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Propagation of nonlinear waves with a weak dispersion via coupled (2 + 1)-dimensional Konopelchenko–Dubrovsky dynamical equation

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Abstract. This work applies the modified extended direct algebraic method to construct some novel exact travelling wave solutions for the coupled (2 + 1)-dimensional Konopelchenko–Dubrovsky (KD) equation. Soliton, periodic, solitary wave, Jacobi elliptic function, new elliptic, Weierstrass elliptic function solutions and so on are obtained, which have several implementations in the field of applied sciences and engineering. In addition, we discuss the dynamics of some solutions like periodic, soliton and dark-singular combo soliton by their evolutionary shapes.

Keywords. (2+1)-Dimensional Konopelchenko–Dubrovsky equation; modified extended direct algebraic method; travelling wave solution; Jacobi elliptic function.

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1. Introduction

In recent years, several researchers have devoted their energy to study the propagation of wave solutions for coupled nonlinear evolution equations (NEEs), which perform a crucial role in explaining the features of nonlinear problems in science and engineering. Several nonlinear coupled wave models are often used to illustrate many of the problems in physics, such as heat flow occurrence, plasma physics, biology, electricity, quantum mechanics, fibre optics etc. [1–6]. In addition, several problems in other areas like epidemiology, ecology, chemical material reactivity, etc. are also described by NEEs. Obtaining exact and numerical solutions for coupled nonlinear wave is crucial and important in the study of physical occurrences. On the other hand, the exact solution of coupled nonlinear problems can help people understand these occurrences better than numerical solutions. Therefore, the study of exact solutions of NEEs is very crucial, and many powerful methods have been developed to construct the exact solutions. Bäcklund transformation, Darboux transformation, Cole–Hopf transformation, various tanh methods, various Jacobi elliptic function methods, variable separation approach, Painlevé method, homogeneous balance method, similarity reduction method, etc. [7–28] are very effective algorithms for constructing exact solutions of many NEEs [29–36].

More periodic wave solutions expressed by Jacobi elliptic functions for the $(2+1)$-dimensional Konopelchenko–Dubrovsky (KD) equations were obtained by using the extended F-expansion method. The solitary wave solutions and trigonometric function solutions for the KD equations were found in [37]. By using the further improved F-expansion method, $(2+1)$-dimensional KD equation were given the Jacobi elliptic function solutions, soliton-like solutions and trigonometric function solutions [38]. The similarity transformation method generated infinite-dimensional Lie algebra and commutation relations of the KD equation [39]. The extended tanh method, the sech–csch ansatz, the Hirota's bilinear formalism combined with the simplified Hereman form and the Darboux transformation method were applied to determine the travelling wave solutions and other kinds of exact solutions for the two-dimensional KD equation and abundant new soliton solutions, kink solutions, periodic wave solutions and complexiton solutions were formally derived [40]. The tanh–sech method, the cosh–sinh method and exponential function method were efficiently employed to handle the $(2+1)$-dimensional KD equation [41].

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In this paper we consider the coupled $(2+1)$-dimensional nonlinear KD equation. By applying the modified extended direct algebraic method, we constructed some novel exact travelling wave solutions for the coupled $(2+1)$-dimensional KD equation. Soliton, periodic, solitary wave, Jacobi elliptic function, new elliptic, Weierstrass elliptic function solutions and so on are obtained, which have several implementations in the field of applied sciences and engineering. The purpose is to find new exact solutions of the equation, and hence we consider the following KD equation:

$$h_\xi - h_{\zeta\zeta\zeta} - 6\gamma h h_\zeta + \frac{3}{2}\beta^2 h^2 h_\zeta - 3g_\iota + 3\beta h_\zeta g = 0,$$
$$g_\zeta = h_\iota, \tag{1}$$

where $h = h(\zeta, \iota, \xi)$ and $g = g(\zeta, \iota, \xi)$, the subscripts represent the partial differentiation, γ and β are the real parameters. For $h_\iota = 0$, eq. (1) is reduced to the Gardner equation and it is also converted to the Kadomtsev–Petviashvili equation, where $\beta = 0$. Furthermore, it becomes modified Kadomtsev–Petviashvili equation with $\gamma = 0$, which is valuable in soliton theory.

Several researchers have studied in detail the KD system and has proposed some efficient methods for its exact solutions. For instance, Wu *et al* [42] successfully obtained exact solutions of the equation by using the Hirota direct method and linear superposition principle, whereas Kumar and Tiwari [39] presented various new group-invariant solutions of the KD equation by using Lie symmetry approach. In addition, Wang and Wei [40] combined the extended tanh method, the sech–csch ansatz, the Hirota bilinear formalism with the simplified Hereman form and the Darboux transformation method to obtain the travelling wave solutions of the KD equations. Wazwaz [41] obtained exact solutions of the KD equation with periodic, soliton and kink wave properties.

Under the impetus of the above literature, the ansatz equation is extended in a new general formula in the modified extended direct algebraic method to find travelling wave solutions of the coupled KD equation [43–45]. Subsequently, more or less novel and further versatile exact travelling wave solutions have been constructed. The following subsection provides a brief overview of the modified extended direct algebraic method.

1.1 *A summary of the method*

We assume that the given NEE of $h(\zeta, \iota, \xi)$ is in the form

$$Q(h, h_\xi, h_\iota, h_\zeta, h_{\xi\xi}, h_{\iota\iota}, h_{\zeta\zeta}, \dots) = 0, \tag{2}$$

where Q is the polynomial in its parameters. The nature of the method can be given in the subsequent stages [46–48]:

Stage 1: To obtain the solutions of eq. (2), take $h(\zeta, \iota, \xi) = H(\phi)$, $\phi = \kappa_1\zeta + \kappa_2\iota - \alpha\xi$ and transform eq. (2) to the ordinary differential equation (ODE)

$$R(H, H', H'', H''', \dots) = 0, \tag{3}$$

where $'$ (prime) represents the derivative with respect to ϕ.

Stage 2: By introducing the solution $H(\phi)$ of eq. (3) in a finite series form given by [50]

$$H(\phi) = \sum_{j=-M}^{M} d_j \theta^j(\phi), \tag{4}$$

where d_j (real constants with $d_M \neq 0$) and M (positive integer) are to be determined. $\theta(\phi)$ represents the solution of the equation below:

$$\theta'(\phi) = \sqrt{f_0 + f_1\theta + f_2\theta^2 + f_3\theta^3 + f_4\theta^4 + f_5\theta^5 + f_6\theta^6}, \tag{5}$$

where f_j are constants.

Stage 3: Calculate M by using the homogeneous balance principle on eq. (3).

Stage 4: By [49,50], using Mathematica to solve the system and based on the value of the parameters f_j we can get the exact solutions of eq. (1).

The structure of this paper is as follows: The introduction is given in §1. In §2, we applied the method to construct more or less novel exact soliton, periodic and Jacobi elliptic solutions of the coupled KD equation. Analysis and discussion of the solution is given in §3. Finally, the conclusion is given §4.

