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Anisotropic bulk viscous string cosmological models of the Universe under a time-dependent deceleration parameter

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Abstract. We investigate a new class of LRS Bianchi type-II cosmological models by revisiting the paper of Mishra *et al* (*Int. J. Theor. Phys*. **52**, 2546 (2013)) by considering a new deceleration parameter (DP) depending on the time in string cosmology for the modified gravity theory suggested by Sáez–Ballester (*Phys. Lett*. **113**, 467 (1986)). We have considered the energy–momentum tensor proposed by Letelier (*Phys. Rev*. **28**, 2414 (1983)) for bulk viscous and perfect fluid under some assumptions. To make our models consistent with recent astronomical observations, we have used the scale factor (Sharma *et al*, *Astron Astrophys*. **19**, 55 (2018), Garg *et al*, *Int. J. Geo. Meth. Mod. Phys*. **16**, 1950007 (2019)) $a(t) = \exp[\frac{1}{\beta}\sqrt{2\beta t + k}]$, where β and k are positive constants and it provides a time-varying DP. By using the recent constraints ($H_0 = 73.8$ and $q_0 = -0.54$) from SN Ia data in combination with BAO and CMB observations (Giostri *et al*, *JCAP* **3**, 27 (2012), arXiv:1203.3213v2[astro-ph.CO]), we affirm $\beta = 0.0062$ and $k = 0.000016$. For these constraints, we have substantiated a new class of cosmological transit models for which the expansion takes place from the early decelerated phase to the current accelerated phase. Also, we have studied some physical, kinematic and geometric behaviour of the models, and have found them consistent with observations and well-established theoretical results. We have also compared our present results with those of Mishra *et al* (*Int. J. Theor. Phys*. **52**, 2546 (2013)) and observed that the results in this paper are much better, stable under perturbation and in good agreement with cosmological reflections.

Keywords. String cosmology; Sáez–Ballester theory; bulk viscosity; transit Universe.

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1. Introduction

The current astronomical reflections, modern experimental data from SNe Ia [1–6], CMBR [7,8], WMAP [9–12] have established two main characteristics of the Universe: (a) the existence of the anisotropic Universe at the early stage of the evolution, which in due course of time attained isotropy and (b) the current Universe is expanding and also the rate of expansion is increasing (i.e. accelerating Universe). The SNe Ia measurements indicate a Universe that undergoes a transition from past decelerating to present accelerating expansion. So, it is a challenge for theoreticians to provide satisfactory theoretical support to these observations.

Friedmann–Robertson–Walker (FRW) space–time describes spatially homogenised and isotropic Universes, which can be appropriate for the contemporary Universe. However, because they have higher symmetries, it does not provide a correct description of the early Universe and presents poor approximations for an early Universe. Therefore, those models are additional applications for the outline of the whole evolution of the Universe, that has an anisotropic nature in early time and approaches isotropy at late times. Bianchi space–times offer a decent framework for this. Out of all, the Bianchi type-II (B-type-II) frame of reference plays a very important role in making models for the measurements of the expansion of the Universe throughout its early phase. Moreover, B-type-II line element yields an anisotropic spatial curvature. Recently, Asseo and Sol [13] and Roy and Banerjee [14] stressed the importance of B-type-II and prescribed LRS

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cosmological model. Kumar and Akarsu [15] mentioned B-type-II Universe with anisotropic dark energy and perfect fluid. Wang [16–18] has investigated the models of Letelier-type within the theoretical account of LRS B-type-II. In the context of massive string, Pradhan *et al* [19] have analysed LRS Bianchi type-II space–time. B-type-II frame of reference is employed to analyse dark energy models within the new role of time-dependent deceleration parameter (DP) by Maurya *et al* [20]. In the present study, we tend to look into LRS B-type-II string models of the Universe for perfect and viscous fluid under three conditions.

Next, although Einstein's general theory of relativity (GR) explains a large number of astrophysical phenomena, it fails to describe some, for instance, the expanding and late time accelerated expansion of the Universe. To deal with these, many alternative theories are proposed, out of which, Brans–Dicke [21] and Sáez–Ballester [22] scalar–tensor theories are of significance. In the present paper, we have studied the Sáez–Ballester modified theory of gravity. In this theory, Einstein's field equations have been modified by incorporating a dimensionless scalar field (ϕ) coupled with the metric g_{ij} in a simple manner. This modification satisfactorily describes the weak field in which an accelerated expansion regime appears. This theory also gives an answer to the question of the disappeared matter in a non-flat FRW Universe.

In recent years, string cosmology is gaining significant interest. Cosmic strings are topologically stable objects, which can be shaped throughout a phase transition within the early Universe. Cosmic strings play a significant role in the study of the early Universe. It is assumed that cosmic strings bring about density perturbations, that help the formation of galaxies or clusters of galaxies [23]. One more necessary feature of the string is that the string tension provides an efficient anisotropic pressure. Also, the stress–energy of the string coupled with the gravitational field is used to elucidate several alternative cosmological phenomena. Pioneer works in string theory were done by many researchers [24,25]. LRS B-type-II cosmological models have been discussed in [26–29] in different contexts. Recently, Pradhan and Jaiswal [30] have looked into string models of accelerated expansion in $f(R, T)$-gravity with a magnetic field.

Also, the dissipation effect together with bulk viscosity presents another model of dark energy. Relaxation processes related to bulk viscosity effectively reduce the pressure in an expanding system. The effective pressure becomes negative for a sufficiently large viscosity that can imitate dark energy behaviour. The idea that the bulk viscosity drives the acceleration of the Universe is mentioned in [31,32]. Some recent and important works of string cosmology in the presence of bulk viscosity have been done by several researchers [33–41] in different contexts. Some researchers have also discussed the physical aspects of massive/cosmic string cosmological models in the presence of magnetic field [42–46].

