Pramana – J. Phys. (2020) 94:28
https://doi.org/10.1007/s12043-019-1894-0

Computational soliton solutions to (2 + 1)-dimensional Pavlov equation using Lie symmetry approach

SACHIN KUMAR[1],*, MUKESH KUMAR[2] and DHARMENDRA KUMAR[3]

[1]Department of Mathematics, Faculty of Mathematical Sciences, University of Delhi, Delhi 110 007, India
[2]Department of Mathematics, Motilal Nehru National Institute of Technology, Allahabad 211 004, India
[3]Department of Mathematics, SGTB Khalsa College, University of Delhi, Delhi 110 007, India
*Corresponding author. E-mail: sachinambariya@gmail.com

MS received 12 June 2019; revised 19 September 2019; accepted 17 October 2019

Abstract. In this work, Lie symmetry analysis and one-dimensional optimal system for Pavlov equation are presented. All the possible vector fields, their commutative and adjoint relations are carried out under invariance property of Lie group theory. On the basis of optimal system, similarity reductions of Pavlov equation are obtained. A repeated process of similarity reductions transforms the Pavlov equation into ordinary differential equations, which generate invariant solutions. The obtained invariant solutions are supplemented by numerical simulation to analyse the physical behaviour. Thus, their parabolic, multisoliton, nonlinear, kink and antikink wave profiles are traced in results and discussions sections.

Keywords. (2 + 1)-Dimensional Pavlov equation; optimal system; invariant solutions.

PACS Nos **02.20.Sv; 02.20.Qs; 05.45.Yv; 02.30.Jr**

1. Introduction

Nonlinear partial differential equations (NPDEs) are widely used to study various phenomena arising in mathematical physics, ion-acoustic waves in plasmas, water surface gravity waves, condensed matter physics etc. [1,2]. The exact solutions of these NPDEs have a significant role to understand the dynamics and in the development of natural phenomena. Therefore, a variety of methods like multiple exp-function method [3], Hirota's method [4], Jacobi elliptic function method [5], tanh–coth method [6], homogeneous balance method [7], Hirota's bilinear method [8], the (G'/G) -expansion method [9], solitary wave ansatz [10], F-expansion method [11], simplest equation method [2] etc. have been developed for obtaining exact solutions of the NPDEs.

The main motive of this research is to attain some group invariant solution of Pavlov equation [1,12–21]

$$\Delta := u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}, \tag{1}$$

where $u(x, y, t)$ is the amplitude of the relevant wave depending upon the space variables x, y and time t. Equation (1) was first derived as a symmetry reduction of the second heavenly equation [13]. The Pavlov equation is a highly nonlinear partial differential equation. Therefore, finding its exact solutions is an arduous task.

Besides the aforementioned references, a rich literature of integrable properties, symmetries and exact solutions of Pavlov equation is presented here. Pavlov [14] presented Benney-type moment chains and constructed new (2 + 1)-dimensional integrable hydrodynamic systems. Also, their hydrodynamical reductions and integrability are described therein. Manakov and Santini [15] derived Pavlov equation from inverse scattering problem for one-parameter families of vector fields and constructed formal solutions of the Cauchy problem. Baran *et al* [16] constructed some symmetry reduction and exact solutions of eq. (1) by employing Lie group theory. Continuing, Baran *et al* [17] used Lax representations to symmetry reductions of Pavlov equation and studied its integrability features under reductions. They also constructed infinite series of conservation laws and proved its nontriviality. Grinevich *et al* [18] solved the scattering and inverse scattering problem for Pavlov equation as an example of integrable dispersionless partial differential equation (PDE) and justified the

Published online: 20 January 2020

existence theorem for globally bounded solutions of the related Cauchy problem with small data. Moreover, Wu [19] used Newtonian iteration scheme to solve nonlinear Riemann–Hilbert problem and proved the unique solvability of the Cauchy problem of the Pavlov equation with large initial data. Lelito and Morozov [20] found three-component nonlocal conservation laws for Pavlov equation, and hence proved that nonlocal conservation laws can be derived from local conservation law of Veronese web equation. Recently, Baran *et al* [21] expanded Lax pairs in terms of spectral parameter and constructed nonlocal symmetry algebras. They also discussed the impact of recursion operators on shadows of nonlocal symmetries.

To fill the gap left in previous researches [1–21], the authors are motivated to derive Lie symmetries [22–36], optimal system and new group invariant solutions. One-dimensional optimal systems of Lie subalgebras are constructed under the invariance property of Lie groups. As an optimal system contains structurally important information about different types of invariant solutions, it provides precise insights into all possible invariant solutions emerging from infinitesimal symmetries. In symmetry analysis, the problem of classifying optimal system [30] is very vital to understand the behaviour of the solutions for a given PDE.

Meanwhile, Pavlov equation is reduced to several homogeneous and non-homogeneous systems of ordinary differential equations (ODEs), which provide analytical solutions. These solutions have wider spectrum of applications because of the existing arbitrary functions and constants.

This article is organised as follows: Section 1 gives a brief introduction. In §2, Lie point symmetry analysis of eq. (1) is performed. The one-dimensional optimal system to classify Lie algebra is constructed in §3. Section 4 contains similarity reductions and the corresponding group-invariant solutions. Finally, the graphical representation of the obtained solutions is performed in §5. Section 6 gives a detailed discussion of this research.

2. Lie point symmetries

A brief review on infinite symmetries under one-parameter group of transformations has been presented in this section. For detailed description, one can review [22,24].

One-parameter (ϵ) transformations for eq. (1) are considered as

$$x^* = x + \epsilon\,\xi^x(x, y, t, u) + O(\epsilon^2),$$

$$y^* = y + \epsilon\,\xi^y(x, y, t, u) + O(\epsilon^2),$$

$$t^* = t + \epsilon\,\tau(x, y, t, u) + O(\epsilon^2),$$

$$u^* = u + \epsilon\,\eta(x, y, t, u) + O(\epsilon^2),$$

where ϵ is the group parameter, ξ^x, ξ^y, τ and η are the infinitesimals depending upon both dependent and independent variables x, y, t and u which are to be determined.

Thus, the associated vector field for Pavlov equation (1) is expressed as

$$\mathbf{V} = \xi^x\frac{\partial}{\partial x} + \xi^y\frac{\partial}{\partial y} + \tau\frac{\partial}{\partial t} + \eta\frac{\partial}{\partial u}, \tag{2}$$

where ξ^x, ξ^y, τ and η are infinitesimals corresponding to x, y, t and u, to be determined.

