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Superposition behaviour between lump solutions and different forms of N-solitons ($N \to \infty$) for the fifth-order Korteweg–de Vries equation

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Abstract. A lump-type solution of the (2 + 1)-dimensional generalised fifth-order Korteweg–de Vries (KdV) equation is obtained from the two-soliton solution by applying the parametric limit method. Some theorems and corollaries about the superposition behaviour between lump solutions and different forms of N-soliton ($N \to \infty$) solutions are constructed, and detailed proofs are given. Besides,we give a large number of examples and spatial evolution graphics to illustrate the effectiveness of the described theorems and corollaries. Some new nonlinear phenomena and superposition behaviour, such as rational-exponential type, rational-cosh-cos type, rational-sin type, rational-logarithmic type etc., are simulated and shown for the first time. Finally, we also illustrate the superposition between high-order lump-type solutions and N-soliton solutions.

Keywords. Fifth-order KdV equation; lump solutions; superposition behaviour; soliton solutions; Hirota's bilinear method.

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1. Introduction

In recent years, the study of lump solution has become a hot topic in the field of nonlinear partial differential equations (NLPDEs). Lump solution, as solitary wave solution in the form of rational functions, shows polynomial decay in all directions of space variables. The first mathematical description of lump solution was reported in 1977 by studying the analytical expression of two-dimensional solitons in KPI equation [1]. Up to now, many effective methods have been proposed to study lump solutions of NLPDEs. Ablowitz and Satsuma [2,3] first proposed the long wave limit method based on exponential function in 1978 to study the N-solitons and M-lump solutions of NLPDEs. Imai [4] constructed Grammian determinant based on binary Darboux transformation to study a class of lump function solutions. Ma *et al* [5,6] proposed direct method based on a class of quadratic polynomial functions to obtain lump solutions. Dai *et al* [7–10] introduced homoclinic (heteroclinic) breather limit techniques based on exponential-trigonometric functions to study rogue waves and lump-type solutions and so on. Very recently, the study of solitons related to lump solutions, such as lump-type solutions, rogue waves, interaction solutions, hybrid solutions, semirational solutions and interaction solitons, has attracted much attention because of their application value and existence theory in the fields of hydrodynamics, nonlinear optics, electromagnetics, optical fibre communications, etc. [11–22]. However, we find that most of the papers contain studies on the interaction solutions between lump solution and N-solitons, where the soliton number N is often less than 3.

It is well known that the superposition of solutions often exists in linear systems, but the superposition of solutions often does not exist for nonlinear equations, especially for NLPDEs. This paper aims to investigate lump-type solution of the (2 + 1)-dimensional generalised fifth-order Korteweg–de Vries (KdV) equation and superposition behaviour of lump solution and different forms of N-solitons ($N \to \infty$). In refs [23,24], the following fifth-order KdV equation is given:

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$$u_t + \alpha u_{xxxxx} + \beta u u_{xxx} + \gamma u_x u_{xx} + \delta u^2 u_x + u_y = 0, \quad (1)$$

which describes motions of long waves in shallow water under gravity field and in a two-dimensional nonlinear lattice, where α, β, γ and δ are some arbitrary constants. As a typical high-order NLPDE, many exact solutions of eq. (1) have been reported by different methods. The lump solution and interaction solution of eq. (1) are studied in refs [23,24] by applying the direct method. Wazwaz [25] studied different forms of soliton solutions by employing extended tanh method. Kim and Lee [26] reported its explicit solutions by using the system technique. In the present paper, we shall study the emergence behaviour from two-soliton solutions to lump-type solutions, and give theoretical proof of superposition property between lump solution and N-solitons ($N \to \infty$) with different forms. Besides, some new nonlinear phenomena are simulated and displayed.

2. From two-soliton solutions to lump-type solutions

According to refs [23,24], we can take the following bilinear transformations when $\beta = 15\alpha$, $\gamma = 45\alpha$ and $\delta = 15\alpha$:

$$u(x, y, t) = 2(\ln f)_{xx}, \quad (2)$$

where $f(x, y, t)$ is an unknown real function which will be determined. We get the bilinear form of eq. (1) by plugging eq. (2) into eq. (1).

$$P(D_x, D_y, D_t) f \cdot f = (D_x D_t + D_x D_y + \alpha D_x^6) f \cdot f = 0, \quad (3)$$

where $D_{\_}$ is the bilinear differential operators defined by [27],

$$D_x^l D_y^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^l \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial y'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x, y, t) \cdot g(x', y', t')|_{(x,y,t)=(x',y',t')}. \quad (4)$$