In recent years, many researchers have investigated the cosmological Universes in Sáez–Ballester modified gravity theory in various contexts [47–52]. Under the above-discussed perspective, the Sáez–Ballester field equations have been solved in an LRS B-type-II space–time in the presence of a cloud of massive string and bulk viscous fluid, under some physically and geometrically viable assumptions. In the present paper, we are revisiting the solutions obtained by Mishra *et al* [53], by assuming a scale factor

$$a(t) = \exp\left[\frac{1}{\beta}\sqrt{2\beta t + k}\right]$$

which results in a time-dependent DP having a transition from the decelerating Universe to the presently accelerating Universe.

The plan of the manuscript is as follows: Section 2 contains definitions and theoretical calculations. Section 2.1 deals with the metric and field equations. Section 2.2 deals with assumptions and under these assumptions, the solutions of the field equations are found. In §3, we have derived the solutions of field equations for three different cases. Results and discussions are given in §4. The stability of the corresponding solutions is analysed in §5. Finally, conclusions are given in §6.

2. Definitions and theoretical calculations

2.1 *Metric and field equations*

We consider an LRS B-type-II space–time [53]:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + X^2\,\mathrm{d}x^2 + Y^2\,\mathrm{d}y^2 + 2Y^2 x\,\mathrm{d}y\,\mathrm{d}z + (Y^2x^2 + X^2)\,\mathrm{d}z^2, \tag{1}$$

where $X = X(t)$ and $Y = Y(t)$.

The field equation (in gravitational units $8\pi G = 1$) proposed by Sáez and Ballester [22] is

$$G_{ij} - \omega\phi^r\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,k}\phi^{,k}\right) = -T_{ij}. \tag{2}$$

Here $G_{ij} = R_{ij} - \frac{1}{2}Rg_{ij}$, T_{ij} stands for the energy–momentum tensor and ϕ stands for the scalar field satisfying the equation

$$r\phi^{r-1}\phi_{,k}\phi^{,k} + 2\phi^r\phi^{;i}_{;i} = 0. \tag{3}$$

Here ω and r stand for the dimensionless coupling and arbitrary constant respectively. The comma denotes the partial derivative whereas the semicolon denotes the partial covariant differentiation with respect to t.

T_{ij}, for a cloud of cosmic string and bulk viscous fluid, reads as

$$T_{ij} = \bar{p} g_{ij} - \lambda x_i x_j + (\rho + \bar{p}) v_i v_j, \tag{4}$$

where

$$\bar{p} = p - 3H\xi. \tag{5}$$

In eqs (4) and (5), the different quantities have the usual meaning as already described in [53]. The four-velocity of the particles $v^i = (0, 0, 0, 1)$ and a unit space-like vector x^i representing the direction of string satisfy $g_{ij} v^i v^j = -g_{ij} x^i x^j = -1$, $v^i x_i = 0$. In LRS B-type-II metric, the mean Hubble parameter (H) can be defined as

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(2\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y}\right) = \frac{1}{3}(2H_1 + H_2). \tag{6}$$

Here $H_1 = \dot{X}/X$ and $H_2 = \dot{Y}/Y$ are directional Hubble parameters along the x- and y-axes respectively. Here $a = a(t)$ is the average scale factor, which, for LRS B-type-II model, is written as

$$a(t) = (X^2 Y)^{1/3}. \tag{7}$$

The particle density denoted by ρ_{p} follows the relation

$$\rho = \rho_{\mathrm{p}} + \lambda. \tag{8}$$

For the metric (1), the Sáez–Ballester field equations (2) and (3), along with the energy–momentum tensor given by (4), we obtain the following system of field equations:

$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}}{X}\frac{\dot{Y}}{Y} + \frac{1}{4}\frac{Y^2}{X^4} - \frac{1}{2}\omega\phi^r\dot{\phi}^2 = \lambda - \bar{p} \tag{9}$$

$$2\frac{\ddot{X}}{X} + \frac{\dot{X}^2}{X^2} - \frac{3}{4}\frac{Y^2}{X^4} - \frac{1}{2}\omega\phi^r\dot{\phi}^2 = -\bar{p} \tag{10}$$

$$\frac{\dot{X}^2}{X^2} + 2\frac{\dot{X}}{X}\frac{\dot{Y}}{Y} - \frac{1}{4}\frac{Y^2}{X^4} + \frac{1}{2}\omega\phi^r\dot{\phi}^2 = \rho \tag{11}$$

$$\ddot{\phi} + \dot{\phi}\left(\frac{2\dot{X}}{X} + \frac{\dot{Y}}{Y}\right) + \frac{r}{2}\frac{\dot{\phi}^2}{\phi} = 0. \tag{12}$$

In the usual notation, expansion scalar (θ) and the shear scalar (σ) are defined and given as

$$\theta = v^i_{;j} = 3\frac{\dot{a}}{a} = 2\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} \tag{13}$$

and

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{2}\left[2\frac{\dot{X}^2}{X^2} + \frac{\dot{Y}^2}{Y^2}\right] - \frac{1}{6}\theta^2, \tag{14}$$

where

$$\sigma_{ij} = v_{i;j} + \frac{1}{2}(v_{i;k} v^k v_j + v_{j;k} v^k v_i) + \frac{1}{3}\theta(g_{ij} + v_i v_j).$$

The anisotropy parameter (A_m) is defined as

$$A_m = 6\left(\frac{\sigma}{\theta}\right)^2 = \frac{2\sigma^2}{3H^2}. \tag{15}$$

2.2 *Assumptions*

Equations (9)–(12) have seven unknowns X, Y, ϕ, p, ρ, ξ and λ. For deterministic solutions of this system, we have to take three more equations, which relate these parameters.

