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Analytical study of D-dimensional fractional Klein–Gordon equation with a fractional vector plus a scalar potential

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Abstract. D-dimensional fractional Klein–Gordon equation with fractional vector and scalar potential has been studied. Both fractional potentials are taken as attractive Coulomb-type with different multiplicative parameters, namely v and s. Jumarie-type definitions for fractional calculus have been used. We have succeeded in achieving Whittaker-type classical differential equation in fractional mode for the required eigenfunction. Fractional Whittaker equation has been manipulated using the behaviour of the eigenfunction at asymptotic distance and origin. This manipulation delivers fractional-type confluent hypergeometric equation to solve. Power series method has been employed to do the task. All the obtained results agree with the existing results in literature when fractional parameter α is unity. Finally, we furnish numerical results with a few eigenfunction graphs for different spatial dimensions and fractional parameters.

Keywords. Fractional Klein–Gordon equation; power series method; fractional Coulomb potential; Mittag–Leffler function.

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1. Introduction

Fractional calculus is almost three centuries old, as old as the formal calculus era of Leibnitz and Newton. But, it was not so popular in the science and engineering community until Lacroix [1] presented a definition of fractional derivative based on the usual expression for the nth derivative of the power function. Soon after that, mathematicians explored that fractional calculus has a sovereign beauty that understands the nature from close. Several useful applications in applied sciences [2–6] have established the fact that 'nature understands the language of fractional calculus'.

From the same understanding, application of fractional calculus in the field of physics have gained considerable popularity and during the last few years, many sparkling results were obtained [7–10]. Fractional calculus is very favourable in describing the evolution of systems with memory. This is obvious from the non-local property of fractional calculus. Non-local effects may happen with space and time. That is why quantum mechanics has a great potential to go with the language of fractional calculus. The use of fractional calculus in quantum mechanics is a very new and fast-developing area. Some of the works are listed in refs [11–16].

There are several definitions for fractional calculus [17–19] and all of them are apparently non-equivalent. The definition by Caputo [20] and Jumarie [21–24] are the most relevant for the physical cases and are being studied a lot. The only drawback of Caputo derivative is that, it collapses for non-differentiable functions. The definition of Jumarie has no such riddle. Recently, we have explicitly developed the fractional Laplacian operator in hyperspherical coordinate system with the Jumarie sense and studied two important molecular potentials, expressed fractionally, via multidimensional fractional Schrödinger equation [25,26]. The general outcome of these studies point out that we need to apply as well as analyse the quantum mechanics in the fractional domain.

Following the same spirit, in this paper, we try to study fractional D-dimensional Klein–Gordon equation for unequal fractional scalar and vector potentials. The selected fractional scalar and vector potentials

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are both attractive Coulomb-type but with different multiplicative coefficients. Klein–Gordon equation is also called relativistic Schrödinger equation and it behaves like Schrödinger equation when the constituent particles are treated as low in energy or velocity. Klein–Gordon equation as well as Dirac equation have been studied a lot in recent years [27–31]. From the mathematical point of view, Klein–Gordon equation uses the same Laplacian operator as Schrödinger equation does. This similarity is the main motivation for our present study. We have tried to monitor the whole study for achieving classical Whittaker-type equation [32] (in fractional mode). As far as the analytical tool is concerned, the power series method has been used to solve fractional differential equation.

The present paper is organised as follows: Next section is a short introduction of Jumarie-type derivative. Section 3 is the main part where the bound-state solution of the selected potential has been done. In §4 we discuss special cases as well as numerical results of the entire model with a few eigenfunction graphs. Lastly, the conclusion is drawn in §5.

2. Jumarie-type fractional derivative – An outline

Jumarie [33–35] defined the fractional-order derivative by modifying the left-Riemann–Liouville (RL) fractional derivative in the following form for a continuous function $f(x)$ (but not necessarily differentiable) in the interval a to x, with $f(x) = 0$ for $x < a$

$$f^{(\alpha)}(x) = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_a^x (x-\xi)^{-\alpha-1} f(\xi)\mathrm{d}\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_a^x (x-\xi)^{-\alpha}(f(\xi) - f(a))\mathrm{d}\xi, & 0 < \alpha < 1, \\ (f^{(\alpha-n)}(x))^{(n)}, & n \le \alpha < n+1. \end{cases} \tag{2.1}$$

It is customary to take the starting point of the interval as $a = 0$ and use the symbol $f^{(\alpha)}(x) \equiv \mathrm{d}^\alpha f(x)/\mathrm{d}x^\alpha$ with Jumarie sense. In the above definition, the first expression is just the RL fractional integration and the second expression is known as the modified RL derivative of order $0 < \alpha < 1$ because of the involvement of $f(a)$. The third line definition is for the range $n \le \alpha < n+1$. Apart from the integral type of definition we can also express fractional derivative via fractional difference. Let $f: \Re \to \Re$ denotes a continuous (but not necessarily differentiable) function such that $x \to f(x)$ for all $x \in \Re$. If $h > 0$ denotes a constant discretisation span with forward operator $\mathrm{FW}(h) f(x) = f(x+h)$, then the right-hand fractional difference of $f(x)$ of order α $(0 < \alpha < 1)$ is defined by the expression

$$\begin{aligned} \triangle^\alpha f(x) &= (\mathrm{FW}(h) - 1)^\alpha f(x) \\ &= \sum_{i=0}^{\infty} (-1)^i \binom{\alpha}{i} f[x + (\alpha - i)h], \end{aligned} \tag{2.2}$$

where the generalised binomial coefficients

$$\frac{\Gamma(\alpha - i)}{\Gamma(-\alpha)\Gamma(i+1)} = \binom{i - \alpha - 1}{i} = (-1)^i \binom{\alpha}{i}.$$

These equalities are readily established from the definition of a binomial coefficient and generalisation of factorials with gamma function

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

Then the Jumarie fractional derivative is defined as

$$f_+^{(\alpha)}(x) = \lim_{h \downarrow 0} \frac{\triangle_+^\alpha [f(x) - f(0)]}{h^\alpha} = \frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha}. \tag{2.3}$$

This definition is close to the standard definition of derivatives for the beginner's study. Following this definition, it is clear that the αth derivative of a constant for $0 < \alpha < 1$ is zero. A few results for Jumarie-type derivative are listed below depending on the characteristics of the given function $(f[u(x)])$ [36].