The second prolongation of V is given by [22]

$$V^{(2)} = V + \eta^x\frac{\partial}{\partial u_x} + \eta^y\frac{\partial}{\partial u_y} + \eta^t\frac{\partial}{\partial u_t} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xt}\frac{\partial}{\partial u_{xt}} + \eta^{yy}\frac{\partial}{\partial u_{yy}}. \tag{3}$$

Employing the above second prolongation formula under invariance condition $\mathrm{Pr}^{(2)}V(\Delta) = 0$ to eq. (1), the invariant surface condition is derived as

$$\eta^{yy} = \eta^{tx} + \eta^y u_{xx} + u_y\eta^{xx} - \eta^x u_{xy} - u_x\eta^{xy} \tag{4}$$

with coefficients

$$\eta^x = D_x(\eta) - u_x D_x(\xi^x) - u_y D_x(\xi^y) - u_t D_x(\tau),$$
$$\eta^y = D_y(\eta) - u_x D_y(\xi^x) - u_y D_y(\xi^y) - u_t D_y(\tau),$$
$$\eta^{xx} = D_x(\eta^x) - u_{xx} D_x(\xi^x) - u_{xy} D_x(\xi^y) - u_{xt} D_x(\tau),$$
$$\eta^{xy} = D_y(\eta^x) - u_{xx} D_y(\xi^x) - u_{xy} D_y(\xi^y) - u_{xt} D_y(\tau),$$
$$\eta^{xt} = D_t(\eta^x) - u_{xx} D_t(\xi^x) - u_{xy} D_t(\xi^y) - u_{xt} D_t(\tau) \tag{5}$$

and the total derivative operators D_x, D_y and D_t are defined as

$$D_x = \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_t} + u_{xx}\frac{\partial}{\partial u_x} + u_{xy}\frac{\partial}{\partial u_y} + \cdots$$

$$D_y = \frac{\partial}{\partial y} + u_y\frac{\partial}{\partial u} + u_{yt}\frac{\partial}{\partial u_t} + u_{xy}\frac{\partial}{\partial u_x} + u_{yy}\frac{\partial}{\partial u_y} + \cdots$$

$$D_t = \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_{xt}\frac{\partial}{\partial u_x} + u_{yt}\frac{\partial}{\partial u_y} + u_{tt}\frac{\partial}{\partial u_t} + \cdots.$$

Substituting eqs (5) into (4) and equating the coefficient of partial derivatives to zero, we obtain the system of determining equations:

$$\eta_u = -2\xi^t_t + 3\xi^y_y, \quad \eta_{xt} = \xi^y_{tt}, \quad \eta_{xx} = 0,$$

$$\eta_{xy} = \tau_{tt}, \quad \eta_{yy} = \xi^y_{tt}, \quad \xi^t_u = 0,$$

$$\xi^t_x = 0, \quad \xi^t_y = 0, \quad \xi^x_t = \eta_y,$$

$$\xi^x_u = 0, \quad \xi^x_x = -\xi^t_t + 2\xi^y_y, \quad \xi^x_y = \frac{1}{2}\eta_x + \frac{1}{2}\xi^y_t,$$

$$\xi^y_u = 0, \quad \xi^y_x = 0, \quad \xi^y_{ty} = \xi^t_{tt}, \quad \xi^y_{yy} = 0 \tag{6}$$

where

$$\eta_u = \frac{\partial \eta}{\partial u}, \quad \eta_{xt} = \frac{\partial^2 \eta}{\partial x \partial t}, \quad \tau_{tt} = \frac{\partial^2 \tau}{\partial t^2}, \quad \xi^y_{tt} = \frac{\partial^2 \xi^y}{\partial t^2},$$

$$\xi^t_u = \frac{\partial \xi^t}{\partial u}, \quad \xi^x_x = \frac{\partial \xi^x}{\partial x}, \quad \xi^y_{ty} = \frac{\partial^2 \xi^y}{\partial t \partial y}, \tag{7}$$

etc. Consequently, the solution of eq. (6) results into the following infinitesimals:

$$\xi^x = (f_1'(t) + 2c_1)x + \frac{1}{2}f_1''(t)y^2 + \left(f_2'(t) + \frac{1}{2}c_2\right)y + f_3(t) + c_3,$$

$$\xi^y = (f_1'(t) + c_1)y + f_2(t), \quad \xi^t = f_1(t),$$

$$\eta^u = (f_1'(t) + 3c_1)u + (f_1''(t)y + f_2'(t) + c_2)x + \frac{1}{6}f_1'''(t)y^3 + \frac{1}{2}f_2''(t)y^2 + f_3'(t)y + f_4(t). \tag{8}$$

Taking $f_1(t) = c_4 + tc_6, f_2(t) = c_5, f_3(t) = 0, f_4(t) = c_7$, the infinitesimals transform to

$$\xi^x = c_6 x + 2c_1 x + \frac{1}{2}yc_2 + c_3, \;\; \xi^y = c_6 y + c_1 y + c_5,$$

$$\xi^t = c_4 + tc_6,$$

$$\eta^u = (c_6 + 3c_1 u) + c_2 x + c_7. \tag{9}$$

Thus, Lie algebra of symmetries of eq. (1) can be spanned by the following vector fields:

$$V_1 = \frac{\partial}{\partial x}, \quad V_2 = \frac{\partial}{\partial y}, \quad V_3 = \frac{\partial}{\partial t}, \quad V_4 = \frac{\partial}{\partial u},$$

$$V_5 = \frac{1}{2}y\frac{\partial}{\partial x} + x\frac{\partial}{\partial u}, \quad V_6 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 3u\frac{\partial}{\partial u},$$

$$V_7 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}.$$

3. Optimal system of subalgebra

The optimal system of subalgebra is of great importance for providing possible invariant solutions under infinitesimal symmetries. It allows the possible invariant solutions into disjoint sets.

3.1 *Construction of invariants*

In order to derive optimal system of Lie algebra, it is necessary to construct the invariants for the selection of representative elements. For the vectors $\mathbf{V} = \sum_{i=1}^{7} a_i V_i$ and $\mathbf{W} = \sum_{i=1}^{7} b_i V_i$, we employ the following adjoint action:

$$\mathrm{Ad}_{\exp(\epsilon \mathbf{W})}(\mathbf{V}) = (a_1 V_1 + \cdots + a_n V_n) - \epsilon(\Theta_1 V_1 + \cdots + \Theta_n V_n), \tag{10}$$

where $\Theta = \Theta(a_1, \ldots, a_n, b_1, \ldots, b_n)$ are obtained with the help of table 1, which are

$$\Theta_1 = -2a_6 b_1 + a_7 b_1 - 2a_1 b_6 - a_1 b_7 + 0.5a_5 b_2 - 0.5a_2 b_5,$$

$$\Theta_2 = a_6 b_2 + a_7 b_2 - a_2 b_6 - a_2 b_7,$$

$$\Theta_3 = a_7 b_3 - a_3 b_7,$$

$$\Theta_4 = a_5 b_1 + 3a_6 b_4 + a_7 b_4 - a_1 b_5 - 3a_4 b_6 - a_4 b_7,$$

$$\Theta_5 = a_6 b_5 - a_5 b_6,$$

$$\Theta_6 = 0, \quad \Theta_7 = 0. \tag{11}$$

Table 1. The commutative table of Lie algebra.