2. The (2+1)-dimensional nonlinear KD equation

Here we consider the coupled $(2+1)$-dimensional nonlinear KD equation discussed in §1 as follows:

$$h_\xi - h_{\zeta\zeta\zeta} - 6\gamma h h_\zeta + \frac{3}{2}\beta^2 h^2 h_\zeta - 3g_\iota + 3\beta h_\zeta g = 0,$$
$$g_\zeta = h_\iota, \tag{6}$$

where γ and β are nonzero real parameters. By considering the travelling wave solution as

$$h(\zeta, \iota, \xi) = h(\phi) = \sum_{j=-M}^{M} d_j \theta^j(\phi),$$

$$g(\zeta,\iota,\xi)=g(\phi)=\sum_{k=-N}^{N}e_k\theta^k(\phi), \tag{7}$$

$$\theta'(\phi)=\sqrt{f_0+f_1\theta+f_2\theta^2+f_3\theta^3+f_4\theta^4+f_5\theta^5+f_6\theta^6},$$
$$\phi=\kappa_1\zeta+\kappa_2\iota-\alpha\xi, \tag{8}$$

where $d_j, e_k, f_0, f_1, f_2, f_3, f_4, f_5, f_6, \gamma, \beta, \kappa_1, \kappa_2$ and α are constants, M, N are positive integers to be determined later. The values of M and N are normally obtained by applying the homogeneous balance principle (i.e. balancing the highest-order linear term together with the highest order of nonlinear terms). Substituting eq. (7) on eq. (6), results

$$-\alpha h'-\kappa_1^3h'''-6\gamma hh'+\frac{3}{2}\beta^2\kappa_1h^2h'$$
$$-3\kappa_2g'+3\beta\kappa_1h'g=0, \tag{9}$$
$$\kappa_1g'=\kappa_2h'. \tag{10}$$

By integrating eq. (10) with respect to ϕ, we get

$$g=\frac{\kappa_2h}{\kappa_1}+A, \tag{11}$$

for which A is the integration constant. By putting eq. (11) into eq. (9) we get

$$-2\kappa_1^4h'''+3\beta^2\kappa_1^2h^2h'+6\kappa_1(\beta\kappa_2-2\gamma\kappa_1)hh'$$
$$+2(3\beta\kappa_1^2A-\alpha\kappa_1-3\kappa_2^2)h'=0. \tag{12}$$

Now by balancing the highest-order derivative h''' and the nonlinear term h^2h', we attain $M=1$ and consider the solution of eq. (12) as

$$h(\phi)=\frac{d_{-1}}{\theta}+d_0+d_1\theta. \tag{13}$$

Now by substituting eqs (8) and (13) into eq. (12) and applying [49,50], the solutions of eq. (6) can be expressed as follows:

In this case, the solutions of eq. (6) are in the following forms:

$$h_1(\phi)=d_0\pm\frac{4f_2\kappa_1\sqrt{f_4}\,\mathrm{sech}(\sqrt{f_2}\phi)}{\beta(\sqrt{\Lambda}-f_4\,\mathrm{sech}(\sqrt{f_2}\phi))},$$
$$g_1(\phi)=\frac{\kappa_2h_1}{\kappa_1}+A,\quad f_2>0,\quad \Lambda>0 \tag{16}$$

$$h_2(\phi)=d_0\mp\frac{4f_2\kappa_1\sqrt{f_4}\,\mathrm{sech}(\sqrt{f_2}\phi)}{\beta(\sqrt{\Lambda}+f_4\,\mathrm{sech}(\sqrt{f_2}\phi))},$$
$$g_2(\phi)=\frac{\kappa_2h_2}{\kappa_1}+A,\quad f_2>0,\quad \Lambda>0 \tag{17}$$

$$h_3(\phi)=d_0\pm\frac{4f_2\kappa_1\sqrt{f_4}\,\mathrm{csch}(\sqrt{f_2}\phi)}{\beta(\sqrt{-\Lambda}-f_4\,\mathrm{csch}(\sqrt{f_2}\phi))},$$
$$g_3(\phi)=\frac{\kappa_2h_3}{\kappa_1}+A,\quad f_2>0,\quad \Lambda<0 \tag{18}$$

$$h_4(\phi)=d_0\mp\frac{4f_2\kappa_1\sqrt{f_4}\,\mathrm{csch}(\sqrt{f_2}\phi)}{\beta(\sqrt{-\Lambda}+f_4\,\mathrm{csch}(\sqrt{f_2}\phi))},$$
$$g_4(\phi)=\frac{\kappa_2h_4}{\kappa_1}+A,\quad f_2>0,\quad \Lambda<0 \tag{19}$$

$$h_5(\phi)=d_0\mp\frac{2f_2\kappa_1}{\beta\sqrt{f_4}}\left(1\pm\tanh\left(\frac{\sqrt{f_2}\phi}{2}\right)\right)$$
$$g_5(\phi)=\frac{\kappa_2h_5}{\kappa_1}+A,\quad f_2>0,\quad \Lambda=0; \tag{20}$$

$$h_6(\phi)=d_0\mp\frac{2f_2\kappa_1}{\beta\sqrt{f_4}}\left(1\pm\coth\left(\frac{\sqrt{f_2}\phi}{2}\right)\right)$$
$$g_6(\phi)=\frac{\kappa_2h_6}{\kappa_1}+A,\quad f_2>0,\quad \Lambda=0, \tag{21}$$

where

$$\phi=\kappa_1\zeta+\kappa_2\iota-\alpha\xi,\quad \Lambda=f_3^2-4f_2f_4,$$

and

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*Case*1 : $f_0=f_1=f_5=f_6=0$

(i) $d_{-1}=0,\quad d_0=\dfrac{-\beta f_3\kappa_1^2-2\beta\kappa_2\sqrt{f_4}+4\gamma\kappa_1\sqrt{f_4}}{2\beta^2\sqrt{f_4}\kappa_1},\quad d_1=\pm\dfrac{2\kappa_1\sqrt{f_4}}{\beta},$

$$A=\frac{8\alpha\beta^2f_4\kappa_1-3\beta^2f_3^2\kappa_1^4+8\beta^2f_2f_4\kappa_1^4+36\beta^2f_4\kappa_2^2-48\beta\gamma f_4\kappa_2\kappa_1+48\gamma^2f_4\kappa_1^2}{24\beta^3f_4\kappa_1^2}. \tag{14}$$

(ii) $d_{-1}=0,\quad d_0=\dfrac{\beta f_3\kappa_1^2-2\beta\kappa_2\sqrt{f_4}+4\gamma\kappa_1\sqrt{f_4}}{2\beta^2\sqrt{f_4}\kappa_1},\quad d_1=\pm\dfrac{2\kappa_1\sqrt{f_4}}{\beta},$

$$A=\frac{8\alpha\beta^2f_4\kappa_1-3\beta^2f_3^2\kappa_1^4+8\beta^2f_2f_4\kappa_1^4+36\beta^2f_4\kappa_2^2-48\beta\gamma f_4\kappa_2\kappa_1+48\gamma^2f_4\kappa_1^2}{24\beta^3f_4\kappa_1^2}. \tag{15}$$

$$A=\frac{8\alpha\beta^2 f_4\kappa_1-3\beta^2 f_3^2\kappa_1^4+8\beta^2 f_2 f_4\kappa_1^4+36\beta^2 f_4\kappa_2^2-48\beta\gamma f_4\kappa_2\kappa_1+48\gamma^2 f_4\kappa_1^2}{24\beta^3 f_4\kappa_1^2}.$$

Case 2: $f_0=f_1=f_3=f_5=f_6=0$

$$d_{-1}=0,\quad d_0=\frac{2\gamma\kappa_1-\beta\kappa_2}{\beta^2\kappa_1},\quad d_1=\pm\frac{2\kappa_1\sqrt{f_4}}{\beta},$$

$$A=\frac{2\alpha\beta^2\kappa_1+9\beta^2\kappa_2^2-12\beta\gamma\kappa_2\kappa_1+12\gamma^2\kappa_1^2+2\beta^2 f_2\kappa_1^4}{6\beta^3\kappa_1^2}. \tag{22}$$