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}(f[u(x)]) = f_u^{(\alpha)}(u)(u_x')^\alpha, \tag{2.4a}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}(f[u(x)]) = (f/u)^{1-\alpha}(f_u'(u))^\alpha u^\alpha(x), \tag{2.4b}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}(f[u(x)]) = (1-\alpha)! u^{\alpha-1} f_u^{(\alpha)}(u) u^\alpha(x), \tag{2.4c}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}(x^\beta) = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} x^{\beta-\alpha}. \tag{2.4d}$$

In fractional calculus, solution of any linear fractional differential equation, composed of Jumarie derivative, can be easily obtained in terms of Mittag–Leffler function of one parameter [37] which is defined as

$$E_\alpha(z) = \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{\Gamma(\alpha\kappa + 1)}, \quad \alpha > 0 \tag{2.5}$$

or the more general form

$$E_{\alpha,\beta}(z) = \sum_{\kappa=0}^{\infty} \frac{z^\kappa}{\Gamma(\alpha\kappa + \beta)}.$$

Clearly, $E_{\alpha,1}(z) = E_\alpha(z)$ and $E_{1,1}(z) = E_1(z) = \mathrm{e}^z$. We provide a few derivative rules [38–40] associated with the Mittag–Leffler function and its trigonometric counterparts.

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}[E_\alpha(ax^\alpha)] = aE_\alpha(ax^\alpha), \tag{2.6a}$$

$$\frac{\mathrm{d}^\beta}{\mathrm{d}x^\beta}[E_\alpha(ax^\alpha)] = x^{\alpha-\beta}E_{\alpha,\alpha-\beta+1}(x^\alpha), \tag{2.6b}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}[\cos_\alpha(ax^\alpha)] = -a\ \sin_\alpha(ax^\alpha), \tag{2.6c}$$

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}[\sin_\alpha(ax^\alpha)] = a\ \cos_\alpha(ax^\alpha), \tag{2.6d}$$

where the one-parameter fractional sine and cosine functions are defined as follows:

$$\cos_\alpha(x^\alpha) = \sum_{\kappa=0}^{\infty}(-1)^\kappa \frac{x^{2\kappa\alpha}}{\Gamma(1 + 2\alpha\kappa)}$$

and

$$\sin_\alpha(x^\alpha) = \sum_{\kappa=0}^{\infty}(-1)^\kappa \frac{x^{(2\kappa+1)\alpha}}{\Gamma(1 + (2\kappa+1)\alpha)}$$

with

$$E_\alpha(ix^\alpha) = \cos_\alpha(x^\alpha) + i\ \sin_\alpha(x^\alpha).$$

3. Spectra for bound states

The motion of a particle governed by D-dimensional fractional Klein–Gordon equation in natural unit may be written as

$$\{-\nabla_D^{2\alpha} + [M_\alpha - V_s(r^\alpha)]^2\}\Psi(r^\alpha, \Omega_D^\alpha) = [\mathcal{E}_\alpha - V_c(r^\alpha)]^2\Psi(r^\alpha, \Omega_D^\alpha), \text{ with } 2\alpha > 1, \tag{3.1}$$

where M_α is the fractional mass and $\mathcal{E}_\alpha$ denotes the fractional energy. The spherically symmetric potential $V_c(r^\alpha)$ may be called as the fractional vector potential and $V_s(r^\alpha)$ as the fractional scalar potential. The fractional energy, fractional potential energy and the fractional mass all have unit GeV^α. We take the form of the potentials similar to the fractional Coulomb-type

$$V_c(r^\alpha) = -\frac{v}{r^\alpha}, \tag{3.2}$$

$$V_s(r^\alpha) = -\frac{s}{r^\alpha}, \tag{3.3}$$

where v (> 0) and s (> 0) are two different multipliers and the sign of v, s indicate that the selected potentials are attractive in nature. These multipliers can be taken equal if one wants to study the case of $V_c(r^\alpha) = V_s(r^\alpha)$. The term Ω_D^α within the argument of Ψ denotes angular variables $\theta_1^\alpha, \theta_2^\alpha, \theta_3^\alpha \cdots \theta_{D-2}^\alpha, \phi^\alpha$. The term $\nabla_D^{2\alpha}$ is called the fractional Laplacian operator in D dimensions. In terms of hyperspherical coordinates, it can be further written as [25,26]

$$\nabla_D^{2\alpha} = \frac{1}{[\Gamma(1+\alpha)]^2}\frac{1}{(r^\alpha)^{D-1}}\frac{\partial^\alpha}{\partial r^\alpha}\left[(r^\alpha)^{D-1}\frac{\partial^\alpha}{\partial r^\alpha}\right] - \frac{\Lambda_{D-1}^{2\alpha}}{r^{2\alpha}}, \tag{3.4}$$

where $\Lambda_{D-1}^{2\alpha}$ is the fractional hyperangular momentum operator. The explicit form is

$$\Lambda_{D-1}^{2\alpha} = -\Bigg[\sum_{k=1}^{D-2}\frac{1}{\sin_\alpha^2\theta_{k+1}^\alpha \sin_\alpha^2\theta_{k+2}^\alpha \cdots \sin_\alpha^2\phi^\alpha} \times\left(\frac{1}{\sin_\alpha^{k-1}\theta_k^\alpha}\frac{\partial^\alpha}{\partial\theta_k^\alpha}\sin_\alpha^{k-1}\theta_k^\alpha\frac{\partial^\alpha}{\partial\theta_k^\alpha}\right) + \frac{1}{\sin_\alpha^{D-2}\phi^\alpha}\frac{\partial^\alpha}{\partial\phi^\alpha}\left(\sin_\alpha^{D-2}\phi^\alpha\frac{\partial^\alpha}{\partial\phi^\alpha}\right)\Bigg]. \tag{3.5}$$