As suggested by Thorne [54] and followed by many researchers [55,56], we first assume that θ is proportional to σ which gives

$$\left(\frac{2\dot{X}}{X} + \frac{\dot{Y}}{Y}\right) = \frac{\ell}{\sqrt{3}}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right), \tag{16}$$

where ℓ is the constant of proportionality. This yields

$$\frac{\dot{X}}{X} = m\frac{\dot{Y}}{Y}, \tag{17}$$

where $m = (\sqrt{3} + \ell)/(\ell - 2\sqrt{3})$. We have selected $m > 0$ for the anisotropic Universe, provided $m \neq 1$, as the study presents a picture of the FRW model for $m = 1$. Integrating eq. (17), we get

$$X = c_1 (Y)^m, \tag{18}$$

where c_1 is the constant of integration. Any loss of generality and for simplicity, $c_1 = 1$ is considered. Hence eq. (18) is reduced to

$$X = (Y)^m. \tag{19}$$

Secondly, we consider q as the linear function of Hubble parameter [57–60]:

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = \beta H + \kappa = \beta\frac{\dot{a}}{a} + \kappa. \tag{20}$$

Here κ and β stand for arbitrary constants. Equation (20) provides

$$\frac{a\ddot{a}}{\dot{a}^2} + \beta\frac{\dot{a}}{a} + \kappa = 0,$$

which by solving proceeds as

$$a = \exp\left[-\frac{(1+\kappa)}{\beta}t - \frac{1}{(1+\kappa)} + \frac{l}{\beta}\right], \tag{21}$$

provided $\kappa \neq -1$.

Here l is a constant of integration.

From the above equation we calculate

$$\dot{a} = -\left(\frac{1+\kappa}{\beta}\right) \exp\left[-\left(\frac{1+\kappa}{\beta}\right)t - \frac{1}{(1+\kappa)} + \frac{l}{\beta}\right],$$
$$\ddot{a} = \left(\frac{1+\kappa}{\beta}\right)^2 \exp\left[-\left(\frac{1+\kappa}{\beta}\right)t - \frac{1}{(1+\kappa)} + \frac{l}{\beta}\right]. \tag{22}$$

Equations (20) and (22) render the value of DP as $q = -1$. We also observed the same value of DP for $\kappa = 0$.

For $\kappa = -1$, eq. (20) is changed to

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \beta H. \tag{23}$$

Equation (23) reproduces the following differential equation:

$$\frac{a\ddot{a}}{\dot{a}^2} + \beta\frac{\dot{a}}{a} - 1 = 0. \tag{24}$$

The solution of the above equation is found to be [59,60]

$$a = \exp\left[\frac{1}{\beta}\sqrt{2\beta t + k}\right], \tag{25}$$

where k is an integrating constant. Equation (25) is recently used by different researchers [57–60] in different contexts.

For the study of cosmic decelerated–accelerated expansion of the Universe, we only consider the case $\kappa = -1$.

For the scale factor (25), the DP (q) and H are given as

$$q = -1 + \frac{\beta}{\sqrt{2\beta t + k}}, \quad H = \frac{1}{\sqrt{2\beta t + k}}. \tag{26}$$

From eq. (26), we observe that $q > 0$ for $t < [(\beta^2 - k)/2\beta]$ and $q < 0$ for $t > [(\beta^2 - k)/2\beta]$.

3. Solution of the field equation

By using eqs (19), (25) and (7), we obtain

$$X = (\mathrm{e}^{(1/\beta)\sqrt{2\beta t+k}})^{3m/(2m+1)}, \tag{27}$$

$$Y = (\mathrm{e}^{(1/\beta)\sqrt{2\beta t+k}})^{3/(2m+1)}. \tag{28}$$

From eqs (12), (27) and (28), we evaluate ϕ as

$$\phi(t) = \left[\frac{r+2}{2}\left(\phi_0 \int \frac{\mathrm{d}t}{(\mathrm{e}^{(3/\beta)\sqrt{2\beta t+k}})} + \phi_1\right)\right]^{2/(r+2)}, \tag{29}$$

where ϕ_0 and ϕ_1 are integrating constants.

Solving eqs (9)–(11) using eqs (27)–(29), we obtain energy density (ρ), effective pressure ($\bar{p}$) and string tension density (λ) as

$$\rho = \left[\frac{9m(m+2)}{(2m+1)^2(2\beta t + k)} - \frac{1}{4}(\mathrm{e}^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)} + \frac{1}{2}\frac{\omega\phi_0^2}{(\mathrm{e}^{(6/\beta)\sqrt{2\beta t+k}})}\right], \tag{30}$$

$$\bar{p} = \left[-\frac{27m^2}{(2m+1)^2(2\beta t+k)} + \frac{6m\beta}{(2m+1)}(2\beta t+k)^{-3/2} + \frac{3}{4}(\mathrm{e}^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)} + \frac{1}{2}\frac{\omega\phi_0^2}{(\mathrm{e}^{(6/\beta)\sqrt{2\beta t+k}})}\right], \tag{31}$$

$$\lambda = \left[\frac{-18m^2 + 9m + 9}{(2m+1)^2(2\beta t + k)} - \frac{3\beta(m-1)}{(2m+1)(2\beta t + k)^{3/2}} + (\mathrm{e}^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)}\right]. \tag{32}$$

Accordingly, ρ_{p} is obtained as

$$\rho_{\mathrm{p}} = \left[\frac{(27^2 + 9m - 9)}{(2m+1)^2(2\beta t + k)} - \frac{5}{4}(\mathrm{e}^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)} + \frac{1}{2}\frac{\omega\phi_0^2}{(\mathrm{e}^{(6/\beta)\sqrt{2\beta t+k}})} - \frac{3\beta(m-1)}{(2m+1)2\beta t + k)^{3/2}}\right]. \tag{33}$$

For calculating the other parameters, we shall consider the following three cases.