In this case, we obtain the following solutions of eq. (6) by putting eq. (22) in eq. (13) together with the solutions of eq. (8):

$$h_1(\phi)=\frac{2\gamma\kappa_1-\beta\kappa_2}{\beta^2\kappa_1}\pm\frac{4f_2\kappa_1\sqrt{f_4}\exp(\sqrt{f_2}\phi)}{\beta(1\pm f_2 f_4\exp(2\sqrt{f_2}\phi))},$$

$$g_1(\phi)=\frac{\kappa_2 h_1}{\kappa_1}+A,\quad f_2>0,\quad f_4<0; \tag{23}$$

$$h_2(\phi)=\frac{2\gamma\kappa_1-\beta\kappa_2}{\beta^2\kappa_1}\pm\frac{2\kappa_1\sqrt{-f_2}}{\beta}\operatorname{csch}(\sqrt{-f_2}\phi),$$

$$g_2(\phi)=\frac{\kappa_2 h_2}{\kappa_1}+A,\quad f_2<0,\quad f_4>0; \tag{24}$$

$$h_3(\phi)=\frac{2\gamma\kappa_1-\beta\kappa_2}{\beta^2\kappa_1}\mp\frac{2\kappa_1\sqrt{-f_2}}{\beta}\csc(\sqrt{-f_2}\phi),$$

$$g_3(\phi)=\frac{\kappa_2 h_3}{\kappa_1}+A,\quad f_2<0,\quad f_4>0; \tag{25}$$

$$h_4(\phi)=\frac{2\gamma\kappa_1-\beta\kappa_2}{\beta^2\kappa_1}\mp\frac{4f_2\kappa_1\sqrt{f_4}\exp(\sqrt{f_2}\phi)}{\beta(f_2 f_4\exp(2\sqrt{f_2}\phi)-1)},$$

$$g_4(\phi)=\frac{\kappa_2 h_4}{\kappa_1}+A,\quad f_2>0,\quad f_4>0, \tag{26}$$

where

$$\phi=\kappa_1\zeta+\kappa_2\iota-\alpha\xi$$

and

$$A=\frac{2\alpha\beta^2\kappa_1+9\beta^2\kappa_2^2-12\beta\gamma\kappa_2\kappa_1+12\gamma^2\kappa_1^2+2\beta^2 f_2\kappa_1^4}{6\beta^3\kappa_1^2}.$$

Case 3 : $f_3=f_4=f_5=f_6=0$

(i) $$d_{-1}=\pm\frac{2\kappa_1\sqrt{f_0}}{\beta},\quad d_0=\frac{4\gamma\kappa_1\sqrt{f_0}-\beta f_1\kappa_1^2-2\beta\kappa_2\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}},\quad d_1=0,$$

$$A=\frac{8\alpha\beta^2 f_0\kappa_1-3\beta^2 f_1^2\kappa_1^4+8\beta^2 f_0 f_2\kappa_1^4+36\beta^2 f_0\kappa_2^2-48\beta\gamma f_0\kappa_2\kappa_1+48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0\kappa_1^2}. \tag{27}$$

(ii) $$d_{-1}=\pm\frac{2\kappa_1\sqrt{f_0}}{\beta},\quad d_0=\frac{4\gamma\kappa_1\sqrt{f_0}+\beta f_1\kappa_1^2-2\beta\kappa_2\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}},\quad d_1=0,$$

$$A=\frac{8\alpha\beta^2 f_0\kappa_1-3\beta^2 f_1^2\kappa_1^4+8\beta^2 f_0 f_2\kappa_1^4+36\beta^2 f_0\kappa_2^2-48\beta\gamma f_0\kappa_2\kappa_1+48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0\kappa_1^2}. \tag{28}$$

In this case, by substituting eq. (27) into eq. (13), eq. (28) into eq. (13) together with the solutions of eq. (8), we respectively obtain the following solitary solutions of eq. (6):

$$h_1(\phi)=\frac{4\gamma\kappa_1\sqrt{f_0}-\beta f_1\kappa_1^2-2\beta\kappa_2\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}}\pm\frac{8f_2\kappa_1\sqrt{f_0}}{\beta(\exp(\sqrt{f_2}\phi)-2f_1+(f_1^2-4f_0f_2)\exp(-\sqrt{f_2}\phi))},$$

$$g_1(\phi)=\frac{\kappa_2 h_1}{\kappa_1}+A,\quad f_0\neq 0,\quad f_1\neq 0,\quad f_2>0; \tag{29}$$

$$h_2(\phi)=\frac{4\gamma\kappa_1\sqrt{f_0}+\beta f_1\kappa_1^2-2\beta\kappa_2\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}}\pm\frac{8f_2\kappa_1\sqrt{f_0}}{\beta(\exp(\sqrt{f_2}\phi)-2f_1+(f_1^2-4f_0f_2)\exp(-\sqrt{f_2}\phi))},$$

$$g_2(\phi)=\frac{\kappa_2 h_2}{\kappa_1}+A,\quad f_0\neq 0,\quad f_1\neq 0,\quad f_2>0, \tag{30}$$

where

$\phi = \kappa_1\zeta + \kappa_2\iota - \alpha\xi$

and

$$A = \frac{8\alpha\beta^2 f_0\kappa_1 - 3\beta^2 f_1^2\kappa_1^4 + 8\beta^2 f_0 f_2\kappa_1^4 + 36\beta^2 f_0\kappa_2^2 - 48\beta\gamma f_0\kappa_2\kappa_1 + 48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0, \kappa_1^2}.$$

Case 4: $f_1 = f_3 = f_4 = f_5 = f_6 = 0$

$$d_{-1} = \pm\frac{2\kappa_1\sqrt{f_0}}{\beta}, \quad d_0 = \frac{2\gamma\kappa_1 - \beta\kappa_2}{\beta^2\kappa_1}, \quad d_1 = 0,$$

$$A = \frac{2\alpha\beta^2\kappa_1 + 9\beta^2\kappa_2^2 - 12\beta\gamma\kappa_2\kappa_1 + 12\gamma^2\kappa_1^2 + 2\beta^2 f_2\kappa_1^4}{6\beta^3\kappa_1^2}. \tag{31}$$

Now by putting eq. (31) into eq. (13) together with the solutions of eq. (8), the following solutions of eq. (6) are obtained:

$$h_1(\phi) = \frac{2\gamma\kappa_1 - \beta\kappa_2}{\beta^2\kappa_1} \pm \frac{4 f_2\kappa_1\sqrt{f_0}}{\beta(\exp(\sqrt{f_2}\phi) - f_0 f_2 \exp(-\sqrt{f_2}\phi))},$$

$$g_1(\phi) = \frac{\kappa_2 h_1}{\kappa_1} + A, \quad f_0 \neq 0, \quad f_2 > 0; \tag{32}$$

$$h_2(\phi) = \frac{2\gamma\kappa_1 - \beta\kappa_2}{\beta^2\kappa_1} \pm \frac{2\kappa_1\sqrt{-f_2}}{\beta \sin(\sqrt{-f_2}\phi)},$$

$$g_2(\phi) = \frac{\kappa_2 h_2}{\kappa_1} + A, \quad f_0 > 0, \quad f_2 < 0; \tag{33}$$

$$h_3(\phi) = \frac{2\gamma\kappa_1 - \beta\kappa_2}{\beta^2\kappa_1} \mp \frac{2\kappa_1\sqrt{f_0 f_2}}{\beta\sqrt{-f_0}\sin(\sqrt{-f_2}\phi)},$$

$$g_3(\phi) = \frac{\kappa_2 h_3}{\kappa_1} + A, \quad f_0 < 0, \quad f_2 < 0, \tag{34}$$

where

$\phi = \kappa_1\zeta + \kappa_2\iota - \alpha\xi$

and

$$A = \frac{2\alpha\beta^2\kappa_1 + 9\beta^2\kappa_2^2 - 12\beta\gamma\kappa_2\kappa_1 + 12\gamma^2\kappa_1^2 + 2\beta^2 f_2\kappa_1^4}{6\beta^3\kappa_1^2}.$$

Case 5 : $f_1 = f_3 = f_5 = f_6 = 0$

(i) $d_{-1} = \pm\frac{2\kappa_1\sqrt{f_0}}{\beta}, \quad d_0 = \frac{2\gamma\kappa_1 - \beta\kappa_2}{\beta^2\kappa_1}, \quad d_1 = 0,$

$$A = \frac{2\alpha\beta^2\kappa_1 + 9\beta^2\kappa_2^2 - 12\beta\gamma\kappa_2\kappa_1 + 12\gamma^2\kappa_1^2 + 2\beta^2 f_2\kappa_1^4}{6\beta^3\kappa_1^2}. \tag{35}$$