Taking the solution by means of separation variable technique $\psi(r^\alpha, \Omega_D^\alpha) = R(r^\alpha)Y(\Omega_D^\alpha)$ and adopting the eigenvalue equation for $Y(\Omega_D^\alpha)$ as

$$\Lambda_{D-1}^{2\alpha}Y(\Omega_D^\alpha) = \ell(\ell + D - 2)|_{D>1}Y(\Omega_D^\alpha), \tag{3.6}$$

where ℓ is the orbital angular momentum quantum number (can take quantised values $0, 1, 2, 3, \ldots$ only), we have the fractional 'radial' equation from eq. (3.1) as

$$\left[\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}r^{2\alpha}} + \frac{\Gamma(1+\alpha(D-1))}{\Gamma(1+\alpha(D-2))}\frac{1}{r^\alpha}\frac{\mathrm{d}^\alpha}{\mathrm{d}r^\alpha} - \frac{\{v^2 - s^2 - \ell(\ell+D-2)\}[\Gamma(1+\alpha)]^2}{r^{2\alpha}} + \frac{2(\mathcal{E}_\alpha v + M_\alpha s)}{r^\alpha}[\Gamma(1+\alpha)]^2 + (\mathcal{E}_\alpha^2 - M_\alpha^2)[\Gamma(1+\alpha)]^2\right]R(r^\alpha) = 0. \tag{3.7}$$

It is easy to remove the second term of the above differential equation by introducing a transformation such as

$$R(r^\alpha) = r^{-\theta}u(r^\alpha), \tag{3.8}$$

where θ satisfies the equation

$$\frac{2\Gamma(1-\theta)}{\Gamma(1-\theta-\alpha)} + \frac{\Gamma(1+\alpha(D-1))}{\Gamma(1+\alpha(D-2))} = 0.$$

Thus, eq. (3.7) reduces to

$$\frac{\mathrm{d}^{2\alpha}u(r^\alpha)}{\mathrm{d}r^{2\alpha}} + \Bigg[-\frac{A}{r^{2\alpha}} + \frac{2(\mathcal{E}_\alpha v + M_\alpha s)}{r^\alpha}[\Gamma(1+\alpha)]^2 + (\mathcal{E}_\alpha^2 - M_\alpha^2)[\Gamma(1+\alpha)]^2\Bigg]u(r^\alpha) = 0, \tag{3.9}$$

where

$$A = \left\{s^2 + \ell(\ell+D-2) - v^2\right\}[\Gamma(1+\alpha)]^2 - \frac{\Gamma(1-\theta)}{\Gamma(1-\theta-\alpha)}\frac{\Gamma(1-\theta-\alpha)}{\Gamma(1-\theta-2\alpha)} - \frac{\Gamma(1-\theta)}{\Gamma(1-\theta-\alpha)}\frac{\Gamma(1+\alpha(D-1))}{\Gamma(1+\alpha(D-2))}. \tag{3.10}$$

Now introducing the new variable

$$y = \{2(M_\alpha^2 - \mathcal{E}_\alpha^2)^{1/2}\Gamma(1+\alpha)\}^{1/\alpha}r = \rho^{1/\alpha}r, \quad (|\mathcal{E}_\alpha| < M), \tag{3.11}$$

and using eq. (2.4a) we have operators

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}r^\alpha} = 2(M_\alpha^2 - \mathcal{E}_\alpha^2)^{1/2}\Gamma(1+\alpha)\frac{\mathrm{d}^\alpha}{\mathrm{d}y^\alpha};$$

$$\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}r^{2\alpha}} = 4(M_\alpha^2 - \mathcal{E}_\alpha^2)[\Gamma(1+\alpha)]^2\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}y^{2\alpha}}.$$

Realising $u(r^\alpha) \to G(y^\alpha)$, it is straightforward to rewrite eq. (3.9) as

$$\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}y^{2\alpha}}G(y^\alpha) + \left[-\frac{1}{4} - \frac{A}{y^{2\alpha}} + \frac{B}{y^\alpha}\right]G(y^\alpha) = 0, \tag{3.12}$$

where

$$B = \frac{\mathcal{E}_\alpha v + M_\alpha s}{\sqrt{(M_\alpha^2 - \mathcal{E}_\alpha^2)}}\Gamma(1+\alpha).$$

It is interesting to note that when $\alpha = 1$, $G(y^\alpha)$ of eq. (3.12) evolves as the perfect Whittaker function and solution comes immediately. In the case of fractional form of eq. (3.12) the situation is not so easy. As $G(y^\alpha)$ represents the eigenfunction, it must be vanishing for very large or very small y^α to correlate the bound-state situation.