·	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	0	0	0	0	V_4	$2V_1$	V_1
V_2	0	0	0	0	$\frac{1}{2}V_1$	V_2	V_2
V_3	0	0	0	0	0	0	V_3
V_4	0	0	0	0	0	$3V_4$	V_4
V_5	$-V_4$	$-\frac{1}{2}V_1$	0	0	0	V_5	0
V_6	$-2V_1$	$-V_2$	0	$-3V_4$	$-V_5$	0	0
V_7	$-V_1$	$-V_2$	$-V_3$	$-V_4$	0	0	0

Table 2. The adjoint table of Lie algebra.

Ad	V_1	V_2	V_3	V_4	V_5	V_6	V_7
V_1	V_1	V_2	V_3	V_4	$V_5-\epsilon V_4$	$V_6-2\epsilon V_1$	$V_7-\epsilon V_1$
V_2	V_1	V_2	V_3	V_4	$V_5-\frac{1}{2}\epsilon V_1$	$V_6-\epsilon V_2$	$V_7-\epsilon V_2$
V_3	V_1	V_2	V_3	V_4	V_5	V_6	$V_7-\epsilon V_3$
V_4	V_1	V_2	V_3	V_4	V_5	$V_6-3\epsilon V_4$	$V_7-\epsilon V_4$
V_5	$V_1+\epsilon V_4$	$V_2+\frac{1}{2}\epsilon V_1+\frac{1}{4}\epsilon^2 V_4$	V_3	V_4	V_5	$V_6-2\epsilon V_5$	V_7
V_6	$V_1 e^{2\epsilon}$	$V_2 e^{\epsilon}$	V_3	$V_4 e^{3\epsilon}$	$V_5 e^{\epsilon}$	V_6	V_7
V_7	$V_1 e^{\epsilon}$	$V_2 e^{\epsilon}$	$V_3 e^{\epsilon}$	$V_4 e^{\epsilon}$	V_5	V_6	V_7

For any b_j, $1 \le j \le 7$,

$$\Theta_1\frac{\partial\phi}{\partial a_1}+\Theta_2\frac{\partial\phi}{\partial a_2}+\Theta_3\frac{\partial\phi}{\partial a_3}+\cdots+\Theta_6\frac{\partial\phi}{\partial a_6} + \Theta_7\frac{\partial\phi}{\partial a_7}=0. \tag{12}$$

Equating the coefficients of b_i, we obtain the following system:

$$b_1 : a_5\frac{\partial\phi}{\partial a_4}+(-2a_6+a_7)\frac{\partial\phi}{\partial a_1}=0,$$

$$b_2 : 2(a_6+a_7)\frac{\partial\phi}{\partial a_2}+a_5\frac{\partial\phi}{\partial a_1}=0,$$

$$b_3 : a_7\frac{\partial\phi}{\partial a_3},$$

$$b_4 : (3a_6+a_7)\frac{\partial\phi}{\partial a_4}=0,$$

$$b_5 : 2a_6\frac{\partial\phi}{\partial a_5}-2a_1\frac{\partial\phi}{\partial a_4}-a_2\frac{\partial\phi}{\partial a_1}=0,$$

$$b_6 : a_5\frac{\partial\phi}{\partial a_5}+3a_4\frac{\partial\phi}{\partial a_4}+a_2\frac{\partial\phi}{\partial a_2}+2a_1\frac{\partial\phi}{\partial a_1}=0,$$

$$b_7 : a_4\frac{\partial\phi}{\partial a_4}+a_3\frac{\partial\phi}{\partial a_3}+a_2\frac{\partial\phi}{\partial a_2}+a_1\frac{\partial\phi}{\partial a_1}=0. \tag{13}$$

The solution of system (13) provides general invariant function $\phi(a_1,a_2,a_3,a_4,a_5,a_6,a_7)=F(a_6,a_7)$ symmetry algebra L^7. Hence, (2 + 1)-dimensional Pavlov equation has two basic invariants.

3.2 *Construction of optimal system*

Now, we construct the following adjoint matrices with the help of table 2:

$$\mathrm{Ad}(e^{\epsilon_1 V_1})=\begin{pmatrix}1&0&0&0&0&-2\epsilon_1&-\epsilon_1\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&-\epsilon_1&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}, \tag{14}$$

$$\mathrm{Ad}(e^{\epsilon_2 V_2})=\begin{pmatrix}1&0&0&0&-\frac{1}{2}\epsilon_1&0&0\\0&1&0&0&0&-\epsilon_1&-\epsilon_1\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}, \tag{15}$$

$$\mathrm{Ad}(e^{\epsilon_3 V_3})=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&-\epsilon_3\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix}, \tag{16}$$

$$\mathrm{Ad}(e^{\epsilon_4 V_4})=\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&-3\epsilon_4&-\epsilon_4\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{pmatrix},$$

$$\mathrm{Ad}(\mathrm{e}^{\epsilon_5 V_5}) = \begin{pmatrix} 1 & \frac{\epsilon_5}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \epsilon_5 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2\epsilon_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{17}$$

$$\mathrm{Ad}(\mathrm{e}^{\epsilon_6 V_6}) = \begin{pmatrix} \mathrm{e}^{2\epsilon_6} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\epsilon_6} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{3\epsilon_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathrm{e}^{\epsilon_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathrm{Ad}(\mathrm{e}^{\epsilon_7 V_7}) = \begin{pmatrix} \mathrm{e}^{\epsilon_7} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathrm{e}^{\epsilon_7} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{e}^{\epsilon_7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{e}^{\epsilon_7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{18}$$