(ii) $d_{-1} = \pm\frac{2\kappa_1\sqrt{f_0}}{\beta}, \quad d_0 = \frac{2\gamma\kappa_1 - \beta\kappa_2}{\beta^2\kappa_1}, \quad d_1 = \pm\frac{2\kappa_1\sqrt{f_4}}{\beta},$

$$A = \frac{2\alpha\beta^2\kappa_1 + 9\beta^2\kappa_2^2 - 12\beta\gamma\kappa_2\kappa_1 + 12\gamma^2\kappa_1^2 + 2\beta^2 f_2\kappa_1^4 - 12\beta^2\kappa_1^4\sqrt{f_0 f_4}}{6\beta^3\kappa_1^2}. \tag{36}$$

(iii) $d_{-1} = \pm\frac{2\kappa_1\sqrt{f_0}}{\beta}, \quad d_0 = \frac{2\gamma\kappa_1 - \beta\kappa_2}{\beta^2\kappa_1}, \quad d_1 = \pm\frac{2\kappa_1\sqrt{f_4}}{\beta},$

$$A = \frac{2\alpha\beta^2\kappa_1 + 9\beta^2\kappa_2^2 - 12\beta\gamma\kappa_2\kappa_1 + 12\gamma^2\kappa_1^2 + 2\beta^2 f_2\kappa_1^4 + 12\beta^2\kappa_1^4\sqrt{f_0 f_4}}{6\beta^3\kappa_1^2}. \tag{37}$$

(iv) $d_{-1} = 0, \quad d_0 = \frac{2\gamma\kappa_1 - \beta\kappa_2}{\beta^2\kappa_1}, \quad d_1 = \pm\frac{2\kappa_1\sqrt{f_4}}{\beta}, \quad A = \frac{2\alpha\beta^2\kappa_1 + 9\beta^2\kappa_2^2 - 12\beta\gamma\kappa_2\kappa_1 + 12\gamma^2\kappa_1^2 + 2\beta^2 f_2\kappa_1^4}{6\beta^3\kappa_1^2}.$ (38)

For Case 5, eq. (6) has the following solutions in terms of the Jacobi elliptic function [50] with different solutions of eq. (8):

$$h_1(\phi) = d_{-1}dc\phi + d_0 + d_1cd\phi,$$
$$g_1(\phi) = \frac{\kappa_2 h_1}{\kappa_1} + A, \quad f_0 = 1,$$
$$f_2 = -(m^2+1), \quad f_4 = m^2; \tag{39}$$
$$h_2(\phi) = d_{-1}ns\phi + d_0 + d_1sn\phi,$$
$$g_2(\phi) = \frac{\kappa_2 h_2}{\kappa_1} + A, \quad f_0 = 1,$$
$$f_2 = -(m^2+1), \quad f_4 = m^2; \tag{40}$$
$$h_3(\phi) = d_{-1}sn\phi + d_0 + d_1ns\phi,$$
$$g_3(\phi) = \frac{\kappa_2 h_3}{\kappa_1} + A, \quad f_0 = m^2,$$
$$f_2 = -m^2-1, \quad f_4 = 1; \tag{41}$$
$$h_4(\phi) = d_{-1}cd\phi + d_0 + d_1dc\phi,$$
$$g_4(\phi) = \frac{\kappa_2 h_4}{\kappa_1} + A, \quad f_0 = m^2,$$
$$f_2 = -m^2-1, \quad f_4 = 1; \tag{42}$$
$$h_5(\phi) = d_{-1}nd\phi + d_0 + d_1dn\phi,$$
$$g_5(\phi) = \frac{\kappa_2 h_5}{\kappa_1} + A, \quad f_0 = m^2-1$$
$$f_2 = -m^2, \quad f_4 = -1; \tag{43}$$
$$h_6(\phi) = d_{-1}nc\phi + d_0 + d_1cn\phi,$$
$$g_6(\phi) = \frac{\kappa_2 h_6}{\kappa_1} + A, \quad f_0 = 1-m^2,$$
$$f_2 = 2m^2-1, \quad f_4 = -m^2; \tag{44}$$
$$h_7(\phi) = d_{-1}cn\phi + d_0 + d_1nc\phi,$$
$$g_7(\phi) = \frac{\kappa_2 h_7}{\kappa_1} + A, \quad f_0 = -m^2,$$
$$f_2 = -1+2m^2, \quad f_4 = 1-m^2; \tag{45}$$
$$h_8(\phi) = d_{-1}dn\phi + d_0 + d_1nd\phi,$$
$$g_8(\phi) = \frac{\kappa_2 h_8}{\kappa_1} + A, \quad f_0 = -1,$$
$$f_2 = 2-m^2, \quad f_4 = m^2-1; \tag{46}$$
$$h_9(\phi) = d_{-1}sc\phi + d_0 + d_1cs\phi,$$
$$g_9(\phi) = \frac{\kappa_2 h_9}{\kappa_1} + A, \quad f_0 = 1-m^2,$$
$$f_2 = 2-m^2, \quad f_4 = 1; \tag{47}$$
$$h_{10}(\phi) = d_{-1}cs\phi + d_0 + d_1sc\phi,$$
$$g_{10}(\phi) = \frac{\kappa_2 h_{10}}{\kappa_1} + A, \quad f_0 = 1,$$
$$f_2 = 2-m^2, \quad f_4 = 1-m^2; \tag{48}$$
$$h_{11}(\phi) = d_{-1}ds\phi + d_0 + d_1sd\phi,$$
$$g_{11}(\phi) = \frac{\kappa_2 h_{11}}{\kappa_1} + A, \quad f_0 = 1,$$
$$f_2 = 2m^2-1, \quad f_4 = m^2(-1+m^2); \tag{49}$$
$$h_{12}(\phi) = d_{-1}sd\phi + d_0 + d_1ds\phi,$$
$$g_{12}(\phi) = \frac{\kappa_2 h_{12}}{\kappa_1} + A, \quad f_0 = m^2(-1+m^2),$$
$$f_2 = 2m^2-1, \quad f_4 = 1; \tag{50}$$
$$h_{13}(\phi) = \frac{d_{-1}}{(ns(\phi) \pm cs(\phi))} + d_0 + d_1(ns(\phi) \pm cs(\phi)),$$
$$g_{13}(\phi) = \frac{\kappa_2 h_{13}}{\kappa_1} + A,$$
$$f_0 = \frac{1}{4}, \quad f_2 = \frac{(1-m^2)}{2}, \quad f_4 = \frac{1}{4}; \tag{51}$$
$$h_{14}(\phi) = \frac{d_{-1}}{(nc(\phi) \pm sc(\phi))} + d_0 + d_1(nc(\phi) \pm sc(\phi)),$$
$$g_{14}(\phi) = \frac{\kappa_2 h_{14}}{\kappa_1} + A,$$
$$f_0 = \frac{(1-m^2)}{4}, \quad f_2 = \frac{(1+m^2)}{2}, \quad f_4 = \frac{1}{4}; \tag{52}$$
$$h_{15}(\phi) = \frac{d_{-1}}{(ns(\phi) \pm ds(\phi))} + d_0 + d_1(ns(\phi) \pm ds(\phi)),$$
$$g_{15}(\phi) = \frac{\kappa_2 h_{15}}{\kappa_1} + A,$$
$$f_0 = \frac{m^2}{4}, \quad f_2 = \frac{(m^2-2)}{2}, \quad f_4 = \frac{1}{4}; \tag{53}$$
$$h_{16}(\phi) = \frac{d_{-1}}{(sn(\phi) \pm icn(\phi))} + d_0 + d_1(sn(\phi) \pm icn(\phi)),$$
$$g_{16}(\phi) = \frac{\kappa_2 h_{16}}{\kappa_1} + A, \quad f_0 = \frac{m^2}{4},$$
$$f_2 = \frac{(m^2-2)}{2}, \quad f_4 = \frac{m^2}{4}; \tag{54}$$
$$h_{17}(\phi) = d_{-1}\left(i\sqrt{1-m^2}sn(\phi) \pm cn(\phi)\right)nd(\phi) + d_0 + d_1\frac{dn(\phi)}{(i\sqrt{1-m^2}sn(\phi) \pm cn(\phi))},$$