The asymptotic solution (*i.e. for* $y^\alpha \to \infty$):

Approximately, we can have

$$\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}y^{2\alpha}}G(y^\alpha) - \frac{1}{4}G(y^\alpha) = 0.$$

There are two possibilities of $G(y^\alpha)$ in terms of Mittag–Leffler function. One is $E_\alpha(\frac{1}{2}y^\alpha)$ and the other is $E_\alpha(-\frac{1}{2}y^\alpha)$. The first one is not acceptable physically as it tries to blow the solution at infinity. The second choice is perfect for our search. Now the complete solution can be expressed as

$$G(y^\alpha) = E_\alpha\left(-\frac{1}{2}y^\alpha\right)H(y^\alpha), \tag{3.13}$$

where $H(y^\alpha)$ is expected to behave like $H(y^\alpha \to 0) \to 0$. Inserting (3.13) in eq. (3.12) we have

$$\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}y^{2\alpha}}H(y^\alpha) - \frac{\mathrm{d}^\alpha}{\mathrm{d}y^\alpha}H(y^\alpha) + \left(-\frac{A}{y^{2\alpha}} + \frac{B}{y^\alpha}\right)H(y^\alpha) = 0. \tag{3.14}$$

It is possible to attain better form of the above equation.

Substitution of $H(y^\alpha) = y^bQ(y)$ $(b > 0)$ into eq. (3.14) yields

$$y^\alpha\frac{\mathrm{d}^{2\alpha}}{\mathrm{d}y^{2\alpha}}Q(y) + (c - y^\alpha)\frac{\mathrm{d}^\alpha}{\mathrm{d}y^\alpha}Q(y) - aQ(y) = 0, \tag{3.15}$$

where we have used the restriction

$$A = \frac{\Gamma(1+b)}{\Gamma(1+b-\alpha)}\frac{\Gamma(1+b-\alpha)}{\Gamma(1+b-2\alpha)}. \tag{3.16}$$

Other symbols have following abbreviations:

$$\frac{\Gamma(1+b)}{\Gamma(1+b-\alpha)} = p + \frac{1}{2},$$

$$c = 2p + 1,$$

$$a = p + \frac{1}{2} - B. \tag{3.17}$$

When $\alpha = 1$, eq. (3.15) is nothing but a confluent hypergeometric equation and one can easily say $Q(y) \sim {}_1F_1(a; c; y)$ where $p = b - \frac{1}{2}$ is to be realised via eqs (3.10) and (3.16). We seek the solution of eq. (3.15) with the help of the power series method. Let the solution be

$$Q(y) = \sum_{m=0}^{\infty}\lambda_m y^{\alpha m}, \quad 0 < \alpha < 1. \tag{3.18}$$

Using the rule (2.4d) we find

$$\frac{\mathrm{d}^{\alpha}Q(y)}{\mathrm{d}y^{\alpha}} = \sum_{m=0}^{\infty}\lambda_m \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha m-\alpha)}y^{\alpha m-\alpha},$$

$$\frac{\mathrm{d}^{2\alpha}Q(y)}{\mathrm{d}y^{2\alpha}} = \sum_{m=0}^{\infty}\lambda_m \frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha m-\alpha)}\frac{\Gamma(1+\alpha m-\alpha)}{\Gamma(1+\alpha m-2\alpha)}y^{\alpha m-2\alpha}. \tag{3.19}$$

Substituting these into (3.15) and equating the coefficient of $y^{\alpha m}$ to zero, the recurrence relation evolves as

$$\lambda_{m+1} = \frac{\left(\frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha m-\alpha)}+a\right)}{\left(\frac{\Gamma(1+\alpha m+\alpha)}{\Gamma(1+\alpha m)}\right)\left(\frac{\Gamma(1+\alpha m)}{\Gamma(1+\alpha m-\alpha)}+c\right)}\lambda_m. \tag{3.20}$$

Selecting $\lambda_0 = 1$ we can easily find

$$\lambda_m = \frac{\left(\frac{\Gamma 1}{\Gamma(1-\alpha)}+a\right)\left(\frac{\Gamma(1+\alpha)}{\Gamma 1}+a\right)\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}+a\right)\cdots\left(\frac{\Gamma(1+(m-1)\alpha)}{\Gamma(1+(m-2)\alpha)}+a\right)}{\Gamma(1+m\alpha)\left(\frac{\Gamma 1}{\Gamma(1-\alpha)}+c\right)\left(\frac{\Gamma(1+\alpha)}{\Gamma 1}+c\right)\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)}+c\right)\cdots\left(\frac{\Gamma(1+(m-1)\alpha)}{\Gamma(1+(m-2)\alpha)}+c\right)}. \tag{3.21}$$

Here $m = 1, 2, 3, \ldots$. In order to express the solution in a compact form, it is useful to introduce a Pochhammer-type notation such as

$$(\delta_\alpha)_q = \prod_{j=0}^{q-1}\left(\frac{\Gamma(1+j\alpha)}{\Gamma(1+j\alpha-\alpha)}+\delta\right)$$

with $(\delta_\alpha)_0 = 1$. Hence, the solution of eq. (3.15) reads as

$$Q(y) = \sum_{m=0}^{\infty}\frac{(a_\alpha)_m}{(c_\alpha)_m}\frac{y^{\alpha m}}{\Gamma(1+\alpha m)}. \tag{3.22}$$

It is interesting to note that when $\alpha = 1$, $(a_\alpha)_m = (a)_m = a(a+1)(a+2)\cdots(a+m-1)$ and $(c_\alpha)_m = (c)_m = c(c+1)(c+2)\cdots(c+m-1)$. Thus, the solution $Q(y)$ becomes $Q(y) \sim {}_1F_1(a;c;y)$. Equation (3.22) shows that the infinite series would become a polynomial if j is chosen properly. This is what we call the quantisation condition of the model. Let $j = n'$ ($n' = 0, 1, 2, 3, \ldots$) be the number where the series terminates. So the quantisation condition reads as

$$a + \frac{\Gamma(1+n'\alpha)}{\Gamma(1+n'\alpha-\alpha)} = 0. \tag{3.23}$$

Usually, in the study of Klein–Gordon equation, a principle quantum number is used rather than n'. We shall follow the same practice here also. Let us define the principle quantum number $n = n' + \ell + 1$. Thus, using (3.17) above the quantisation condition takes the following form:

$$B = \frac{\Gamma(1+(n-\ell-1)\alpha)}{\Gamma(1+(n-\ell-2)\alpha)} + \frac{\Gamma(1+b)}{\Gamma(1+b-\alpha)} = \beta(>0). \tag{3.24}$$