3.1 *Case* I: *Bulk viscous model with* $p = \alpha\rho$

Consider the perfect gas equation of state as

$$p = \alpha\rho, \tag{34}$$

where α ($0 \le \alpha \le 1$) is a constant. For various values of α, we shall get three types of models:

(i) if $\alpha = 0$, we tend to get matter-dominant model.
(ii) if $\alpha = \frac{1}{3}$, we tend to get radiation-dominant model.
(iii) if $\alpha = 1$, we get $\rho = p$ which is termed as Zel'dovich fluid or stiff fluid model [61].

Therefore, by eqs (5) and (34), we can directly calculate p and ξ:

$$p = \left[\frac{9m\alpha(m+2)}{(2m+1)^2(2\beta t+k)} - \frac{\alpha}{4}(e^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)} + \frac{\alpha}{2}\frac{\omega\phi_0^2}{(e^{(6/\beta)\sqrt{2\beta t+k}})}\right] \quad (35)$$

$$\xi = \left[\frac{(3m^2\alpha+6m\alpha+9m^2)}{(2m+1)^2\sqrt{2\beta t+k}} - \left(\frac{\alpha+3}{12}\right) \times (e^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)}\sqrt{2\beta t+k} - \frac{2m\beta}{(2m+1)(2\beta t+k)} + \left(\frac{\alpha-1}{6}\right)\frac{\omega\phi_0^2\sqrt{2\beta t+k}}{(e^{(6/\beta)\sqrt{2\beta t+k}})}\right]. \quad (36)$$

3.2 *Case* II: *Bulk viscous model with* $\xi = \xi_0\rho^n$

For most of the investigations, we found that the coefficient of bulk viscosity (ξ) is considered as a simple power function of energy density and it depends on time. It is assumed that

$$\xi = \xi_0\rho^n, \quad (37)$$

where ξ_0 and n are real constants [62–64]. For small density and radiative fluid, n may be equal to 1 [65,66]. The assumption $0 \le n \le 1/2$ is a good assumption to obtain realistic results, as given by Belinskii and Khalatnikov [67].

Using eqs (5), (30), (31) and (37), the expressions for ξ and p are given as

$$\xi = \xi_0\left[\frac{9m(m+2)}{(2m+1)^2(2\beta t+k)} - \frac{1}{4}(e^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)} + \frac{1}{2}\frac{\omega\phi_0^2}{(e^{(6/\beta)\sqrt{2\beta t+k}})}\right]^n \quad (38)$$

$$p = \left[\frac{3\xi_0}{\sqrt{2\beta t+k}}\left[\frac{9m(m+2)}{(2m+1)^2(2\beta t+k)} - \frac{1}{4}(e^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)} + \frac{1}{2}\frac{\omega\phi_0^2}{(e^{(6/\beta)\sqrt{2\beta t+k}})}\right]^n - \frac{27m^2}{(2m+1)^2(2\beta t+k)} + \frac{6m\beta}{(2m+1)}(2\beta t+k)^{-3/2} + \frac{3}{4}(e^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)} + \frac{1}{2}\frac{\omega\phi_0^2}{(e^{(6/\beta)\sqrt{2\beta t+k}})}\right]. \quad (39)$$

3.3 *Case* III: *Perfect fluid model with* $\xi = 0$

For a perfect fluid, ξ is assumed to be zero. The rest of the six unknowns X, Y, ϕ, p, ρ and λ can be directly calculated from the field equations (9)–(12). For $\xi = 0$, eq. (5) gives $\bar{p} = p$, i.e. the effective pressure equals isotropic pressure, and the expression is given by

$$p = \left[-\frac{27m^2}{(2m+1)^2(2\beta t+k)} + \frac{6m\beta}{(2m+1)}(2\beta t+k)^{-3/2} + \frac{3}{4}(e^{(1/\beta)\sqrt{2\beta t+k}})^{(6-12m)/(2m+1)} + \frac{1}{2}\frac{\omega\phi_0}{(e^{(6/\beta)\sqrt{2\beta t+k}})}\right]. \quad (40)$$

4. Interpretation of the results

From eq. (26), the present value of DP can be taken as

$$q_0 = -1 + \beta H_0 = -1 + \frac{\beta}{\sqrt{2\beta t_0 + k}},$$

where H_0 and t_0 have their usual meaning.

By using the recent constraints ($H_0 = 73.8$ and $q_0 = -0.54$) from SN Ia data in combination with BAO and CMB observations [68], we take the values of $\beta = 0.0062$ and $k = 0.000016$. We have used these values in formulating and drawing different figures to analyse the nature of physical quantities.