Therefore, the global adjoint matrix can be obtained as

$$A = \mathrm{Ad}(\mathrm{e}^{\epsilon_1 V_1}) * \mathrm{Ad}(\mathrm{e}^{\epsilon_2 V_2}) \cdots \mathrm{Ad}(\mathrm{e}^{\epsilon_7 V_7})$$

$$= \begin{pmatrix} \mathrm{e}^{2\epsilon_6+\epsilon_7} & \frac{1}{2}\mathrm{e}^{\epsilon_6+\epsilon_7}\epsilon_5 & 0 & 0 & -\frac{1}{2}\mathrm{e}^{\epsilon_6}\epsilon_2 & \epsilon_2\epsilon_5 - 2\epsilon_1 & -\epsilon_1 \\ 0 & \mathrm{e}^{\epsilon_6+\epsilon_7} & 0 & 0 & 0 & -\epsilon_2 & -\epsilon_2 \\ 0 & 0 & \mathrm{e}^{\epsilon_7} & 0 & 0 & 0 & -\epsilon_3 \\ \mathrm{e}^{2\epsilon_6+\epsilon_7}\epsilon_5 & 0 & 0 & \mathrm{e}^{3\epsilon_6+\epsilon_7} & -\mathrm{e}^{\epsilon_6}\epsilon_1 & 2\epsilon_1\epsilon_5 - 3\epsilon_4 & -\epsilon_4 \\ 0 & 0 & 0 & 0 & e^{\epsilon_6} & -2\epsilon_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{19}$$

To determine the increments $\epsilon \mathrm{Ad}_g$, the adjoint transformation is

$$(a_1, \ldots, a_7) * A = (\beta_1, \ldots, \beta_7).$$

Then, we have the following system of equations:

$$\beta_1 = a_1 + a_4\epsilon_5,$$

$$\beta_2 = a_2 + 0.5a_1\epsilon_5,$$

$$\beta_3 = a_3,$$

$$\beta_4 = a_4,$$

$$\beta_5 = a_5 - a_4\epsilon_1 - \frac{1}{2}a_1\epsilon_2,$$

$$\beta_6 = a_6 - a_2\epsilon_2 - 2a_5\epsilon_5 + a_4(2\epsilon_1\epsilon_5 - 3\epsilon_4) + a_1(\epsilon_2\epsilon_5 - 2\epsilon_1),$$

$$\beta_7 = a_7 - a_1\epsilon_1 - a_2\epsilon_2 - a_3\epsilon_3 - a_4\epsilon_4, \tag{20}$$

which must have solutions for ϵ_i's, for $i = 1, 2, \ldots, 7$ (assuming $\epsilon_6 = \epsilon_7 = 0$).

Now, five cases are discussed by using invariants (a_6, a_7) as follows:

Case 1: $a_1 = 1, a_6 = 0, a_7 = 0$.
Select a representative element $\tilde{V} = V_1$. Substituting $\beta_1 = 1, \beta_i = 0, 2 \le i \le 7$ and $a_1 = 1, a_2 = a_3 = a_4 = a_6 = a_7 = 0$ into eq. (20), one can get the solution

$$\epsilon_1 = 0, \quad \epsilon_2 = 2a_5, \quad \epsilon_5 = -2a_2, \quad \epsilon_6 = 0, \quad \epsilon_7 = 0. \tag{21}$$

Therefore, the vector $V_1 + a_2V_2 + a_3V_3 + a_4V_4 + a_5V_5 + a_6V_6 + a_7V_7$ is equivalent to V_1.

Case 2: $a_3 = 1, a_6 = 0, a_7 = 0$.
For the representative element $\tilde{V} = V_3$, substituting $\beta_3 = 1, \beta_i = 0, 1, 2, 4, 5, 6, 7$ and $a_3 = 1, a_1 = a_2 = a_4 = a_6 = 0$ into eq. (20), we have

$$\epsilon_3 = a_7, \quad \epsilon_6 = 0, \quad \epsilon_7 = 0. \tag{22}$$

Case 3: For the representative element $\tilde{V} = a_2V_2 + a_4V_4$, the substitution $\beta_2 = a_2, \beta_4 = a_4\beta_i = 0, i = 1, 3, 5, 6, 7$ into eq. (20) results as

$$\epsilon_1 = \frac{a_5}{a_4}, \quad \epsilon_2 = \frac{-a_6 + 3a_7}{2a_2}, \quad \epsilon_4 = \frac{a_6 - a_7}{2a_4},$$

$$\epsilon_5 = 0, \quad \epsilon_6 = 0, \quad \epsilon_7 = 0. \tag{23}$$

Case 4: Taking representative element $\tilde{V} = V_5$ and putting $\beta_5 = 1, \beta_i = 0, i = 1, 2, 3, 4, 6, 7$ into eq. (20), the solution is

$$\epsilon_1 = \frac{a_5}{a_4}, \quad \epsilon_5 = \frac{a_6}{2a_5}, \quad \epsilon_6 = 0, \quad \epsilon_7 = 0. \tag{24}$$

Case 5: $a_1 \neq 0, a_2 \neq 0, a_3 \neq 0, a_4 \neq 0$. Adopt one representative element $\tilde{V} = a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4$. Substituting $\beta_5 = \beta_6 = \beta_7 = 0$ into eq. (20), we obtain the solution

$$\epsilon_2 = \frac{2(a_5 - a_4\epsilon_1)}{a_1},$$

$$\epsilon_3 = \frac{-4a_2a_5 - a_1a_6 + 3a_1a_7 - a_1^2\epsilon_1 + 4a_2a_4\epsilon_1}{3a_1a_3},$$

$$\epsilon_4 = \frac{-2a_2a_5 + a_1a_6 - 2a_1^2\epsilon_1 + 2a_2a_4\epsilon_1}{3a_1a_4},$$

$$\epsilon_5 = 0, \quad \epsilon_6 = 0, \quad \epsilon_7 = 0. \tag{25}$$

Hence, the one-dimensional optimal system of subalgebra for Pavlov equation is given by

$$\{V_1,\ V_3,\ a_2V_2 + a_4V_4,\ V_5,\ a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4\}. \tag{26}$$

4. Some certain exact solutions

4.1 *Subalgebra* $V_1 = \partial/\partial x$

For V_1, eq. (1) is transformed to $F_{YY} = 0$, with $T = t$, $Y = y$ and $u = F(X, Y)$. Therefore, the solution is given as

$$F(Y, T) = f_1(T) + Yf_2(T). \tag{27}$$

Hence, the solution of eq. (1) is

$$u = f_1(t) + yf_2(t). \tag{28}$$

4.2 *Subalgebra* $V_3 = \partial/\partial t$

Using V_3, eq. (1) is transformed into

$$F_{YY} + F_XF_{XY} - F_YF_{XX} = 0, \tag{29}$$

where

$$u = F(X, Y), \quad X = x, \quad Y = y.$$

Again, eq. (29) admits infinitesimals given as

$$\xi_X = \frac{1}{2}(A_2 + A_5)X + \frac{1}{2}A_1Y + \frac{1}{2}A_4F,$$

$$\xi_Y = A_4X + A_5Y + A_6,\ \ \eta_F = FA_2 + A_1X + A_3, \tag{30}$$

where A_i's $(1 \leq i \leq 7)$ are arbitrary constants.