To search for two-soliton solutions and lump solution of eq. (1), according to refs [10,28], we choose the following test functions:

$$f(x, y, t) = \mathrm{e}^{-\xi_1} + \delta_2 \cos(\xi_2) + \delta_1 \mathrm{e}^{\xi_1}, \quad (5)$$

where

$$\xi_i = k_i(x + p_i y + q_i t + r_i)$$

and $\delta_i, k_i, p_i, q_i, r_i (i = 1, 2)$ are arbitrary constants to be determined later. Collecting all coefficients of $\mathrm{e}^{-\xi_1}, \mathrm{e}^{\xi_1}, \cos(\xi_2), \sin(\xi_2)$ and letting the constant term to zero with the help of Maple 17, we get

$$p_1 = -\alpha(k_1^4 - 10k_1^2k_2^2 + 5k_2^4) - q_1,$$

$$p_2 = -\alpha(5k_1^4 - 10k_1^2k_2^2 + k_2^4) - q_2,$$

$$\delta_2 = \pm\sqrt{\frac{4k_1^2(3k_1^2 - k_2^2)\delta_1}{k_2^2(3k_2^2 - k_1^2)}}, \quad (6)$$

where the parameters k_1 and k_2 satisfy

$$\frac{1}{3} < \frac{k_2^2}{k_1^2} < 3 \quad \text{or} \quad \frac{1}{3} < \frac{k_1^2}{k_2^2} < 3.$$

Substituting eq. (5) with relations (6) into eq. (2), we obtain the following two-soliton solution (see figure 1a):

$$u = \frac{4\sqrt{\delta_1}\delta_2((k_1^2 - k_2^2)\Phi + 2k_1k_2\Psi) - 2\delta_2^2k_2^2 + 8\delta_1k_1^2}{\left(2\sqrt{\delta_1}\cosh(\xi_1 + \ln\sqrt{\delta_1}) + \delta_2\cos\xi_2\right)^2}, \quad (7)$$

where

$$\Phi = \cosh(\xi_1 + \ln\sqrt{\delta_1})\cos\xi_2$$

and

$$\Psi = \sinh(\xi_1 + \ln\sqrt{\delta_1})\sin\xi_2.$$

In order to obtain the lump-type solution of eq. (1), taking $k_2 = k_1$, $\delta_1 = 1$, $\delta_2 = -2\sqrt{\delta_1}$ in eq. (7), and letting k_1 tends to 0, we obtain lump-type solution in the following form (see figure 1b):

$$u(x, y, t)_{\text{lump-type}} = \frac{-16(x - q_1y + q_1t + r_1)(x - q_2y + q_2t + r_2)}{\left((x - q_1y + q_1t + r_1)^2 + (x - q_2y + q_2t + r_2)^2\right)^2}. \quad (8)$$

Figure 1 shows the spatial structure of two-soliton solution (eq. (7)) and lump-type solution (eq. (8)), in which the values of parameters are as follows: (a) $r_1 = r_2 = 0$, $k_1 = k_2 = \frac{1}{2}$, $q_1 = q_2 = \delta_1 = \alpha = 1$; (b) $r_1 = r_2 = 0$, $q_1 = -1$, $q_2 = 1$. The two-soliton solution degenerates into a rational function solution when $k_1 \to 0$. Note that eq. (8) is decaying as $O(x^{-2}, y^{-2})$ for $|x|, |y| \to \infty$. This means that the lump solution (eq. (8)) is rationally localised in all directions in the space, but eq. (8) has singularity according to the expression. Therefore, this form of rational solution is called lump-type solution [29,30].