$$g_{17}(\phi) = \frac{\kappa_2 h_{17}}{\kappa_1} + A, \quad f_0 = \frac{m^2}{4},$$
$$f_2 = \frac{(m^2-2)}{2}, \quad f_4 = \frac{m^2}{4}; \tag{55}$$

$$h_{18}(\phi) = d_{-1} cn(\phi) ns(\phi) nd(\phi) + d_0 + d_1 sn(\phi) dn(\phi) nc(\phi),$$
$$g_{18}(\phi) = \frac{\kappa_2 h_{18}}{\kappa_1} + A,$$
$$f_0 = 1, \quad f_2 = 2 - 4m^2, \quad f_4 = 1; \tag{56}$$

$$h_{19}(\phi) = d_{-1} B_1 (1 + sn(\phi))(1 + m sn(\phi)) nd(\phi) nc(\phi) + d_0 + d_1 \frac{dn(\phi) cn(\phi)}{B_1 (1 + sn(\phi))(1 + m sn(\phi))},$$
$$g_{19}(\phi) = \frac{\kappa_2 h_{19}}{\kappa_1} + A,$$
$$f_0 = \frac{(m-1)^2}{4B_1^2}, \quad f_2 = \frac{(1 + m^2 + 6m)}{2},$$
$$f_4 = \frac{B_1^2(-1+m)^2}{4}; \tag{57}$$

$$h_{20}(\phi) = d_{-1} B_1 (1 + sn(\phi))(1 - m sn(\phi)) nd(\phi) nc(\phi) + d_0 + d_1 \frac{dn(\phi) cn(\phi)}{B_1 (1 + sn(\phi))(1 - m sn(\phi))},$$
$$g_{20}(\phi) = \frac{\kappa_2 h_{20}}{\kappa_1} + A, \quad f_0 = \frac{(m+1)^2}{4B_1^2},$$
$$f_2 = \frac{(1 + m^2 + 6m)}{2},$$
$$f_4 = \frac{B_1^2(1+m)^2}{4}; \tag{58}$$

$$h_{21}(\phi) = d_{-1} \frac{(1 + m sn^2(\phi)) nd(\phi) nc(\phi)}{m} + d_0 + d_1 \frac{m dn(\phi) cn(\phi)}{(1 + m sn^2(\phi))},$$
$$g_{21}(\phi) = \frac{\kappa_2 h_{21}}{\kappa_1} + A,$$
$$f_0 = -2m^3 + m^4 + m^2, \quad f_2 = \frac{-4}{m},$$
$$f_4 = 6m - m^2 - 1; \tag{59}$$

$$h_{22}(\phi) = d_{-1} \frac{(m sn^2(\phi) - 1) nd(\phi) nc(\phi)}{m} + d_0 + d_1 \frac{m dn(\phi) cn(\phi)}{\left(m sn^2(\phi) - 1\right)},$$
$$g_{22}(\phi) = \frac{\kappa_2 h_{22}}{\kappa_1} + A,$$
$$f_0 = 2m^3 + m^4 + m^2, \quad f_2 = -6m - m^2 - 1,$$
$$f_4 = \frac{-4}{m}; \tag{60}$$

$$h_{23}(\phi) = d_{-1} \frac{(\sqrt{1-m^2} - dn^2(\phi)) ns(\phi) nc(\phi)}{m^2} + d_0 + d_1 \frac{m^2 sn(\phi) cn(\phi)}{(\sqrt{1-m^2} - dn^2(\phi))},$$
$$g_{23}(\phi) = \frac{\kappa_2 h_{23}}{\kappa_1} + A, \quad f_0 = 2 + 2\sqrt{1-m^2} - m^2,$$
$$f_2 = 6\sqrt{1-m^2} - m^2 + 2,$$
$$f_4 = 4\sqrt{1-m^2}; \tag{61}$$

$$h_{24}(\phi) = -\left(d_{-1} \frac{(\sqrt{1-m^2} + dn^2(\phi)) ns(\phi) nc(\phi)}{m^2}; + d_0 + d_1 \frac{m^2 sn(\phi) cn(\phi)}{(\sqrt{1-m^2} + dn^2(\phi))}\right),$$
$$g_{24}(\phi) = \frac{\kappa_2 h_{24}}{\kappa_1} + A, \quad f_0 = 2 - 2\sqrt{1-m^2} - m^2,$$
$$f_2 = 6\sqrt{1-m^2} - m^2 + 2, \quad f_4 = -4\sqrt{1-m^2}; \tag{62}$$

$$h_{25}(\phi) = d_{-1} \frac{(B_2 cn(\phi) + B_3 dn(\phi))\, ns(\phi)}{\sqrt{(B_2^2 - B_3^2)/(B_2^2 - B_3^2 m^2)}} + d_0 + d_1 \frac{\sqrt{[(B_2^2 - B_3^2)/(B_2^2 - B_3^2 m^2)]} sn(\phi)}{B_2 cn(\phi) + B_3 dn(\phi)},$$
$$g_{25}(\phi) = \frac{\kappa_2 h_{25}}{\kappa_1} + A,$$
$$f_0 = \frac{m^2 - 1}{4(B_3^2 m^2 - B_2^2)}, \quad f_2 = \frac{m^2+1}{2},$$
$$f_4 = \frac{(B_3^2 m^2 - B_2^2)(m^2 - 1)}{4}; \tag{63}$$

$$h_{26}(\phi) = d_{-1} \frac{(B_2 sn(\phi) + B_3 cn(\phi)) nd(\phi)}{\sqrt{(B_2^2 + B_3^2 - B_3^2 m^2)/(B_2^2 + B_3^2)}} + d_0 + d_1 \frac{\sqrt{[(B_2^2 + B_3^2 - B_3^2 m^2)/(B_2^2 + B_3^2)]} dn(\phi)}{B_2 sn(\phi) + B_3 cn(\phi)},$$
$$g_{26}(\phi) = \frac{\kappa_2 h_{26}}{\kappa_1} + A, \quad f_0 = \frac{m^2}{4(B_3^2 + B_2^2)},$$
$$f_2 = \frac{m^2 - 2}{2}, \quad f_4 = \frac{(B_3^2 + B_2^2)}{4}; \tag{64}$$

$$h_{27}(\phi) = d_{-1}\frac{B_2(msn^2(\phi)+1)}{\left(msn^2(\phi)-1\right)} + d_0 + d_1\frac{(msn^2(\phi)-1)}{B_2\left(msn^2(\phi)+1\right)},$$
$$g_{27}(\phi) = \frac{\kappa_2 h_{27}}{\kappa_1} + A,$$
$$f_0 = \frac{2m - m^2 - 1}{B_2^2}, \quad f_2 = 2m^2 + 2,$$
$$f_4 = -B_2^2 m^2 - B_2^2 - 2B_2^2 m; \tag{65}$$

$$h_{28}(\phi) = d_{-1}\frac{B_2(msn^2(\phi)-1)}{\left(msn^2(\phi)+1\right)} + d_0 + d_1\frac{(msn^2(\phi)+1)}{B_2\left(msn^2(\phi)-1\right)},$$
$$g_{28}(\phi) = \frac{\kappa_2 h_{28}}{\kappa_1} + A,$$
$$f_0 = -\frac{2m + m^2 + 1}{B_2^2}, \quad f_2 = 2m^2 + 2,$$
$$f_4 = -B_2^2 m^2 + B_2^2 + 2B_2^2 m; \tag{66}$$

$$h_{29}(\phi) = d_{-1}(mcn(\phi) \pm i\sqrt{1-m^2})nd(\phi) + d_0 + d_1\frac{dn(\phi)}{(mcn(\phi) \pm i\sqrt{1-m^2})},$$
$$g_{29}(\phi) = \frac{\kappa_2 h_{29}}{\kappa_1} + A,$$
$$f_0 = f_4 = \frac{1}{4}, \quad f_2 = \frac{1-2m^2}{2}; \tag{67}$$