We have plotted the variation of q with respect to cosmic time (t) in figure 1a, and observed that DP is positive at early time and negative at present time indicating that our models are evolving from decelerating phase ($q > 0$) to accelerating phase ($q < 0$), and the models show a phase transition from positive to negative for q for $k = 0.000016$ and $\beta = 0.0062$. The critical time at which the phase transition took place is given by

$$t_c = \frac{\beta^2 - k}{2\beta}.$$

Also, when $t \to \infty$, $q \to -1$. According to SNe Ia observation, the Universe is accelerating at present and the value of DP lies in the range $-1 < q < 0$. So our models show consistency with recent observations.

Figure 1b shows the variation of H with t as per eq. (26). We see that H is a positive, decreasing function

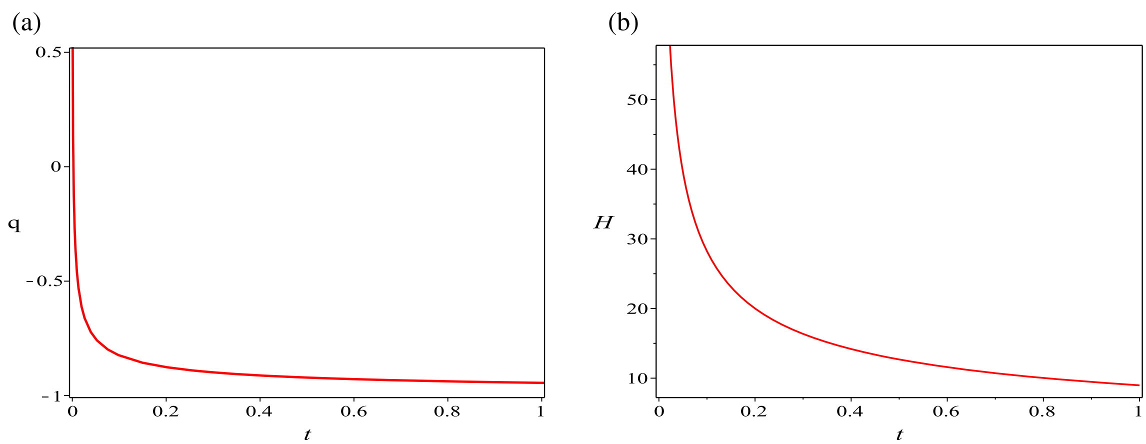

Figure 1. (**a**) Plot of q vs. t and (**b**) plot of H vs. t. Here $\beta = 0.0062$ and $k = 0.000016$.

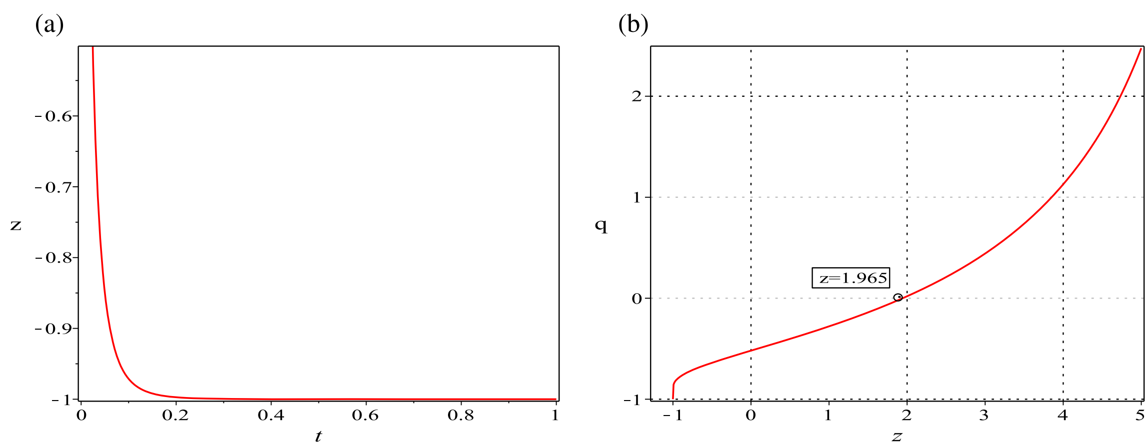

Figure 2. (**a**) Plot of z vs. t and (**b**) plot of q vs. z.

of time, and tends to zero as $t \to \infty$, which agrees with the established theories.

The average scale factor $a(t)$ in terms of red-shift (z) is given by

$$a(t) = \frac{a_0}{1+z},$$

where a_0 is the present value of $a(t)$.

From eq. (25), we can get

$$a_0 = \exp\left[\frac{1}{\beta}\sqrt{2\frac{\beta}{H_0} + k}\right].$$

Using the values $\beta = 0.0062$, $k = 0.000016$ and $H_0 = 73.8$ we get $a_0 = 8.6021$. We have used these values to draw the graph.

From eq. (25), we get

$$\frac{\sqrt{2\beta t + k}}{\beta} = \ln(a).$$

Also from

$$a(t) = \frac{a_0}{1+z},$$

we have

$$\ln(a) = \ln(a_0) - \ln(1+z).$$

Substituting the above in eq. (26), we get

$$q(z) = -1 + \frac{1}{\ln(a_0) - \ln(1+z)}. \quad (41)$$

Figure 2a shows the fluctuation of z with t for our derived models. From the figure, we see that z is a monotonic decreasing function of t for $a_0 = 8.6021$. Also, z starts with a small positive value (5.16) at $t = 0$ and $z \to -1$ as $t \to \infty$ for our derived models. So, we can say that $t \to \infty$ corresponds to $z \to -1$.