4.2.1 $A_1 \neq 0$ *and rest of the* A_i*'s are zero.* Then, we obtain the vector

$$X_1 = \frac{Y}{2}\frac{\partial}{\partial X} + X\frac{\partial}{\partial F}$$

for which the required ODE is given as

$$wR'' - 2R' = 0, \tag{31}$$

where

$$w = y, \quad F = \frac{x^2}{y} + R(w).$$

The solution of eq. (31) is

$$R(y) = \frac{c_1w^3}{3} + c_2. \tag{32}$$

Hence, invariant solution of Pavlov equation is

$$u = \frac{c_1y^3}{3} + c_2 + \frac{x^2}{y}. \tag{33}$$

4.2.2 $A_2 \neq 0$ *and rest of the* A_i*'s are zero.* Now, we have the vector

$$X_2 = \frac{X}{2}\frac{\partial}{\partial X} + F\frac{\partial}{\partial F},$$

which results in the following ODE:

$$R'' + 2RR' = 0, \tag{34}$$

where $w = y$, $F = x^2R(w)$. Therefore, the solution of eq. (34) is given as

$$R(y) = \alpha_1 \tanh(\alpha_1 y + \alpha_1\alpha_2), \tag{35}$$

where α_1 and α_2 are arbitrary constants. Hence, the solution of eq. (1) is

$$u = \alpha_1x^2 \tanh(\alpha_1 y + \alpha_1\alpha_2). \tag{36}$$

4.2.3 $A_5 \neq 0$ *and rest of the* A_i*'s are zero.* In this case, we find the vector

$$X_5 = \frac{X}{2}\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y},$$

which provides the following ODE:

$$2R'^2 - w(3R' + wR'') = 0, \tag{37}$$

where

$$w = \frac{x}{\sqrt{y}}, \quad F(X, Y) = R(w).$$

The solution of eq. (37) is

$$R(w) = \frac{1}{\alpha_3} \tan^{-1}(w^2\alpha_3) + \alpha_4, \tag{38}$$

where α_3 and α_4 are arbitrary constants. Consequently, the desired solution of Pavlov equation is

$$u = \frac{1}{\alpha_3} \tan^{-1}\left(\frac{\alpha_3 x^2}{y}\right) + \alpha_4. \tag{39}$$

4.3 *Subalgebra* $a_2V_2 + a_4V_4 = a_2(\partial/\partial y) + a_4(\partial/\partial u)$

For $a_2V_2 + a_4V_4$, eq. (1) is transformed to

$$a_2F_{XT} + a_4F_{XX} = 0, \tag{40}$$

where

$$T = t, \quad X = x, \quad u = \frac{a_4}{a_2}y + F(X, T).$$

Then, the solution of eq. (40) is

$$F(X, T) = f_1(a_4T - a_2X) + f_2(T). \tag{41}$$

Hence, the solution of the test equation is

$$u = \frac{a_4}{a_2}y + f_1(a_4t - a_2x) + f_2(t). \tag{42}$$

4.4 *Subalgebra* $V_5 = (y/2)(\partial/\partial x) + x(\partial/\partial u)$

Using V_5, eq. (1) is reduced to $YF_{YY} - 2F_Y = 0$, where

$$T = t, \quad Y = y, \quad u = \frac{x^2}{y} + F(Y, T). \tag{43}$$

Here, the solution of the reduced equation is

$$F(Y, T) = \frac{1}{3}Y^3 f_1(T) + f_2(T). \tag{44}$$

Thus, the exact solution of eq. (1) is

$$u = \frac{x^2}{y} + \frac{1}{3}y^3 f_1(t) + f_2(t). \tag{45}$$

4.5 *Subalgebra* $a_1V_1 + a_2V_2 + a_3V_3 + a_4V_4 = a_1(\partial/\partial x) + a_2(\partial/\partial y) + a_3(\partial/\partial t) + a_4(\partial/\partial u)$

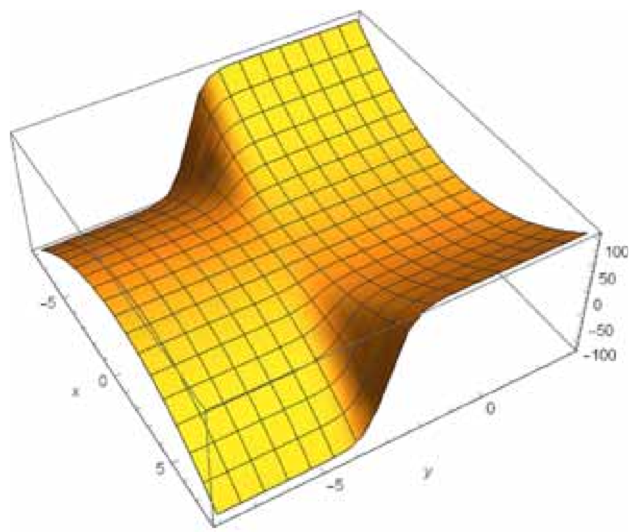
Then, the characteristic equation is

$$\frac{\mathrm{d}x}{a_1} = \frac{\mathrm{d}y}{a_2} = \frac{\mathrm{d}t}{a_3} = \frac{\mathrm{d}u}{a_4}. \tag{46}$$

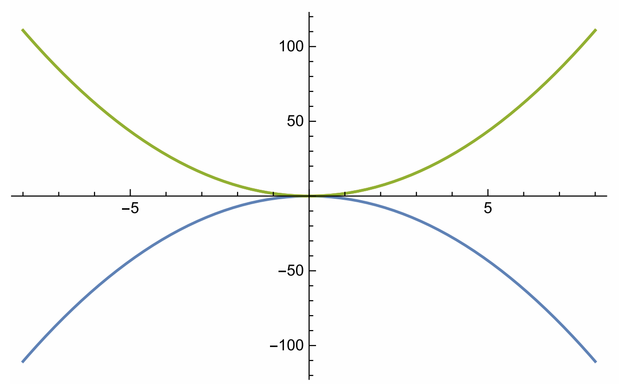
Omitting the intermediate steps, we obtain invariant solution of eq. (1) as

$$u = \alpha_5 \tanh(\chi) + \alpha_6 \tanh^3(\chi) + \frac{a_4t}{a_3} + \alpha_7, \tag{47}$$

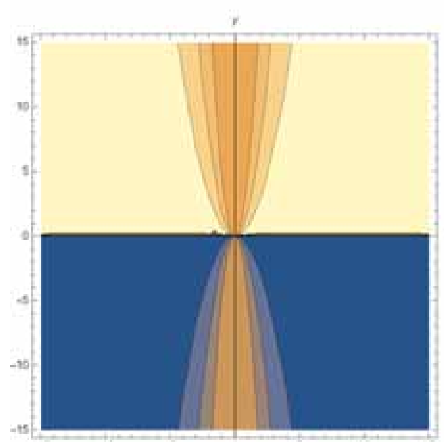
where



(i)