$$h_{30}(\phi) = d_{-1}\frac{1}{msn(\phi) \pm idn(\phi)} + d_0 + d_1\left(msn(\phi) \pm idn(\phi)\right),$$
$$g_{30}(\phi) = \frac{\kappa_2 h_{30}}{\kappa_1} + A,$$
$$f_0 = f_4 = \frac{1}{4}, \quad f_2 = \frac{1-2m^2}{2}; \tag{68}$$

$$h_{31}(\phi) = d_{-1}\frac{1}{mns(\phi) \pm cs(\phi)} + d_0 + d_1(mns(\phi) \pm cs(\phi)),$$
$$g_{31}(\phi) = \frac{\kappa_2 h_{31}}{\kappa_1} + A,$$
$$f_0 = f_4 = \frac{1}{4}, \quad f_2 = \frac{1-2m^2}{2}; \tag{69}$$

$$h_{32}(\phi) = d_{-1}(1 \pm cn(\phi))\,ns(\phi) + d_0 + d_1\frac{sn(\phi)}{(1 \pm cn(\phi))},$$
$$g_{32}(\phi) = \frac{\kappa_2 h_{32}}{\kappa_1} + A,$$
$$f_0 = f_4 = \frac{1}{4}, \quad f_2 = \frac{1-2m^2}{2}; \tag{70}$$

$$h_{33}(\phi) = d_{-1}(1 \pm msn(\phi))\,nd(\phi) + d_0 + d_1\frac{dn(\phi)}{(1 \pm msn(\phi))},$$
$$g_{33}(\phi) = \frac{\kappa_2 h_{33}}{\kappa_1} + A,$$
$$f_0 = f_4 = \frac{m^2-1}{4}, \quad f_2 = \frac{1+m^2}{2}; \tag{71}$$

$$h_{34}(\phi) = d_{-1}\frac{1}{msd(\phi) \pm nd(\phi)} + d_0 + d_1(msd(\phi) \pm nd(\phi)),$$
$$g_{34}(\phi) = \frac{\kappa_2 h_{34}}{\kappa_1} + A,$$
$$f_0 = f_4 = \frac{m^2-1}{4}, \quad f_2 = \frac{1+m^2}{2}; \tag{72}$$

$$h_{35}(\phi) = d_{-1}(1 \pm sn(\phi))\,nc(\phi) + d_0 + d_1\frac{cn(\phi)}{(1 \pm sn(\phi))},$$
$$g_{35}(\phi) = \frac{\kappa_2 h_{35}}{\kappa_1} + A,$$
$$f_0 = f_4 = \frac{1-m^2}{4}, \quad f_2 = \frac{1+m^2}{2}; \tag{73}$$

$$h_{36}(\phi) = d_{-1}\frac{1}{(nc(\phi) \pm sc(\phi))} + d_0 + d_1(nc(\phi) \pm sc(\phi)),$$
$$g_{36}(\phi) = \frac{\kappa_2 h_{36}}{\kappa_1} + A,$$
$$f_0 = f_4 = \frac{1-m^2}{4}, \quad f_2 = \frac{1+m^2}{2}; \tag{74}$$

$$h_{37}(\phi) = d_{-1}\frac{1}{(mcn(\phi) \pm dn(\phi))} + d_0 + d_1(mcn(\phi) \pm dn(\phi)),$$
$$g_{37}(\phi) = \frac{\kappa_2 h_{37}}{\kappa_1} + A,$$
$$f_0 = -\frac{(1-m^2)^2}{4}, \quad f_2 = \frac{1+m^2}{2}, \quad f_4 = \frac{1}{4}; \tag{75}$$

$$h_{38}(\phi) = d_{-1}(dn(\phi) \pm cn(\phi))\, ns(\phi) + d_0 + d_1 \frac{sn(\phi)}{(dn(\phi) \pm cn(\phi))},$$

$$g_{38}(\phi) = \frac{\kappa_2 h_{38}}{\kappa_1} + A, \quad f_0 = \frac{1}{4},$$

$$f_2 = \frac{1+m^2}{2}, \quad f_4 = \frac{(1-m^2)^2}{4}; \tag{76}$$

$$h_{39}(\phi) = d_{-1}(\sqrt{1-m^2} \pm dn(\phi))nc(\phi) + d_0 + d_1 \frac{cn(\phi)}{(\sqrt{1-m^2} \pm dn(\phi))},$$

$$g_{39}(\phi) = \frac{\kappa_2 h_{39}}{\kappa_1} + A, \quad f_0 = \frac{1}{4},$$

$$f_2 = \frac{m^2-2}{2}, \quad f_4 = \frac{m^4}{4}; \tag{77}$$

$$h_{40}(\phi) = d_{-1}\,(1 \pm dn(\phi))\, ns(\phi) + d_0 + d_1 \frac{sn(\phi)}{(1 \pm dn(\phi))},$$

$$g_{40}(\phi) = \frac{\kappa_2 h_{40}}{\kappa_1} + A, \quad f_0 = \frac{1}{4},$$

$$f_2 = \frac{m^2-2}{2}, \quad f_4 = \frac{m^4}{4}, \tag{78}$$

where $\phi = \kappa_1\zeta + \kappa_2\iota - \alpha\xi$ fulfills eqs (35)–(78), m $(0 < m < 1)$ is a modulus, B_1, B_2, B_3 $(B_1B_2B_3 \neq 0)$ and B_4 are arbitrary constants. The Jacobi elliptic functions are doubly periodic and have the following characteristics of triangular functions:

$$sn^2(\phi) + cn^2(\phi) = 1, \quad dn^2(\phi) = 1 - m^2 sn^2(\phi),$$

$$sn'(\phi) = cn(\phi)dn(\phi), \quad cn'(\phi) = -sn(\phi)dn(\phi),$$

$$dn'(\phi) = -m^2 sn(\phi)cn(\phi).$$

When $m \to 1$, the Jacobi functions reprobate to the hyperbolic functions. Thus,

$$sn(\phi) \to \tanh(\phi), \quad cn(\phi) \to \text{sech}(\phi).$$

When $m \to 0$, the Jacobi functions reprobate to the trigonometric functions. Thus,

$$sn(\phi) \to \sin(\phi), \quad \cos(\phi) \to \text{sech}(\phi).$$

Case 6 : $f_5 = f_6 = 0$

(i) $d_{-1} = -\dfrac{2\kappa_1\sqrt{f_0}}{\beta}, \quad d_0 = \dfrac{-\beta f_1\kappa_1^2 - 2\beta\kappa_2\sqrt{f_0} + 4\gamma\kappa_1\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}}, \quad d_1 = 0,$

$$A = \frac{8\alpha\beta^2 f_0\kappa_1 - 3\beta^2 f_1^2\kappa_1^4 + 8\beta^2 f_0 f_2\kappa_1^4 + 36\beta^2 f_0\kappa_2^2 - 48\beta\gamma f_0\kappa_2\kappa_1 + 48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0\kappa_1^2}. \tag{79}$$

(ii) $d_{-1} = \dfrac{2\kappa_1\sqrt{f_0}}{\beta}, \quad d_0 = \dfrac{\beta f_1\kappa_1^2 - 2\beta\kappa_2\sqrt{f_0} + 4\gamma\kappa_1\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}}, \quad d_1 = 0,$

$$A = \frac{8\alpha\beta^2 f_0\kappa_1 - 3\beta^2 f_1^2\kappa_1^4 + 8\beta^2 f_0 f_2\kappa_1^4 + 36\beta^2 f_0\kappa_2^2 - 48\beta\gamma f_0\kappa_2\kappa_1 + 48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0\kappa_1^2}. \tag{80}$$

