first example of applying trilinear forms to explore nonlinear partial differential equations which possess lump solutions. The study consists of a kind of symbolic computations with Maple. It is particularly interesting that the dispersion relations in the obtained lump solutions involve higher-order rational dependences on the wave numbers. All existing examples only involve lower-order rational dispersion relations. Contour plots and three-dimensional plots were drawn for a specific example by using Maple.

It is known from recent studies that many (2 + 1)-dimensional nonlinear equations possess lump solutions, and those equations contain the generalised KP equation, the generalised BKP equation, the KP–Boussinesq equation, the generalised Bogoyavlensky–Konopelchenko equation, the generalised Calogero–Bogoyavlenskii–Schiff equation and the Sawada–Kotera equation [35–39]. It has also been demonstrated that linear partial differential equations can possess abundant lump solutions [27,40,41], besides nonlinear partial differential equations in (2+1) dimensions (see, e.g., [42–46]) and in (3+1) dimensions (see, e.g., [47–52]). Furthermore, there exist interaction solutions [53] to (2 + 1)-dimensional integrable equations, and such solutions include lump soliton solutions of homoclinic type (see, e.g., [54–56]) and lump-kink solutions of heteroclinic type (see, e.g., [57–60]). It should be interesting to look for interaction solutions with lump solutions whose dispersion relations involve higher-order rational dependences on wave numbers. Lump and interaction solutions also exhibit diversity of exact solutions constructed from other kinds of combinations (see, e.g., [61–63]), and they can lead to many symmetries of Lie–Bäcklund type, from which one can formulate conservation laws (see [64–66] for conservation laws by pairs of symmetries and adjoint symmetries).

We also remark that under a reduction of $\alpha=\sqrt{6}/6$, the second member of the commuting KP hierarchy (see [19]) gives the second KPII equation

$$P(u,v)=u_t+2u_xv_x+4uu_y+\frac{1}{6}v_{yy}+u_{xxy}=0,$$

where $v_{xx}=u_y$. This equation has a trilinear form

$$B(f)=-f^2f_{yyy}+3ff_yf_{yy}-6f^2f_{xxt}$$
$$+6ff_tf_{xx}-12f_tf_x^2$$
$$+12ff_xf_{tx}-6f^2f_{xxxxy}+6ff_yf_{xxxx}$$
$$+24ff_xf_{xxxy}-12ff_{xx}f_{xxy}$$
$$-2f_y^3-24f_xf_yf_{xxx}+12f_yf_x^2$$
$$-24f_x^2f_{xxy}+24f_xf_{xx}f_{xy}=0,$$

under the logarithmic transformations

$$u=2(\ln f)_{xx},\quad v=2(\ln f)_y.$$

Clearly, there is the same relation between the two equations as before: $B(f)=-3f^3P(u,v)$. Also, we can similarly determine that a kind of polynomial solutions to the above trilinear equation is given by

$$f=(a_1x+a_2y+a_3t+a_4)^2$$
$$+(a_5x+a_6y+a_7t+a_8)^2+a_9,$$

where

$$\begin{cases} a_3 = -\dfrac{a_1^2a_2^3 - 3a_1^2a_2a_6^2 + 6a_1a_2^2a_5a_6 - 2a_1a_5a_6^3 - a_2^3a_5^2 + 3a_2a_5^2a_6^2}{6(a_1^2+a_5^2)^2}, \\ a_7 = -\dfrac{3a_1^2a_2^2a_6 - a_1^2a_6^3 - 2a_1a_2^3a_5 + 6a_1a_2a_5a_6^2 - 3a_2^2a_5^2a_6 + a_5^2a_6^3}{6(a_1^2+a_5^2)^2}, \\ a_9 = -\dfrac{6(a_1^2+a_5^2)^3}{(a_1a_6 - a_2a_5)^2}. \end{cases}$$

Here the three signs in the frequencies and the translation were just changed. Due to the negative translation parameter a_9, the polynomial solutions of the above kind never yield any lump solution to the second KPII equation above. This leads us to the conjecture that no lump solution would exist for the second KPII equation. Finally, we remark that we could consider other members in the KP hierarchy, which could exhibit even higher-order rational dispersion relations. It would also be important to explore what relation could exist between dispersion relations of different members in the commuting KP hierarchy.

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