In figure 2b, we have shown the fluctuation of q with z as per eq. (41). From this figure, we see that as z decreases, q changes its phase from positive (decelerating phase) to negative (accelerating phase) and $q \to -1$ as $z \to -1$. Recently, Capazziello *et al* [69,70] have

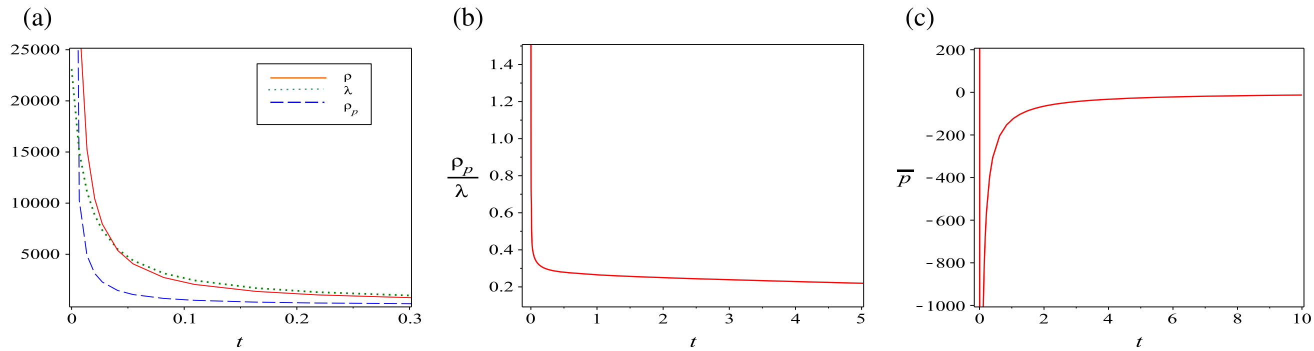

Figure 3. (**a**) Plot of energy densities vs. t, (**b**) plot of $\rho_{\rm p}/\lambda$ vs. t and (**c**) plot of $\bar{p}$ vs. t. Here $\omega = \phi_0 = 1$, $m = 0.5$, $\beta = 0.0062$ and $k = 0.000016$.

studied the transition red-shift in $f(T)$ cosmology and observational constraints and cosmographic bounds on the cosmological deceleration–acceleration transition redshift in $F(R)$ gravity respectively.

It was found in its analysis that the SNe data favour current acceleration ($z < 0.5$) and past deceleration ($z > 0.5$). Recently, according to the high-z supernova search (HZSNS) team $z_t = 0.46 \pm 0.13$ at (1σ) confidence level (c.l.) [3] which has been further analysed to $z_t = 0.43 \pm 0.07$ at (1σ) c.l. [3]. According to SNLS [71], as well as the one recently compiled in [72], the transition red-shift $z_t \sim 0.6(1\sigma)$ is in better agreement with the flat ΛCDM model ($z_t = (2\Omega_\Lambda/\Omega_m)^{1/3} - 1 \sim 0.66$). Another limit is $0.60 \le z_t \le 1.18$ (2σ, joint analysis) [73]. Further, the transition red-shift for our derived model comes to be $z_t \cong 1.965$ (see figure 2b) which is in good agreement with the Type- Ia supernovae observations, including the farthest known supernova SNI997ff at $z \approx 1.7$ [2] and [74]. We see that the variation of q with z obtained in our model is compatible with the results obtained in the above references.

In figure 3a, we have shown three curves of ρ, $\rho_{\rm p}$ and λ. We see that all the energy densities are positive decreasing functions of time showing the expanding Universe. All the energy densities approach zero as $t \to \infty$, indicating that the Universe will keep on expanding forever. Also, we see that $\lambda < \rho_{\rm p}$ for the early phase of evolution i.e., particle dominates over the string, and then $\lambda > \rho_{\rm p}$ in due course of evolution, i.e., the string dominates over the particle thereafter.

The comparative behaviour of $\rho_{\rm p}$ and λ is also studied in figure 3b. From figure 3b we see that the ratio $(\rho_{\rm p}/\lambda) > 0$ throughout. At the early phase of evolution, the ratio $(\rho_{\rm p}/\lambda) > 1$, indicating that $\rho_{\rm p} > \lambda$, i.e. the particle-dominated phase. But, as time progresses, the ratio falls below 1 indicating the string-dominated phase. These observations are supported by Krori *et al* [75] and Kibble [76].

In figure 3c, we have plotted the variation of $\bar{p}$ with t as per eq. (31). We see that $\bar{p}$ is negative at present, which may support for the current accelerated expansion of the Universe.

In figure 4a we have plotted the behaviour of p for Case I when $p = \alpha\rho$ for three scenarios, $\alpha = 0$ (dust-filled), $1/3$ (radiation-dominated) and 1 (stiff matter filled) Universe. In all the cases we find that p is a positive decreasing function of time.

In figure 4b we consider Case II when $\xi = \xi_0\rho^n$ and plotted $p(t)$ for three values of n (=0, 1/2 and 1). We observed that for all the three values of n, p is again a positive decreasing function of time. We also observed in both Cases I and II that $p \to 0$ when $t \to \infty$.

In Case III, when $\xi = 0$ (i.e., in the absence of viscous effect) p and $\bar{p}$ become equal. The behaviour of $\bar{p}$ is shown in figure 4c. We see that in the absence of viscosity, $\bar{p}$ becomes highly negative at the early time, then increases and tends to a small negative value at late time.