(iii) $d_{-1} = 0, \quad d_0 = \dfrac{-\beta f_1\kappa_1^2 - 2\beta\kappa_2\sqrt{f_0} + 4\gamma\kappa_1\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}}, \quad d_1 = -\dfrac{2\kappa_1\sqrt{f_0}}{\beta},$

$$A = \frac{8\alpha\beta^2 f_0\kappa_1 - 3\beta^2 f_1^2\kappa_1^4 + 8\beta^2 f_0 f_2\kappa_1^4 + 36\beta^2 f_0\kappa_2^2 - 48\beta\gamma f_0\kappa_2\kappa_1 + 48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0\kappa_1^2}. \tag{81}$$

(iv) $d_{-1} = 0, \quad d_0 = \dfrac{\beta f_1\kappa_1^2 - 2\beta\kappa_2\sqrt{f_0} + 4\gamma\kappa_1\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}}, \quad d_1 = \dfrac{2\kappa_1\sqrt{f_0}}{\beta},$

$$A = \frac{8\alpha\beta^2 f_0\kappa_1 - 3\beta^2 f_1^2\kappa_1^4 + 8\beta^2 f_0 f_2\kappa_1^4 + 36\beta^2 f_0\kappa_2^2 - 48\beta\gamma f_0\kappa_2\kappa_1 + 48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0\kappa_1^2}. \tag{82}$$

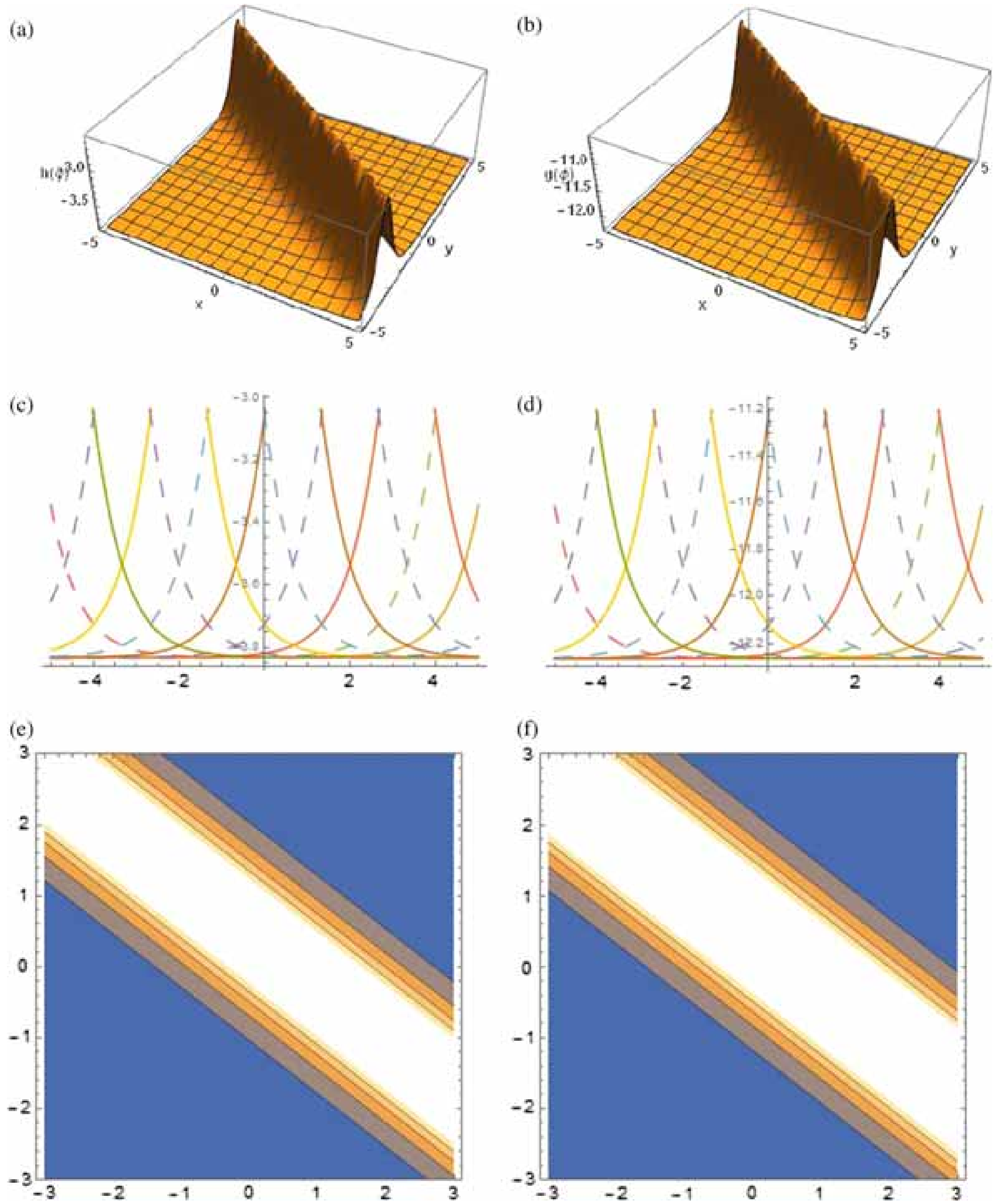

Figure 1. Solutions of eq. (16) with different plotted shapes at $\beta = 1, \gamma = 1, \alpha = 1, \kappa_1 = 1.5, \kappa_2 = 2, f_2 = 1, f_3 = 6, f_4 = 1$: (**a**), (**c**) solitary wave and (**e**) contour plot of h, (**b**), (**d**) solitary wave and (**f**) contour plot of g.

Case 7 : $f_2 = f_4 = f_5 = f_6 = 0$

(i) $d_{-1} = -\dfrac{2\kappa_1\sqrt{f_0}}{\beta}, \quad d_0 = \dfrac{-\beta f_1\kappa_1^2 - 2\beta\kappa_2\sqrt{f_0} + 4\gamma\kappa_1\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}}, \qquad d_1 = 0,$

$$A = \frac{8\alpha\beta^2 f_0\kappa_1 - 3\beta^2 f_1^2\kappa_1^4 + 36\beta^2 f_0\kappa_2^2 - 48\beta\gamma f_0\kappa_2\kappa_1 + 48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0\kappa_1^2}. \tag{83}$$