Finally, the expression of B helps to derive the explicit form of the energy eigenvalue as

$$\mathcal{E}_\alpha = \mathcal{E}_\alpha(n,\ell,D) = M_\alpha\left[-\frac{vs[\Gamma(1+\alpha)]^2}{v^2[\Gamma(1+\alpha)]^2+\beta^2} \pm \Lambda(s,v,\alpha,\beta)\right], \tag{3.25}$$

where

$$\Lambda(s,v,\alpha,\beta) = \left[\frac{s^2v^2[\Gamma(1+\alpha)]^4}{(v^2[\Gamma(1+\alpha)]^2+\beta^2)^2} - \frac{s^2[\Gamma(1+\alpha)]^2-\beta^2}{v^2[\Gamma(1+\alpha)]^2+\beta^2}\right]^{1/2}.$$

Last but not the least, the overall eigenfunctions of the system in y-space can be expressed as

$$G(y^\alpha) \sim E_\alpha\left(-\frac{1}{2}y^\alpha\right)y^b\sum_{m=0}^{\infty}\frac{(a_\alpha)_m}{(c_\alpha)_m}\frac{y^{\alpha m}}{\Gamma(1+\alpha m)} \tag{3.26}$$

or in r-space the radial eigenfunctions are

$$R(r^\alpha) = R_{n\ell D}(r^\alpha) = \mathbb{C}r^{b-\theta}E_\alpha\left(-\frac{1}{2}\rho r^\alpha\right)\sum_{n=0}^{\infty}\frac{(a_\alpha)_n}{(c_\alpha)_n}\frac{r^{\alpha n}}{\Gamma(1+\alpha n)}, \tag{3.27}$$

where without loss of meaning m has been replaced by n and $\mathbb{C}$ may be realised as normalisation constant which contains a few multiplicative parameters originated from the scaling transformation (3.11).

4. Discussion

In this section, we shall discuss a few special cases for both $\alpha = 1$ and $0 < \alpha < 1$.

4.1 $\alpha = 1$

1. $\mathbf{s \neq v \neq 0}$
   $\alpha = 1$ corresponds to $\theta = (D-1)/2$. Hence

$$A = s^2 + \ell(\ell + D - 2) - v^2 - \theta(\theta + 1) + \theta(D-1) = s^2 - v^2 + \left(k^2 - \frac{1}{4}\right), \tag{4.1}$$

   where $k = |\ell - 1 + (D/2)|$. Further, using (3.16) and (3.24)

$$b = \frac{1}{2} + \sqrt{s^2 + k^2 - v^2}, \tag{4.2}$$

$$\beta = n - \ell - \frac{1}{2} + \sqrt{s^2 + k^2 - v^2}, \tag{4.3}$$

   where we have assumed that $\sqrt{s^2 + k^2 - v^2} > 0$. The energy eigenvalue is

$$\mathcal{E} = \mathcal{E}(n, \ell, D) = M\left\{ - \frac{vs}{v^2 + \beta^2} \pm \left[\frac{s^2 v^2}{(v^2 + \beta^2)^2} - \frac{s^2 - \beta^2}{v^2 + \beta^2} \right]^{1/2} \right\}. \tag{4.4}$$

   It is easy to extract eigenfunctions from eq. (3.27) by realising the different parameters properly. The solution reads as

$$R(r) = R_{n\ell D}(r) = \mathbb{C} r^{b-\theta} \mathrm{e}^{-(1/2)\rho r}\, {}_1F_1(a; c; r), \tag{4.5}$$

   where $\rho = 2\sqrt{M^2 - \mathcal{E}^2}$ and a and c are given by eq. (3.17).
2. $\mathbf{s = 0, v \neq 0}$
   Here the positive energy eigenvalue is

$$\mathcal{E} = \mathcal{E}(n, \ell, D) = M\left\{ 1 + \frac{v^2}{(n - \ell - \frac{1}{2} + \sqrt{k^2 - v^2})^2} \right\}^{-1/2}, \tag{4.6}$$

   where $b = \frac{1}{2} + \sqrt{k^2 - v^2}$ is used to evaluate β. The eigenfunctions are

$$R(r) = R_{n\ell D}(r) = \mathbb{C} r^{b-\theta} \mathrm{e}_1^{-(1/2)\rho r}\, {}_1F_1(a; c; r), \tag{4.7}$$

   where $\theta = (D-1)/2$, $\rho = 2\sqrt{M^2 - \mathcal{E}^2}$ and a and c are given by eq. (3.17).
3. $\mathbf{v = 0, s \neq 0}$
   Here the energy eigenvalue is

$$\mathcal{E} = \mathcal{E}(n, \ell, D) = \pm M\left\{ 1 - \frac{s^2}{(n - \ell - \frac{1}{2} + \sqrt{s^2 + k^2})^2} \right\}^{1/2}, \tag{4.8}$$

Table 1. Energy spectrum $\mathcal{E}_\alpha(2, 1, D)$ in (GeV$^\alpha$). $s = 2, v = 1, M_\alpha = 1$. $|^-\mathcal{E}_\alpha| \Rightarrow$ second term is negative in (3.25), $|^+\mathcal{E}_\alpha| \Rightarrow$ second term is positive in (3.25).