(ii) $y = -5, 0, 5$



(iii)

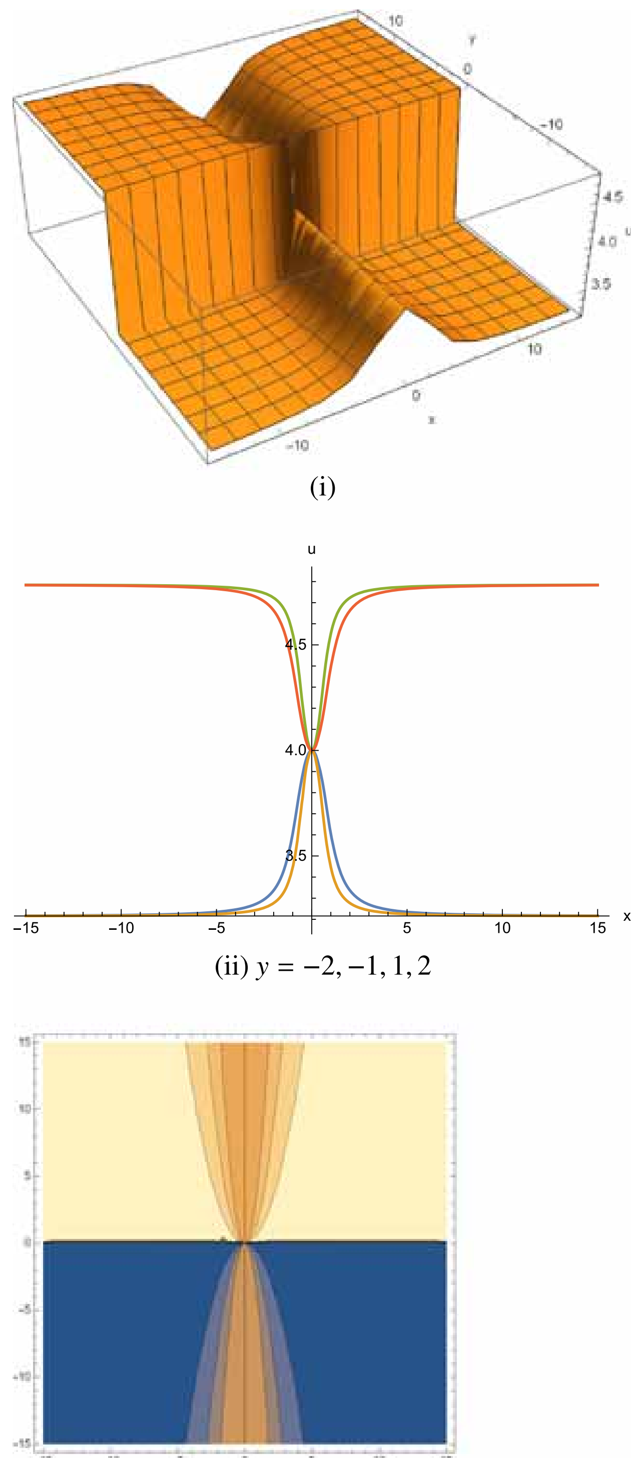
Figure 1. Profile for the solution listed in eq. (36).

$$\chi = \frac{\alpha_8(\sqrt{a_2^2 - 4a_1a_3} - a_2)(y - (a_2t/a_3))}{2a_3} + \alpha_8\left(x - \frac{a_1t}{a_3}\right) + \alpha_9$$

and $\alpha_5, \alpha_6, \alpha_7, \alpha_8$ and α_9 are arbitrary constants.

5. Results and discussions

Mathematical expressions of exact solutions are much efficient to exhibit the physical behaviour of the

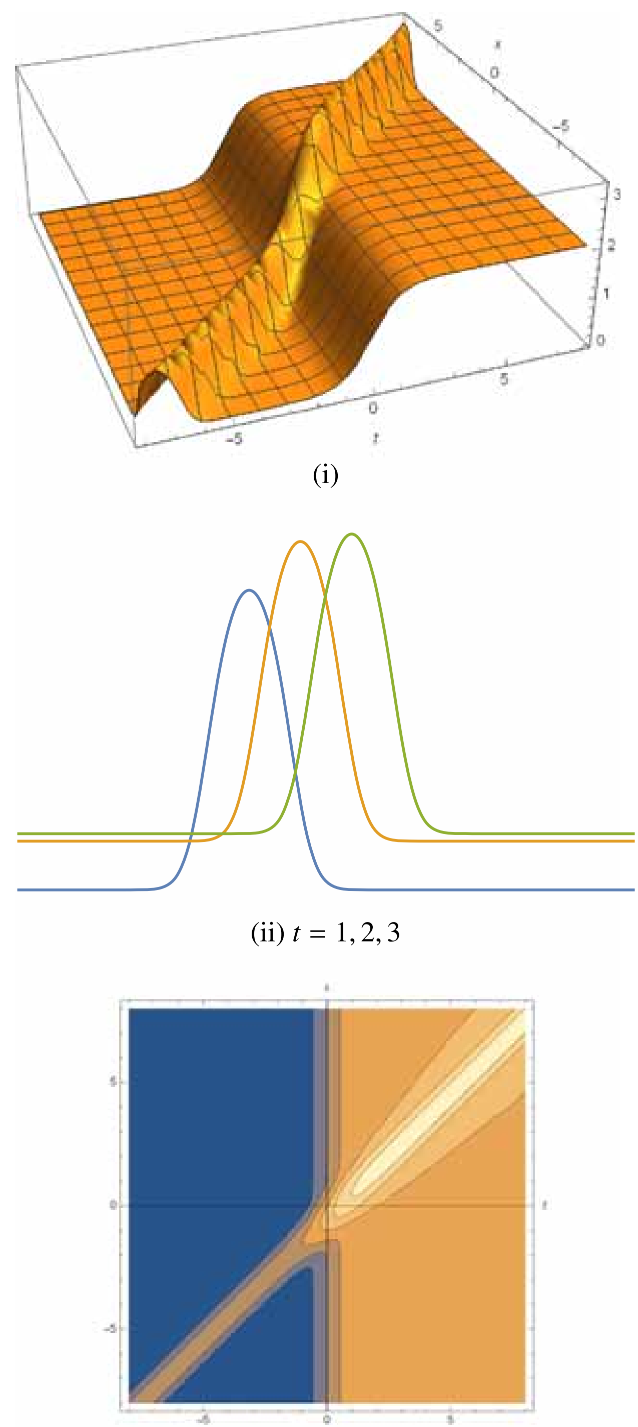


(i)

(ii) $y = -2, -1, 1, 2$

(iii)

Figure 2. Profile showing the interaction of kink–antikink waves for eq. (39).