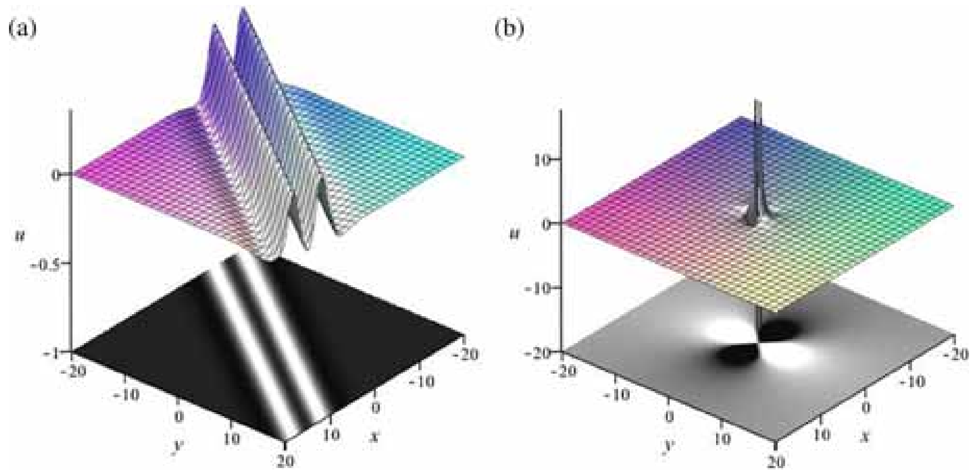

Figure 1. Spatial structure evolution diagram from two-soliton solution to lump-type solution as $t = 0$.

3. Superposition between the lump solution and N-solitons

In this section, we shall study the existence theorems of superposition between the lump solution and N-solitons ($N \to \infty$) with different forms, and give a detailed proof. Here, we choose a test function consisting of a quadratic function and N exponential functions,

$$f(x, y, t) = a_0 + \sum_{i=1}^{I}(a_i x + b_i y + c_i t + d_i)^2 + \sum_{n=1}^{N} \delta_n e^{\nu_n(p_n x + q_n y + r_n t + s_n)} \overset{\text{def}}{=} h(x, y, t) + g(x, y, t), \tag{9}$$

where a_0, a_i, b_i, c_i, d_i $(i = 1, 2, \ldots, I)$ and p_n, q_n, $\nu_n, r_n, s_n, \delta_n$ $(n = 1, 2, \ldots, N)$ are some arbitrary constants to be determined. $h(x, y, t)$ denotes a polynomial function and $g(x, y, t)$ is an exponential function.

Theorem 1. *If $c_i = -b_i$, $p_n = 0$ and $r_n = -q_n$ in eq. (9), then $h(x, y, t)$, $g(x, y, t)$ and $f(x, y, t)$ are solutions of eq. (3).*

Proof. By direct computation

$$P(D_x, D_y, D_t) f \cdot f = ((D_t + D_y + \alpha D_x^5) D_x) \times (h \cdot h + h \cdot g + g \cdot h + g \cdot g). \tag{10}$$

Note that $p_n = 0$, then $g_x = 0$, and we have

$$P(D_x, D_y, D_t) g \cdot g = ((D_t + D_y + \alpha D_x^5) D_x)(g \cdot g) = 0. \tag{11}$$

Then $g(x, y, t)$ is the solution of bilinear equation (3). According to the expression of h and $c_i = -b_i$, we get

$$\begin{aligned} P(D_x, D_y, D_t) h \cdot h &= 2(h_{xt}h - h_x h_t + h_{xy}h - h_x h_y \\ &\quad + h_{xxxxxx}h - 6h_{xxxxx}h_x \\ &\quad + 15h_{xxxx}h_{xx} - 10h_{xxx}^2) \\ &= 2((h_{xt} + h_{xy})h - h_x(h_y + h_t)) \end{aligned} \tag{12}$$

and

$$\begin{aligned} h_{xt} + h_{xy} &= \sum_{i=1}^{I} 2a_i c_i + \sum_{i=1}^{I} 2a_i b_i \\ &= \sum_{i=1}^{I} 2a_i(c_i + b_i) = 0, \\ h_y + h_t &= \sum_{i=1}^{I} 2(c_i + b_i)(a_i x + b_i y + c_i t + d_i) = 0. \end{aligned} \tag{13}$$

Thus

$$P(D_x, D_y, D_t)(h \cdot h) = 0. \tag{14}$$

Then, $h(x, y, t)$ is also the solution of eq. (3). Substituting eqs (11) and (14) into eq. (10), and combining the expressions of h and g, we deduce that

$$\begin{aligned} P(D_x, D_y, D_t) f \cdot f &= 2((D_t + D_y + D_x^5) D_x)(h \cdot g) \\ &= 2\big(D_t + D_y + D_x^5\big)(h_x g - h g_x) \\ &= 4\big(h_{xt}g - h_x g_t + h_{xy}g - h_x g_y + h_{xxxxxx}g \end{aligned}$$