In figure 5a we have plotted ξ for Case I when $p = \alpha\rho$ for three scenarios ($\alpha = 0$, $1/3$ and 1). In all the cases we find that ξ is a positive decreasing function of time. In the early Universe, it was high and after that, it reduces gradually and tends to zero as $t \to \infty$. So, we can say that the nature of the fluid was highly viscous at the time of the early Universe which tends to reduce and vanish in due course of time. In figure 5b we consider Case II when $\xi = \xi_0\rho^n$ and plotted $\xi(t)$ for $n = 0$, $1/2$ and 1. Here also, we observe the same behaviour for $n = 1/2$ and 1, whereas for $n = 0$ the viscous effect vanishes throughout the evolution of the Universe.

Other physical parameters, such as expansion scalar (θ), volume scalar (V), shear scalar (σ), anisotropy parameter (A_m) and directional Hubble parameters (H_1) and (H_2) are obtained as

$$\theta = 3H = \frac{3}{\sqrt{2\beta t + k}} \tag{42}$$

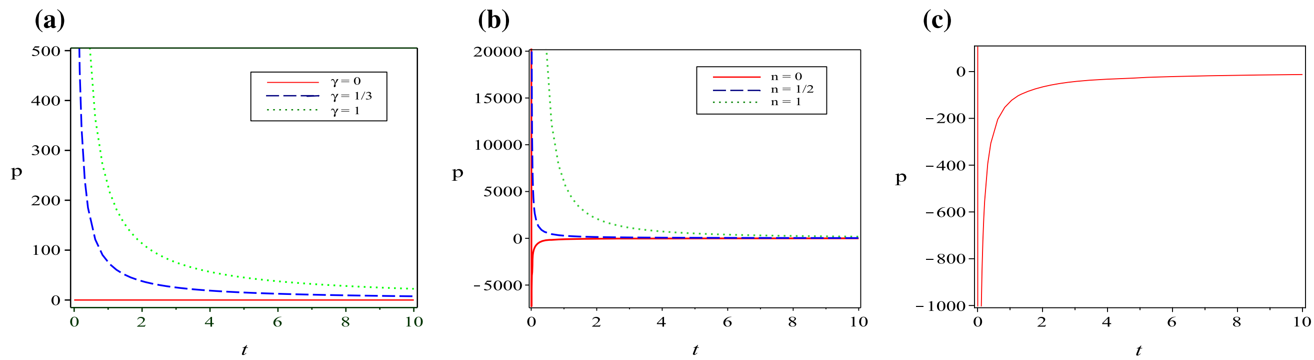

Figure 4. (**a**) Plot of p vs. t for Case I, (**b**) plot of p vs. t for Case II and (**c**) plot of p vs. t for Case III. Here $\omega = \phi_0 = 1, m = 0.5, \beta = 0.0062$ and $k = 0.000016$.

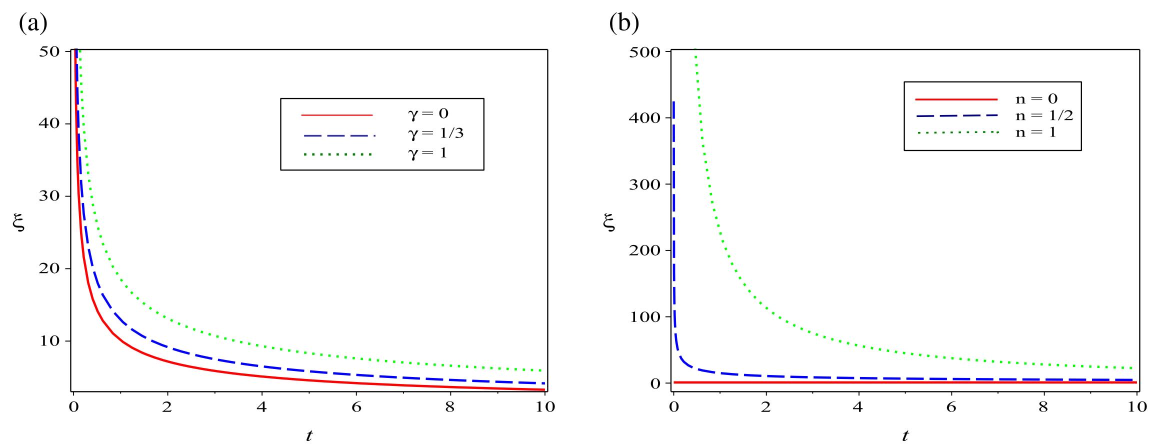

Figure 5. (**a**) Plot of ξ vs. t for Case I and (**b**) plot of ξ vs. t for Case II. Here $\omega = \phi_0 = 1, m = 0.5, \beta = 0.0062$ and $k = 0.000016$.

$$V = A^2B = \mathrm{e}^{(3/\beta)\sqrt{2\beta t+k}} \tag{43}$$

$$\sigma^2 = -\frac{1}{2}\left[\frac{9(2m^2+1)}{(2m+1)^2(2\beta t+k)}\right] - \frac{3}{2\sqrt{2\beta t+k}} \tag{44}$$

$$A_m = \frac{2m^2-4m+2}{(2m+1)^2} \tag{45}$$

$$H_1 = mH_2 = \frac{3m}{(2m+1)\sqrt{2\beta t+k}}. \tag{46}$$

In Big-Bang scenario the parameters like σ, θ and H are finite. From eq. (43) V is zero at $t = 0$. As $t \to \infty$, V becomes infinite whereas θ, H and σ approach zero.

5. The model stability

We have tested the stability of the background solution with respect to the perturbations of the metric. For the study, we adopt notation a_i for the metric potentials (i.e. $a_1 = X$ and $a_2 = Y$).