(i)

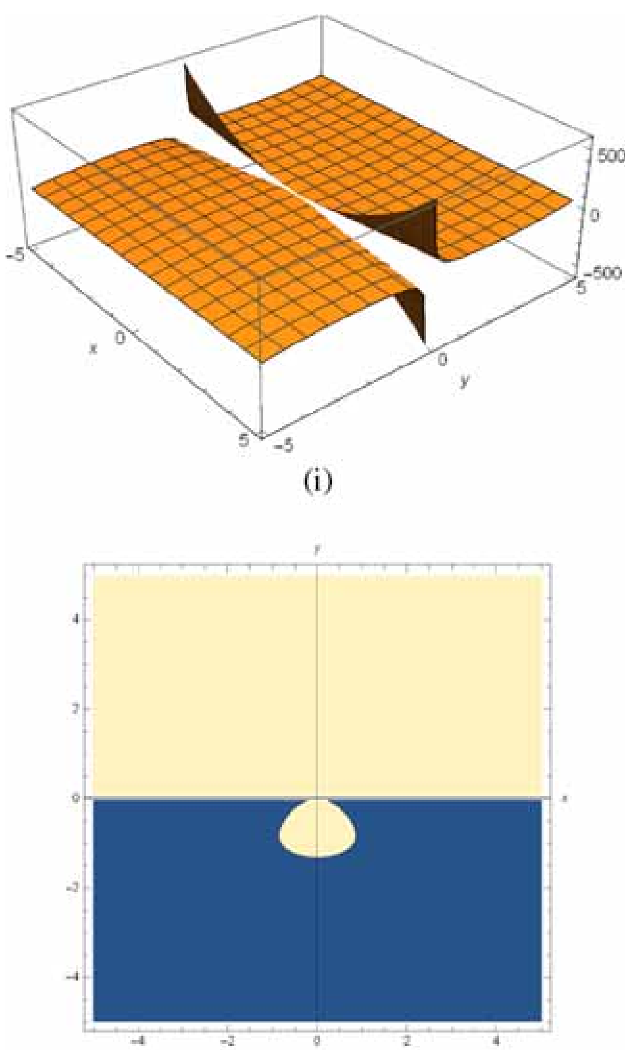
(ii) $t = 1, 2, 3$

(iii)

Figure 3. Multisoliton profile for the solutions expressed in eq. (42).

phenomena depending upon their graphical representation. In the present section, we interpret solutions (36), (39), (42), (45) and (47) physically by using numerical simulation on MAPLE while the behaviour of solutions (28) and (33) is self-explanatory. The obtained solutions contain arbitrary constants and arbitrary functions, which help to describe rich physically meaningful structures. The simulation is performed to obtain suitable value of these arbitrary constants. It is well known that solitons are solitary waves, which are completely elastic during mutual collision. The graphical behaviour of solutions is analysed as follows:

The solution given by eq. (36) is traced graphically via figure 1, which shows parabolic nature. Suitable values to arbitrary constants are given as $\alpha_1 = 1.732$ and $\alpha_2 = 3$. The dynamics of wave propagation along the x-axis is analysed at $y = -5, 0$ and 5.

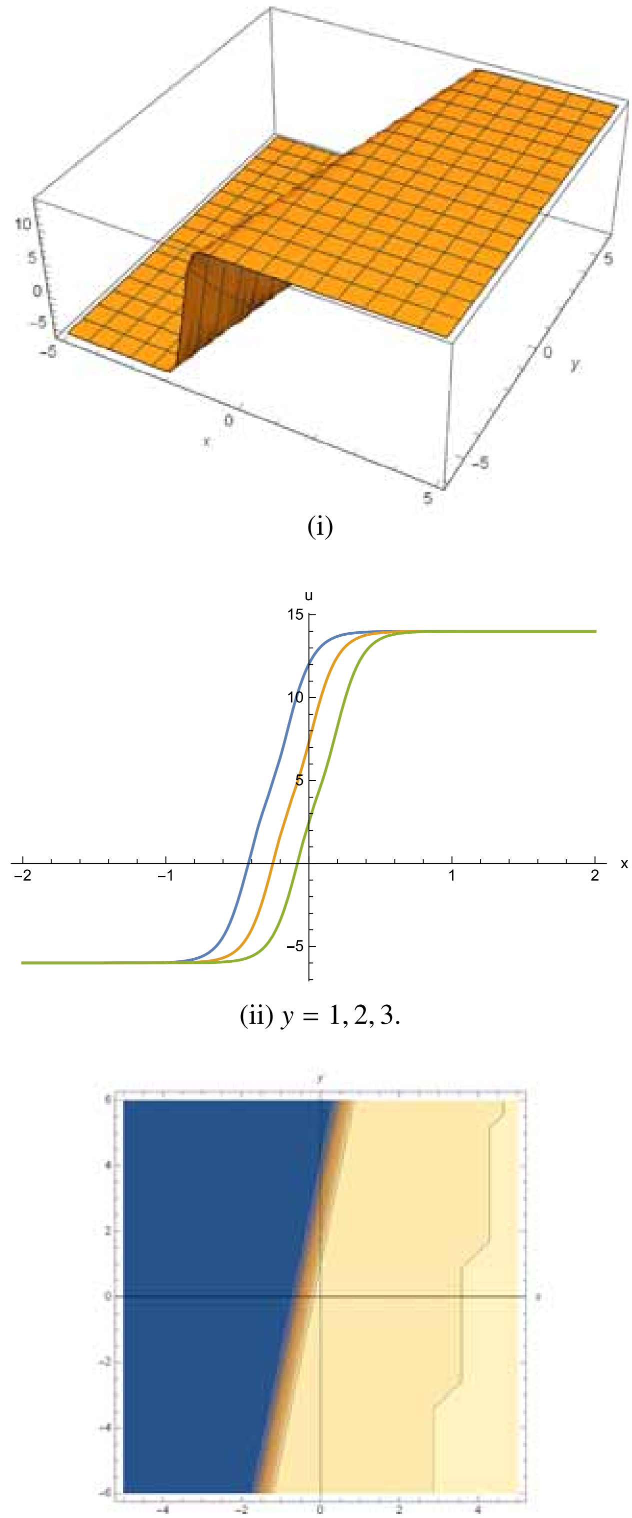


(i)

(ii)

Figure 4. Nonlinear behaviour of the solutions given by (45).