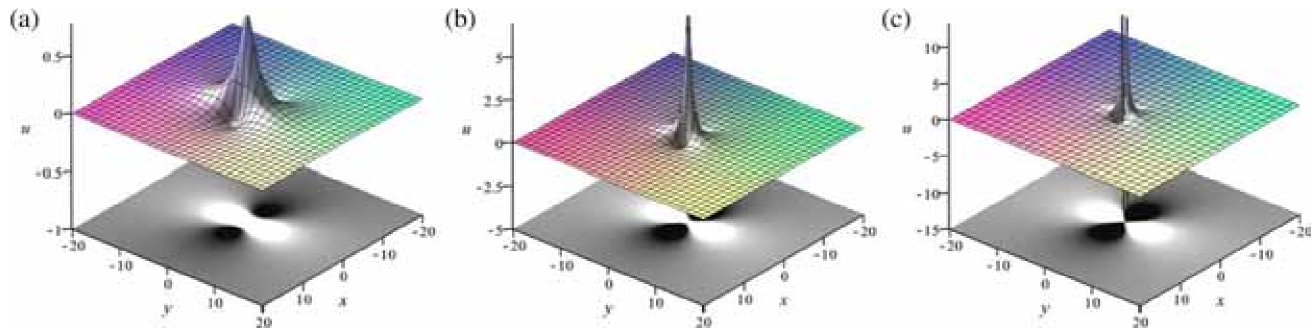

Figure 2. Evolutionary behaviour of spatial structure of lump solution $u(x, y, t)$ when $N = 0$. (**a**) $a_0 = 10$, (**b**) $a_0 = 1$ and (**c**) $a_0 = 0$.

$$-6h_{xxxxx}g_x + 15h_{xxxx}g_{xx} - 10h_{xxx}g_{xxx}\Big) = 4\big(g(h_t + h_y)_x - h_x(g_y + g_t)\big) \quad (15)$$

and

$$g_y + g_t = \sum_{n=1}^{N} \delta_n(q_n + r_n)e^{\nu_n(p_n x + q_n y + r_n t + s_n)} = 0. \quad (16)$$

Then $P(D_x, D_y, D_t) f \cdot f = 0$. The theorem is thus proved.

Therefore, some new superposition solutions of eq. (1) are obtained by substituting eq. (9) into eq. (2).

$$u = 2\left(\frac{2\sum_{i=1}^{I} a_i^2}{h + g} - \frac{\left(2\sum_{i=1}^{I} a_i(a_i x + b_i y - b_i t + d_i)\right)^2}{(h + g)^2}\right). \quad (17)$$

Now, we study and show the evolution behaviour of the spatial structure of the superposition solution by taking different parameter values and soliton numbers N. When $N = 0$, superposition solution (eq. (17)) represents a lump solution for which the spatial structure is mainly determined by the value of the parameter a_0. Figure 2 shows the spatial structure of lump solution when $t = 0$ and the parameter a_0 takes different values, but there is no smooth graphic structure when $a_0 < 0$. The parameter values in figure 2 are $(I, a_1, b_1, d_1, a_2, b_2, d_2) = (2, 1, -1, 0, 1, 1, 0)$, i.e., eq. (17) when $f(x, y, 0) = a_0 + 2x^2 + 2y^2$ and $g(x, y, t) = 0$. Figure 3 shows the structure evolution behaviour of the superposition solution equation (17) with $f(x, y, t)$ given in eq. (9) and with the increase of soliton number N. By choosing appropriate parameter values in eq. (9), we get the corresponding expressions of figure 3 as follows:

(a) $f(x, y, 0) = 10 + 2x^2 + 2y^2 + 2e^{2y+1}$;

(b) $f(x, y, 0) = 10 + 2x^2 + 2y^2 + 2e^{2y+1} + 2e^{-2y-1}$;

(c) $f(x, y, 0) = 10 + 2x^2 + 2y^2 + 2e^{2y+1} + 2e^{-2y-1} + 2e^{10y+2}$.

Note that

$$\cosh(\nu_n(q_n y - q_n t + s_n)) = \frac{1}{2}\left(e^{\nu_n(q_n y - q_n t + s_n)} + e^{-\nu_n(q_n y - q_n t + s_n)}\right),$$

and according to Theorem 1, we have the following corollary.

COROLLARY 1

Let

$$f(x, y, t) = a_0 + \sum_{i=1}^{I}(a_i x + b_i y - b_i t + d_i)^2 + \sum_{l=1}^{L} \epsilon_l \cosh(\nu_l(q_l y - q_l t + s_l)). \quad (18)$$

Then $f(x, y, t)$ is a solution of eq. (3).