The stability analysis is performed against the perturbations of all possible fields. The stability of the solution has been first discussed by Chen and Kao [77]. Here perturbation will be considered for the two-expansion factor a_i via

$$a_i \to a_{B_i} + \delta a_i = a_{B_i}(1+\delta b_i), \tag{47}$$

where $\delta a_i = a_{B_i}\delta b_i$.

Accordingly, the perturbations of the volume scalar, directional Hubble factors and the mean Hubble parameter are shown as follows:

$$V \to V_B + V_B\sum_{i=1}^{3}\delta b_i, \quad H_i \to H_{B_i} + \delta\dot{b}_i,$$

$$H \to H_B + \frac{1}{3}\sum_{i=1}^{3}\delta\dot{b}_i,$$

$$\sum_{i=1}^{3} H_i^2 \to \sum_{i=1}^{3} H_{B_i}^2 + 2\sum_{i=1}^{3} H_{B_i}.\delta b_i. \tag{48}$$

Hence, the metric perturbations δb_i obey the following equations:

$$\Sigma\delta\ddot{b_i} + 2\Sigma H_{B_i}\delta\dot{b_i} = 0 \tag{49}$$

$$\Sigma\delta\ddot{b_i} + 2\frac{\dot{V}_B}{V_B}\delta\dot{b_i} + \Sigma\delta\dot{b}_j H_{B_i} = 0 \tag{50}$$

$$\Sigma\delta\dot{b}_j = 0. \tag{51}$$

From the above three equations, it can easily be seen that

$$\delta\ddot{b_i} + \frac{\dot{V}_B}{V_B}\delta\dot{b_i} = 0, \tag{52}$$

where V_B is the background volume scalar and in this model, it is given by

$$V_B = \mathrm{e}^{(3/\beta)\sqrt{2\beta t + k}}. \tag{53}$$

We can calculate δb_i with the help of eq. (51). Then we get

$$\delta b_i = \mathrm{e}^{(-3/\beta)\sqrt{2\beta t+k}}\left(-\frac{1}{3}\sqrt{2\beta t + k} - \frac{1}{9}\beta\right) + c, \tag{54}$$

where c is the constant of integration. Therefore, the actual fluctuations for each expansion factor $\delta a_i = a_{B_i}\delta b_i$ are given by

$$\delta a_i = a_{B_i}\mathrm{e}^{(-3/\beta)\sqrt{2\beta t+k}}\left(-\frac{1}{3}\sqrt{2\beta t + k} - \frac{1}{9}\beta\right) + c. \tag{55}$$

From eq. (55) and figure 6, it is observed that for positive values of β and k, δa_i approaches zero for large t, i.e. $t \to \infty$, $\delta a_i \to 0$. Consequently, the background solution is stable against the metric perturbation.

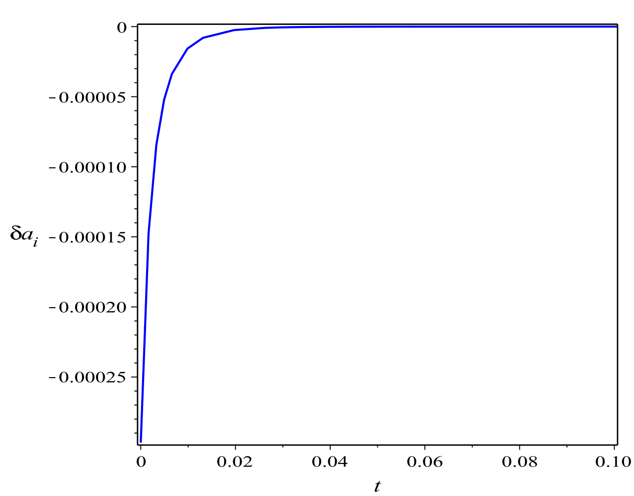

Figure 6. Plot of metric perturbation δa_i vs. t. Here $\beta = 0.0062$, $k = 0.000016$ and $c = 0$.

If δb_i tends to zero, from eq. (48), we see that $V \to V_B$, $H_i \to H_{B_i}$, $H \to H_B$ and so we can say that our solution is stable against the perturbation of scale volume factor, directional Hubble and average Hubble parameters also.

6. Conclusion

The present study contributes to the exact solutions of anisotropic bulk viscous string cosmological models in the scalar–tensor theory of gravitation described by Sáez–Ballester. It is worth mentioning here that ϕ plays a significant role in expressing the physical quantities $\bar{p}$, ρ, p, ξ and ρ_{p}. We find a point-type singularity [78] in the derived models as p, ρ, λ and ρ_{p} diverge at $t \to \infty$.

The model shows a phase transition from an early decelerating to present the accelerating expansion of the Universe. The phase transition took place at $z = 1.965 \approx 2$. Recently, Hayes *et al* [79] and Dunlop [80] used the comparison of Lyman-α and H-α luminosity functions to deduce the range of red-shift, which currently is feasible at $z \approx 2$. Thus, $z = 1.965$ in our derived models is consistent with the observational value [79,80].

Also, our derived models are stable under perturbations. So, we may conclude that our models are improved from earlier works and they present a better picture of the Universe. So it deserves attention.

From figures 4a–4c, it is observed that the isotropic pressure, in the presence of the bulk viscosity, is a decreasing function of time (t) and approaches zero at late time but in the absence of bulk viscosity the pressure is always negative and tends to zero at the present time. Thus, we see that bulk viscosity plays a role in the evolution of the Universe.

It is important to mention here that our present work is more relevant than that of Mishra *et al* [53] as we have considered advanced observational constraints. We obtain more advanced results in a cosmic string with bulk viscosity.

Lastly, we conclude that our derived models show a better shape of the Universe.

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