D	α	θ	A	b	β	$\lvert^-\mathcal{E}_\alpha\rvert < M_\alpha$	$\lvert^+\mathcal{E}_\alpha\rvert < M_\alpha$
3	0.80	0.67289	2.9644	2.30456	2.2395	−0.6055	+0.0938
	0.85	0.69243	2.8280	2.23504	2.2018	−0.5969	+0.0399
	0.90	0.72806	2.6728	2.17948	2.1657	−0.5896	−0.0198
	0.95	0.79226	2.5208	2.14205	2.1379	−0.5859	−0.0808
	1.00	1.00	5.00	2.79128	2.1791	−0.5992	−0.0967
4	0.80	1.49258	3.7484	2.61182	2.4427	−0.6500	+0.2096
	0.85	1.47243	3.3726	2.43226	2.3485	−0.6298	+0.1308
	0.90	1.45864	2.8787	2.25011	2.2242	−0.6031	+0.0202
	0.95	1.45852	2.3175	2.07443	2.0761	−0.5717	−0.1280
	1.00	1.50	6.75	3.14575	3.1458	−0.7584	+0.3913
5	0.80	1.57425	3.5748	2.54594	2.3995	−0.6411	+0.1867
	0.85	1.58350	3.1854	2.36597	2.2994	−0.6191	+0.1017
	0.90	1.61569	2.4115	2.08652	2.0885	−0.5713	−0.0760
	0.95	1.69493	1.4501	1.75180	1.7799	−0.5001	−0.3932
	1.00	2.00	9.00	3.54138	3.5414	−0.8003	+0.5049

where $b = \frac{1}{2} + \sqrt{s^2 + k^2}$ is used for evaluating β. The eigenfunctions are

$$R(r) = R_{n\ell D}(r) = \mathbb{C} r^{b-\theta} \mathrm{e}^{-(1/2)\rho r}\, {}_1F_1(a; c; r), \tag{4.9}$$

where $\theta = (D-1)/2$, $\rho = 2\sqrt{M^2 - \mathcal{E}^2}$ and a and c are given by eq. (3.17).

4. $\mathbf{s} = \mathbf{v} = \frac{\gamma}{2}$

In this case, $\beta = n - \ell - \frac{1}{2} + k = n + (D-3)/2$. So, the energy eigenvalue is

$$\mathcal{E} = \mathcal{E}(n, \ell, D) = M\left\{1 - \frac{2v^2}{v^2 + (n + [(D-3)/2])^2}\right\} \tag{4.10}$$

and the eigenfunctions are

$$R(r) = R_{n\ell D}(r) = \mathbb{C} r^{b-\theta} \mathrm{e}^{-(1/2)\rho r}\, {}_1F_1(a; c; r), \tag{4.11}$$

and all the parameters are the same as given earlier. It is interesting to see here that with the mapping $\mathcal{E} + M \to 2M, \mathcal{E} - M \to \mathcal{E}$, as well as $v = s = \gamma/2$ we can find the results of Schrödinger equation from (3.21). It is easy to have

$$B^2 = v^2 \frac{M + \mathcal{E}}{M - \mathcal{E}} = \beta^2.$$

Thus, Schrödinger sense eigenvalue is

$$\mathcal{E}_{\mathrm{Sch}} = \mathcal{E}_{\mathrm{Sch}}(n, \ell, D) = -\frac{M}{2}\left(\frac{\gamma}{n + (D-3)/2}\right)^2. \tag{4.12}$$

All the results of this section are consistent with ref. [41].

4.2 $0 < \alpha < 1$

It is possible to have the Schrödinger sense energy eigenvalue for $\alpha \in [0, 1]$. To do that we set $s = v = \eta/2$. For low-energy particle, the mapping $\mathcal{E}_\alpha + M_\alpha \to 2M_\alpha, \mathcal{E}_\alpha - M_\alpha \to \mathcal{E}_\alpha$ transfer the Klein–Gordon equation into non-relativistic wave equation, i.e. Schrödinger equation. The Schrödinger sense energy eigenvalue for the fractional Coulomb potential comes as

$$(\mathcal{E}_\alpha)_{\mathrm{Sch}} = -\frac{M_\alpha}{2}\frac{\eta^2}{\beta^2}, \tag{4.13}$$

where β is given by eq. (3.24). This result is consistent with our previous work [25].

Besides the theoretical aspects, it is interesting to go through the numerical results of the entire model in table 1. We have closely monitored that, under the chosen parameter values with $s > v$, there is no bound states for $(n, \ell, D) = (1, 1, D)$ very near to $\alpha = 0.95, 0.90$. As we go to the higher state $(2, 1, D)$ bound-state solutions are available. We have examined $(2, 1, D)$ state eigenfunctions for different fractional parameters $\alpha = 0.85, 0.90, 0.95$ and 1.00. Figures 1–3 are for $D = 3-5$ with the second term negative in eq. (3.25). On the other hand, figures 4–6 are the same except that the second term of the previously quoted equation is positive. The eigenfunctions are not well behaved for most

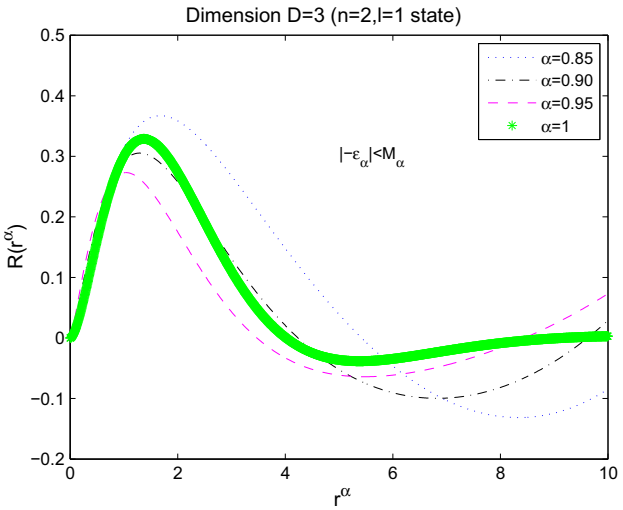

Figure 1. $(n = 2, \ell = 1, D = 3)$ state eigenfunctions for $\alpha = 0.85, 0.90, 0.95$ and 1.00 when the second term is negative in eq. (3.25).