Letting $\mu_k = I\nu_l$, where I denotes imaginary units, and according to Corollary 1, we have the following corollary.

COROLLARY 2

Take

$$f(x, y, t) = a_0 + \sum_{i=1}^{I}(a_i x + b_i y - b_i t + d_i)^2 + \sum_{k=1}^{K} \lambda_k \cos(\mu_k(q_k y - q_k t + s_k)). \quad (19)$$

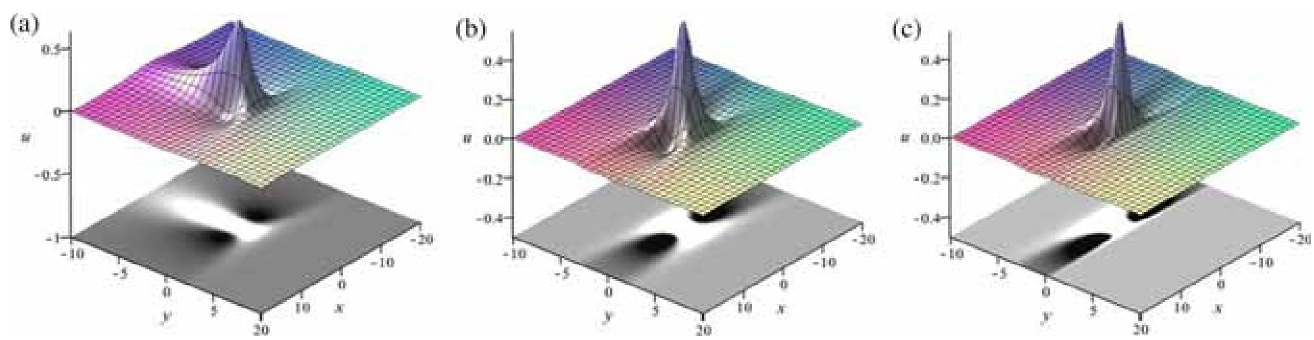

Figure 3. Spatial superposition behaviour of solution $u(x, y, t)$ with $f(x, y, t)$ given in eq. (9) and with increasing soliton number. (**a**) $N = 1$, (**b**) $N = 2$ and (**c**) $N = 3$.

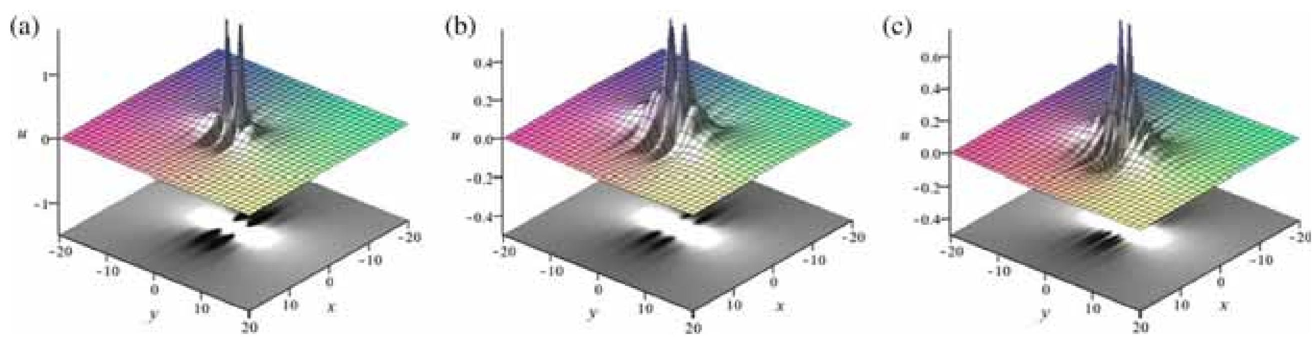

Figure 4. Spatial superposition behaviour of solution $u(x, y, t)$ with $f(x, y, t)$ given in eq. (19) and with increasing soliton number. (**a**) $K = 1$, (**b**) $K = 2$ and (**c**) $K = 3$.

Then $f(x, y, t)$ is a solution of eq. (3).