The interaction of kink and antikink waves for the solution listed in eq. (39) is exhibited in figure 2 by taking $\alpha_3 = 2$ and $\alpha_4 = 4$. To view asymptotic nature



(i)

(ii) $y = 1, 2, 3$.

(iii)

Figure 5. Kink wave profile for solution (47).

of the kink and antikink waves, 2D graph is traced at $y = -2, -1, 1$ and 2.

Figure 3 reveals the multisoliton profile for the wave component listed in eq. (42). Adequate values of arbitrary functions and constants are provided as $f_1(t) = f_2(t) = \tanh t$ and $a_2 = 6$, $a_4 = 1$. The amplitude of solitary wave is shown in $x - u$ view. Also, the intensive behaviour of multisoliton is observed.

The nonlinear behaviour of the wave component u listed in eq. (45) is observed in figure 4. The arbitrary functions are chosen as $f_1(t) = \sinh t$ and $f_2(t) = \cosh t$.

The kink wave profile for solution (47) is traced in figure 5 at $t = 2$. The asymptotic behaviour of the kink waves is observed along the x-axis at $y = 1, 2$ and 3. The adequate choice of arbitrary constants is taken as $a_1 = 1$, $a_2 = 6$, $a_3 = 1$, $a_4 = 1$, $\alpha_1 = 2$, $\alpha_2 = 5$, $\alpha_3 = 5$, $\alpha_4 = 2$, $\alpha_5 = 5$, $\alpha_6 = 2$ and $\alpha_7 = 5$. It is concluded that the dynamics of the solution depends on the value of arbitrary constants. For instance, the value of $c_5 = c_7 = -5$ results in kink wave profile.

6. Conclusion

In the present work, the authors have employed classical approach of Lie groups to derive infinitesimals, generators, commutative and adjoint relations. Thereafter, one-dimensional optimal system for Pavlov equation has been constructed by using invariants functions. Thus, the Pavlov equation is reduced to ODEs corresponding to the optimal system. These ODEs are solved exactly and provide invariant solutions. The obtained solutions are of rich physical structures due to the existing arbitrary constants and function. Moreover, some of the obtained solutions are plotted in the symbolic computation software Mathematica 11.0, which show kink, antikink, multisoliton and asymptotic nature. Thus, these results may be helpful to understand the propagation processes for nonlinear waves in the existing phenomena.

Acknowledgements

The authors sincerely acknowledge Dig Vijay Tanwar, Department of Mathematics, MNNIT Allahabad, for helpful suggestions to enhance the quality of the paper. The third author thanks Department of Mathematics, SGTB Khalsa College, University of Delhi, India for financial support.

References

[1] G M Kuz'mina, *Proc. Moscow State Pedagog. Inst.* **271**, 67 (1967) (in Russian)
[2] N A Kudryashov, *Chaos Solitons Fractals* **24**, 1217 (2005)
[3] A R Adem, Y Yildirim and E Yaşar, *Pramana – J. Phys.* **92**: 36 (2019)
[4] J Manafian and M Lakestani, *Pramana – J. Phys.* **92**: 41 (2019)
[5] Z T Fu, S K Liu, S D Liu and Q Zhao, *Phys. Lett. A* **290(1–2)**, 72 (2001)
[6] H Kumar, A Malik, F Chand and S C Mishra, *Indian J. Phys.* **86**, 819 (2012)
[7] M Wang, Y Zhou and Z Li, *Phys. Lett. A* **216**, 67 (1996)
[8] R Hirota, *The direct method in soliton theory* (Cambridge University Press, Cambridge, 2004)
[9] A Malik, F Chand, H Kumar and S C Mishra, *Comput. Math. Appl.* **64**, 2850 (2012)
[10] H Kumar, A Malik and F Chand, *Pramana – J. Phys.* **80**, 361 (2013)
[11] H Kumar, A Malik and F Chand, *J. Math. Phys.* **53**, 103704 (2012)
[12] V G Mikhalev, *Funct. Anal. Appl.* **26**, 140 (1992)
[13] M Dunajski, *J. Geom. Phys.* **51**, 126 (2004)
[14] M V Pavlov, *J. Math. Phys.* **44**, 4134 (2003)
[15] S V Manakov and P M Santini, *J. Phys. A* **42**, 404013 (2009)
[16] H Baran, I S Krasischik, O I Morozov and P Vojčák, *J. Non. Math. Phys.* **21**, 671 (2014)
[17] H Baran, I S Krasischik, O I Morozov and P Vojčák, *J. Non. Math. Phys.* **22**, 210 (2015)
[18] P G Grinevich, P M Santini and D Wu, *Nonlinearity* **28**, 3709 (2015)
[19] D Wu, *J. Diff. Equ.* **263(3)**, 1874 (2017)
[20] A Lelito and O I Morozov, *J. Geom. Phys.* **131**, 89 (2018)
[21] H Baran, I S Krasi'schik, O I Morozov and P Vojčák, *Theor. Math. Phys.* **196**, 1089 (2018)
[22] P J Olver, *Applications of Lie groups to differential equations* (Springer, New York, 1993)
[23] J Volkmann and G Baumann, *Proc. Institute of Mathematics of NAS of Ukraine* **50(1)**, 282 (2004)
[24] G W Bluman and J D Cole, *Similarity methods for differential equations* (Springer, New York, 1974)
[25] M Kumar, D V Tanwar and R Kumar, *Nonlinear Dyn.* **94**, 2547 (2018)
[26] M Kumar, D V Tanwar and R Kumar, *Comput. Math. Appl.* **75**, 218 (2018)
[27] M Kumar and D V Tanwar, *Comput. Math. Appl.* **76**, 2535 (2018)
[28] M Kumar and D V Tanwar, *Commun. Nonlinear Sci. Numer. Simul.* **69**, 45 (2019)
[29] S Kumar and D Kumar, *Int. J. Dynam. Control* **7**, 496 (2019)
[30] S Kumar and D Kumar, *Comput. Math. Appl.* **77**, 2096 (2019)

[31] S Kumar, D Kumar and A M Wazwaz, *Phys. Scr.* **94**, 065204 (2019)
[32] D Kumar and S Kumar, *Comput. Math. Appl.* **78**, 857 (2019)
[33] S Kumar, A M Wazwaz, D Kumar and A Kumar, *Phys. Scr.* **94**, 115202 (2019)
[34] M Kumar and A K Tiwari, *Comput. Math. Appl.* **75**, 1434 (2018)
[35] M Kumar and Y K Gupta, *Pramana – J. Phys.* **74**, 883 (2010)
[36] M Singh and R K Gupta, *Pramana – J. Phys.* **92**: 1 (2019)