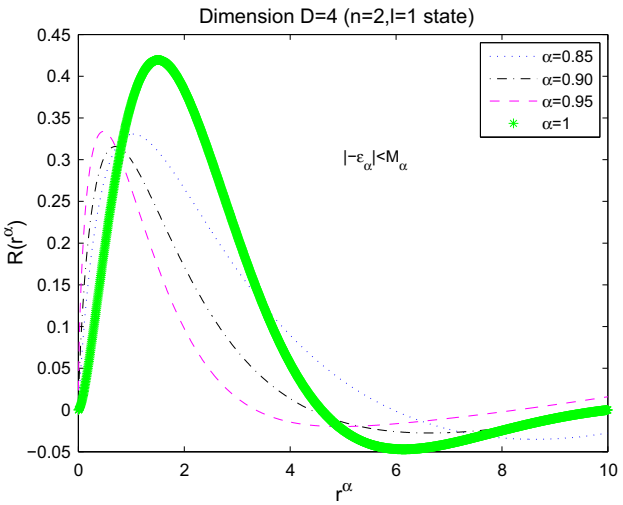

Figure 2. $(n = 2, \ell = 1, D = 4)$ state eigenfunctions for $\alpha = 0.85, 0.90, 0.95$ and 1.00 when second term is negative in eq. (3.25).

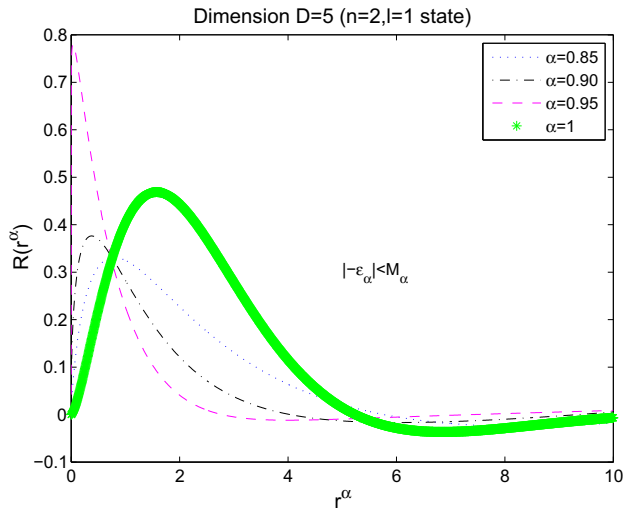

Figure 3. ($n = 2, \ell = 1, D = 5$) state eigenfunctions for $\alpha = 0.85, 0.90, 0.95$ and 1.00 when the second term is negative in eq. (3.25).

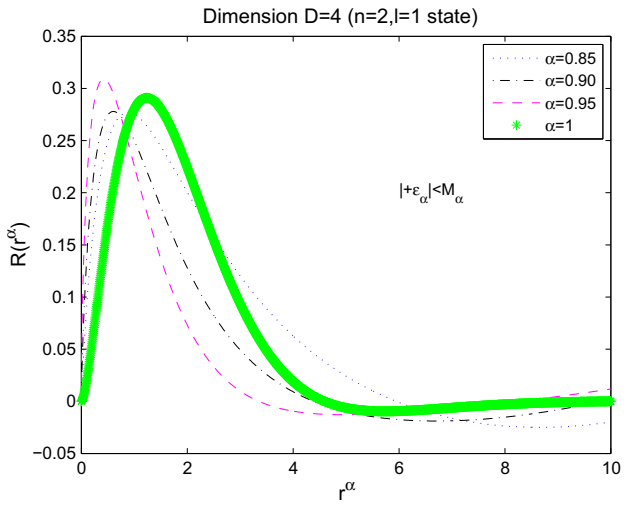

Figure 5. ($n = 2, \ell = 1, D = 4$) state eigenfunctions for $\alpha = 0.85, 0.90, 0.95$ and 1.00 when the second term is positive in eq. (3.25).

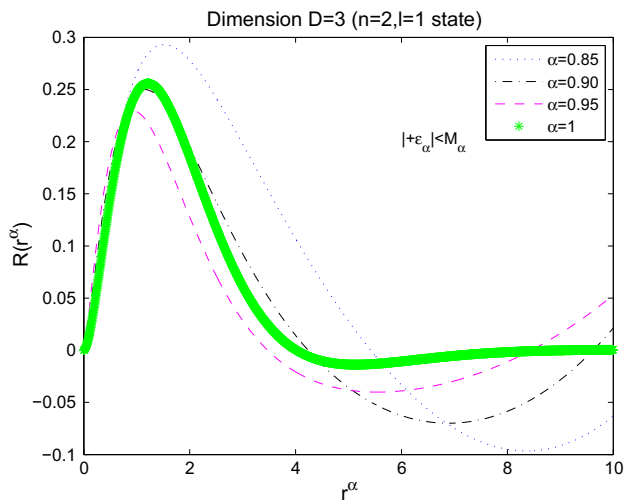

Figure 4. ($n = 2, \ell = 1, D = 3$) state eigenfunctions for $\alpha = 0.85, 0.90, 0.95$ and 1.00 when the second term is positive in eq. (3.25).