By choosing appropriate parameter values in eq. (19), we get the corresponding expressions of figure 4 as follows:

(a) $f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\cos(2y)$;

(b) $$f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\cos(2y) + 10\cos\left(\frac{1}{4}y\right);$$

(c) $$f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\cos(2y) + 10\cos\left(\frac{1}{4}y\right) + 10\cos(10y).$$

COROLLARY 3

Take

$$f(x, y, t) = a_0 + \sum_{i=1}^{I}(a_i x + b_i y - b_i t + d_i)^2 + \sum_{j=1}^{J}\omega_j \sin(\tau_j(q_j y - q_j t + s_j)). \qquad (20)$$

Then $f(x, y, t)$ is a solution of eq. (3).

Similarly, by choosing appropriate parameter values in eq. (20), we get the corresponding expressions of figure 5 as follows:

(a) $f(x, y, 0) = 10 + 2x^2 + 2y^2 - 10\sin(y - 1)$;

(b) $$f(x, y, 0) = 10 + 2x^2 + 2y^2 - 10\sin(y - 1) + 10\sin\left(\frac{1}{4}y + 1\right);$$

(c) $$f(x, y, 0) = 10 + 2x^2 + 2y^2 - 10\sin(y - 1) + 10\sin\left(\frac{1}{4}y + 1\right) + 10\sin(3y + 2).$$

COROLLARY 4

Let

$$f(x, y, t) = a_0 + \sum_{i=1}^{I}(a_i x + b_i y - b_i t + d_i)^2 + \sum_{n=1}^{N}\delta_n e^{\nu_n(q_n y - q_n t + s_n)} + \sum_{l=1}^{L}\epsilon_l \cosh(\nu_l(q_l y - q_l t + s_l))$$

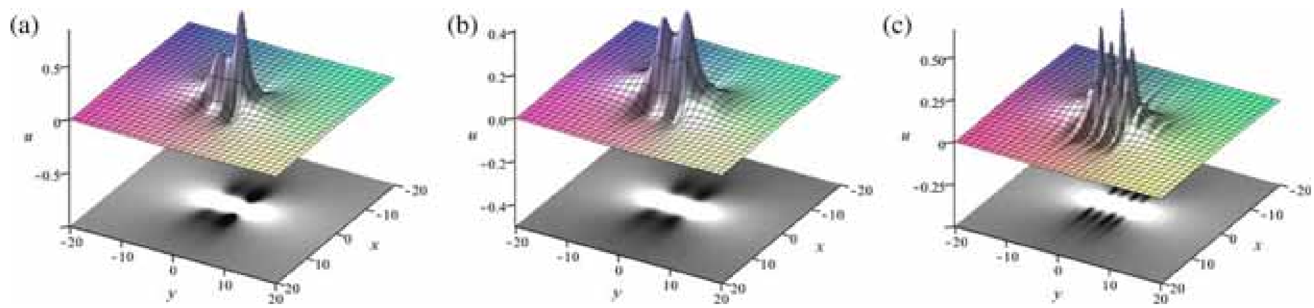

Figure 5. Spatial superposition behaviour of solution $u(x, y, t)$ with $f(x, y, t)$ given in eq. (20) and with increasing soliton number. (**a**) $J = 1$, (**b**) $J = 2$ and (**c**) $J = 3$.

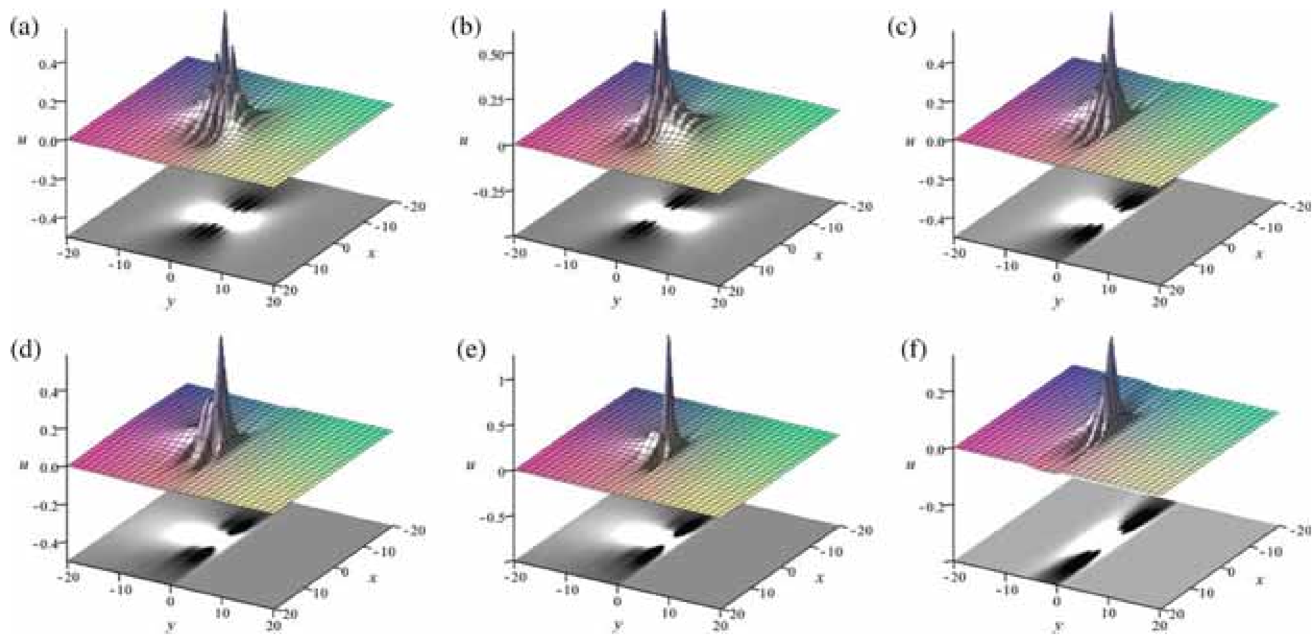

Figure 6. Spatial superposition behaviour of solution $u(x, y, t)$ with $f(x, y, t)$ given in eq. (21): (**a**) rational-cos-cosh type, (**b**) rational-cos-cosh-sin type, (**c**) rational-cos-cosh-exponential type, (**d**) rational-cos-sin-exponential type, (**e**) rational-sin-exponential type and (**f**) rational-cos-cosh-sin-exponential type.

$$+\sum_{k=1}^{K}\lambda_k\ \cos(\mu_k(q_k y - q_k t + s_k))$$
$$+\sum_{j=1}^{J}\omega_j\ \sin(\tau_j(q_j y - q_j t + s_j))$$
$$+\sum_{m=1}^{M}\sigma_m\ \sinh(\kappa_m(q_m y - q_m t + s_m)). \tag{21}$$

Then $f(x, y, t)$ is a solution of eq. (3).

By choosing appropriate parameter values in eq. (21), we get the corresponding expressions of figure 6 as follows:

(a) $f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\cos(4y + 1) + 10\cosh\left(\frac{1}{2}y + 1\right);$

(b) $f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\cos(4y + 1) + 10\cosh\left(\frac{1}{2}y + 1\right) + 40\sin\left(\frac{1}{2}y + 1\right);$

(c) $f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\cos(4y + 1) + 10\cosh\left(\frac{1}{2}y + 1\right) + \mathrm{e}^{4y+2};$

(d) $f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\cos(4y + 1) + 10\sin(2y + 2) + 10\mathrm{e}^{4y+2};$

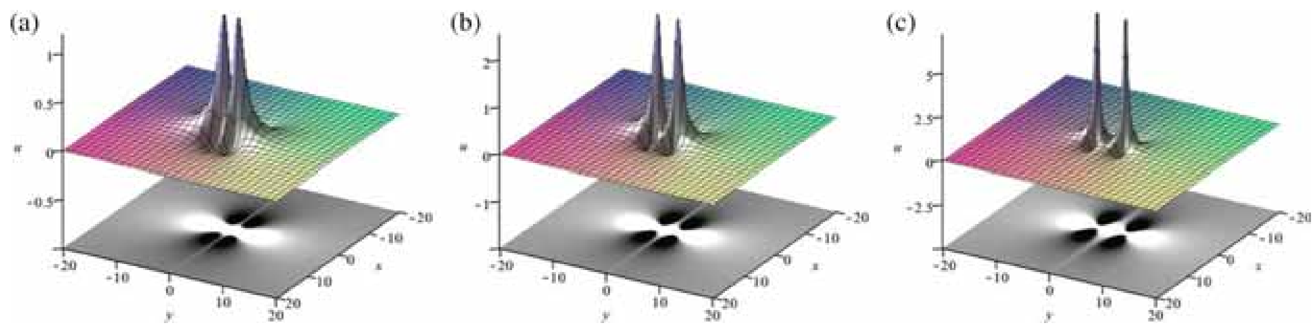

Figure 7. Evolutionary behaviour of the spatial structure of rational-logarithmic type solutions at $t = 0$. (**a**) $J = 1$, (**b**) $J = 2$ and (**c**) $J = 3$.