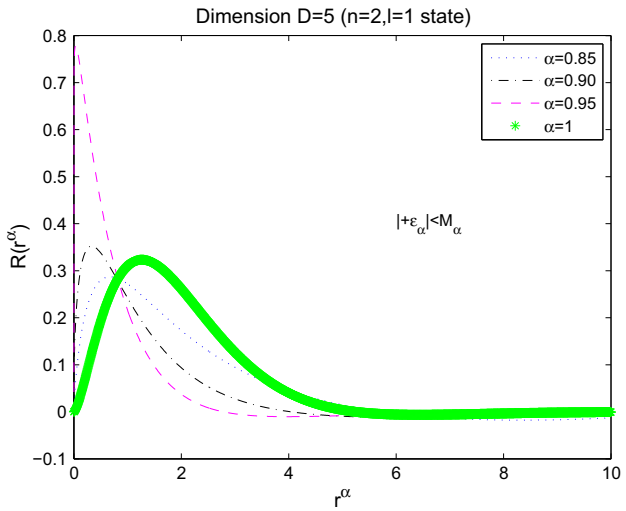

Figure 6. ($n = 2, \ell = 1, D = 5$) state eigenfunctions for $\alpha = 0.85, 0.90, 0.95$ and 1.00 when the second term is positive in eq. (3.25).

of the cases when α takes the value 0.80. Thus, it is clear that if the value of α is very far from 1, we cannot study the bound-state spectra of the model. This result is the same as the result we achieved in our previous works [25,26] where analytical method such as Laplace transform was used instead of the power series method. Furthermore, for higher dimensions ($D \geq 4$), the peak of the eigenfunctions tend to shift towards the vertical axis. For $\alpha = 0.95$, $D = 5$ the eigenfunctions almost loose its well-behaved nature both for $|^{-}\mathcal{E}_\alpha| < M_\alpha$ and $|^{+}\mathcal{E}_\alpha| < M_\alpha$ cases. Figures 7 and 8 are for the energy spectrum. These two figures indicate that the energy spectrum is quite opposite for cases $|^{-}\mathcal{E}_\alpha| < M_\alpha$ and $|^{+}\mathcal{E}_\alpha| < M_\alpha$.

The general structure of the eigenfunctions, depicted from figures 1–6, show that the spatial width of the eigenfunctions is smaller than that for the normal study ($\alpha = 1$). In other words, it reveals that the probability of finding a constituent particle in a specified range for the fractional case is higher than the normal case ($\alpha = 1$). Thus, we may predict that relativistic or non-relativistic quantum mechanical wave equations have a vast scope in fractional language. We need more study to explore the hidden physics behind it.

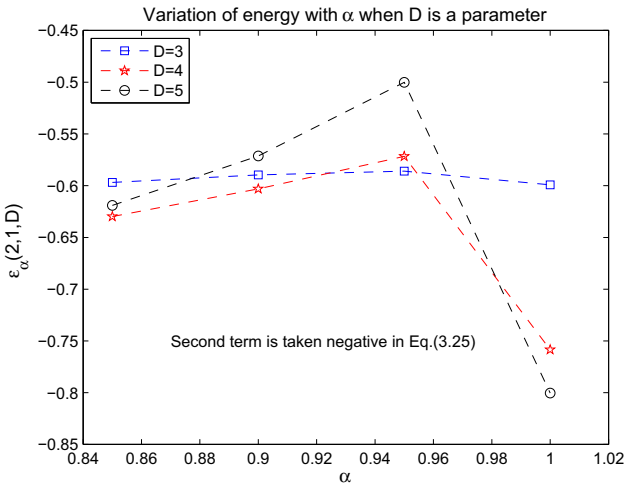

Figure 7. Bound-state energy variation with α taking D as a parameter when the second term is negative in eq. (3.25).

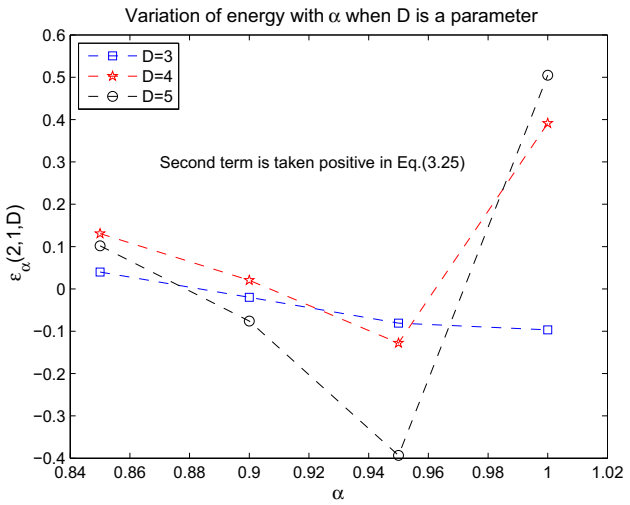

Figure 8. Bound-state energy variation with α taking D as a parameter when the second term is positive in eq. (3.25).

5. Conclusion

In this paper, we have studied the bound-state solution of the fractional Klein–Gordon equation for unequal fractional vector and scalar potential. Both the potentials are fractional Coulomb-type but attractive in nature. The important fact of the study is, it goes with the same mathematical craft as the usual classical quantum mechanical problems follow. For example, we have obtained a confluent hypergeometric-type differential equation in fractional mode. Power series method has been used to solve that portion with the same spirit of the classical process. We have introduced Pochhammer-type notation in fractional mode to express the solution in a compact form. Finally, the mathematical outputs have been verified with the existing results. They are remarkably consistent with each other.

Furthermore, we have furnished numerical data as well as eigenfunction graphs for $(2, 1, D = 3{-}5)$ states for different fractional parameter $\alpha \in (0, 1)$. Clearly, these reflect that fractional differential equations are quite interesting in conventional quantum mechanics. We need extra research on the uncertainty principle, scattering state solutions of the eigenfunctions and more to construct a concrete model of fractional quantum mechanics. This work may be called as the generalisation of all the previous works that are based on the attractive Coulomb potential. In the near future, we shall try to revisit the study for very weak Coulomb potential with $s < v$ and other spherically symmetric fractional molecular potentials.

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