(e) $$f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\sin(2y + 2) + 10e^{4y+2};$$

(f) $$f(x, y, 0) = 10 + 2x^2 + 2y^2 + 10\cos(4y + 1) + 10\cosh\left(\frac{1}{2}y + 1\right) + 10\sin\left(\frac{1}{2}y + 1\right) + 10e^{4y+2}.$$

In fact, according to the proof of Theorem 1, and examples in the Corollaries, we have the following theorems.

Theorem 2. *Let $f(x, y, t) = a_0 + \sum_{i=1}^{I}(a_i x + b_i y - b_i t + d_i)^2 + g(x, y, t)$. Assume that function $g(x, y, t)$ satisfies $g_x = 0$ and $g_y + g_t = 0$, then $f(x, y, t)$ is the solution of eq.* (3).

Here, to illustrate the effectiveness of Theorem 2, we give a new example: Take

$$g(x, y, t) = \sum_{j=1}^{J}\delta_j \log_a(m_j y - m_j t)^2.$$

By choosing appropriate parameter values, we get some new superposition structures as shown in figure 7. This is a new interaction, which has been reported for the first time, where

(a) $$f(x, y, t) = 10 + (x - y + t)^2 + (x + y - t)^2 - 5\log_4(2y - 2t)^2;$$

(b) $$f(x, y, t) = 10 + (x - y + t)^2 + (x + y - t)^2 - 5\log_4(2y - 2t)^2 - 5\log_4(y - t)^2;$$

(c) $$f(x, y, t) = 10 + (x - y + t)^2 + (x + y - t)^2 - 5\log_4(2y - 2t)^2 - 5\log_4(y - t)^2 - 5\log_4\left(\frac{y - t}{2}\right)^2.$$

Theorem 3. *Let $f(x, y, t) = a_0 + \sum_{i=1}^{I}(a_i x + b_i y - b_i t + d_i)^3 + g(x, y, t)$. Assume that $\sum_{i=1}^{I} a_i^3 = 0$, $g_x = 0$ and $g_y + g_t = 0$. Then $f(x, y, t)$ is the solution of eq.* (3).

The proof of Theorem 3 is similar to Theorem 1. Here, note that eq. (12) is replaced by $P(D_x, D_y, D_t)(h \cdot h) = 2\big((h_{xt} + h_{xy})h - h_x(h_y + h_t) - 10h_{xxx}^2\big)$. Such as $f(x, y, t) = a_0 + 8(x + \epsilon y - \epsilon t + \gamma_1)^3 + (-2x + \varepsilon y - \varepsilon t + \gamma_2)^3 + g(x, y, t)$, but there is no smooth spatial structure because the function $f(x, y, t)$ contains an odd polynomial, that is, the superposition solution u has singular points.

In fact, according to the proof of Theorem 1, we get the superposition between high-order lump-type solutions and different forms of N-soliton solutions. For example, we take the test function $f(x, y, t) = h(x, y, t) + g(x, y, t) + \sum_{j=1}^{J}\omega_j(q_j y - q_j t + s_j)^m$ in eq. (9), where m is a positive integer. From figures 2–7, we can clearly see the spatial structure of superposition behaviour between lump solutions and solitons of different forms and numbers. Actually, many NLPDEs such as potential Boiti–Leon–Manna–Pempinelli equation [31], Burgers equation [32] and so on, obtained results similar to the results obtained in this paper.

4. Conclusions

In summary, a lump-type solution from two-soliton solutions is obtained by employing parameter limit. Some theorems and corollaries about the superposition between lump solutions and N-soliton solutions have been proved and explained. To the best of the author's knowledge, this is the first time that the superposition between different forms of N-solitons ($N \to \infty$) is reported for NLPDEs. Some specific examples and the corresponding spatial structure diagrams are given to illustrate the effectiveness and correctness of the theorems and corollaries obtained in this paper. Besides, the influence of parameter a_0 on the spatial structure of lump solution is discussed and simulated (see figure 2). Some spatial structures of superposition solutions with different forms, such as rational-exponential type, rational-sin type, rational-cosh-cos type, especially rational-logarithmic type, are shown and simulated for the first time. We hope that these new nonlinear phenomena and spatial structures obtained in this work can provide abundant information for related research in the field of physics.

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