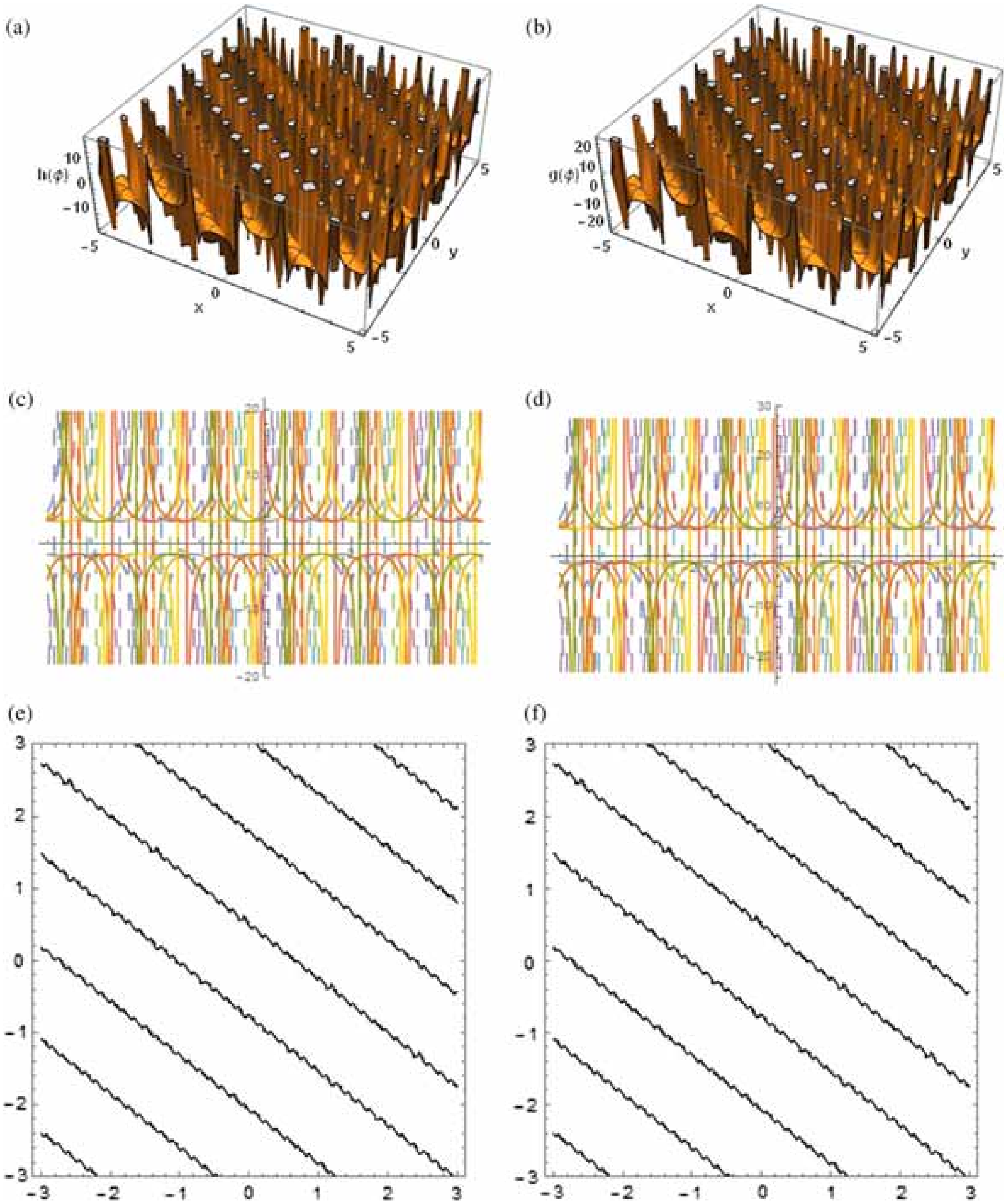

Figure 2. The solutions of eq. (25) with different plotted shapes at $\beta = 1.5$, $\gamma = 2$, $\alpha = 1$, $\kappa_1 = 1.5$, $\kappa_2 = 2$, $f_2 = -1.5$, $f_4 = 2$: (**a**), (**c**) periodic solitary wave and (**e**) contour plot of h, (**b**), (**d**) periodic solitary wave and (**f**) contour plot of g.

$$\textbf{(ii)} \quad d_{-1} = \frac{2\kappa_1\sqrt{f_0}}{\beta}, \quad d_0 = \frac{\beta f_1 \kappa_1^2 - 2\beta\kappa_2\sqrt{f_0} + 4\gamma\kappa_1\sqrt{f_0}}{2\beta^2\kappa_1\sqrt{f_0}}, \qquad d_1 = 0,$$

$$A = \frac{8\alpha\beta^2 f_0\kappa_1 - 3\beta^2 f_1^2\kappa_1^4 + 36\beta^2 f_0\kappa_2^2 - 48\beta\gamma f_0\kappa_2\kappa_1 + 48\gamma^2 f_0\kappa_1^2}{24\beta^3 f_0\kappa_1^2}. \tag{84}$$

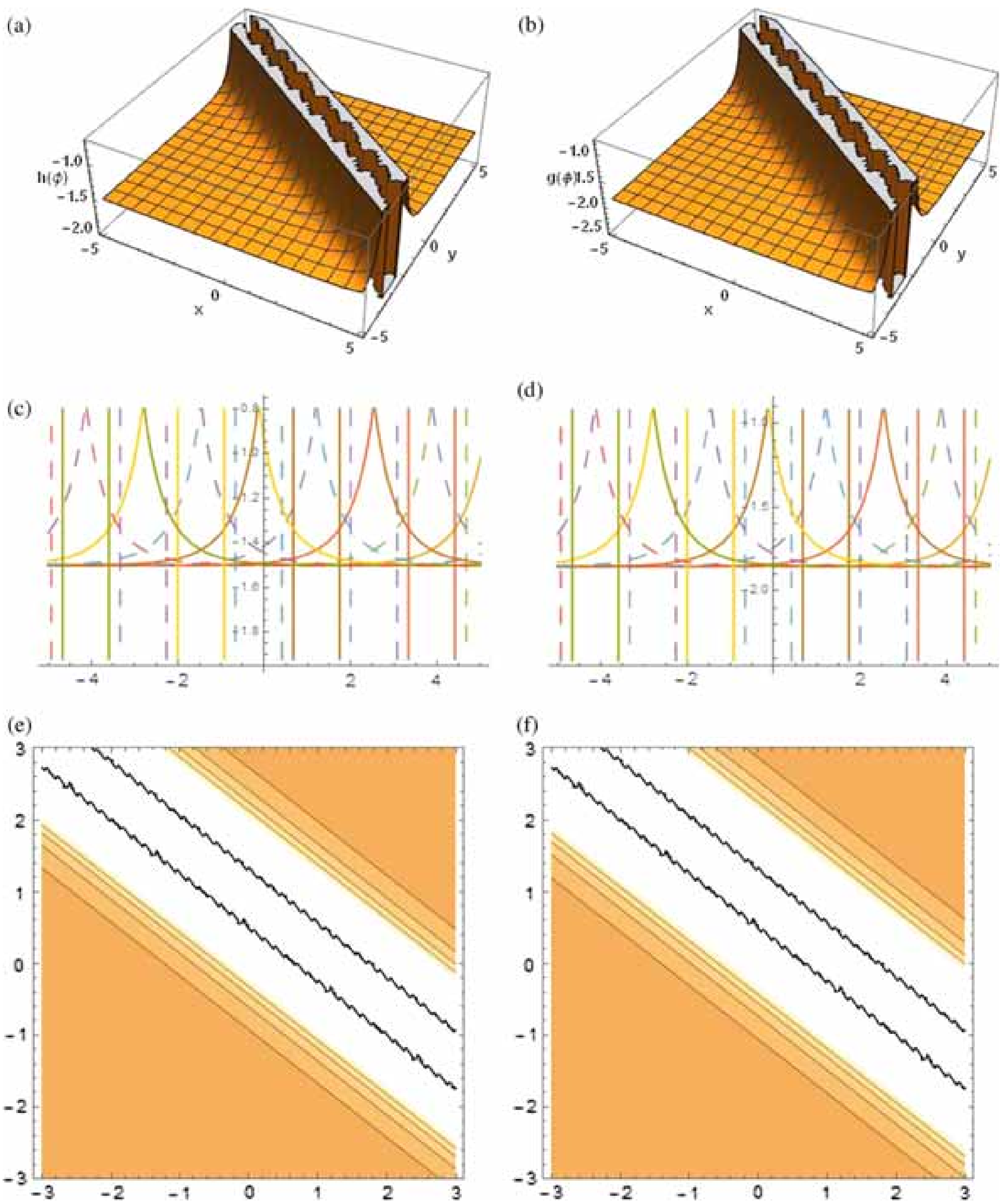

Figure 3. Solutions of eq. (29) with different plotted shapes at $\beta = 1.5, \gamma = 1, \alpha = 1, \kappa_1 = 1.5, \kappa_2 = 2, f_0 = 1, f_2 = 1, f_1 = 3$: (**a**), (**c**) periodic solitary wave and (**e**) contour plot of h, (**b**), (**d**) periodic solitary wave and (**f**) contour plot of g.

With [50], one can obtain the travelling wave solutions of eq. (6) from Cases 6 and 7 in the new elliptic and Weierstrass elliptic function solutions.

3. Analysis and discussion

The dynamic behaviour of the results, for example periodic, solitons, periodic solitary and singular combo soliton wave solutions, are found by their graphical natures. The soliton theory is very useful for explaining various phenomena in nonlinear evolution, such as the propagation of light in fibres, fluid dynamics and many other occurrences in plasma. The plots of the results were tracked between the intervals $-5 \leq \zeta, \iota \leq 5$ and the values of the arbitrary constants were randomly obtained. The process has been obtained by

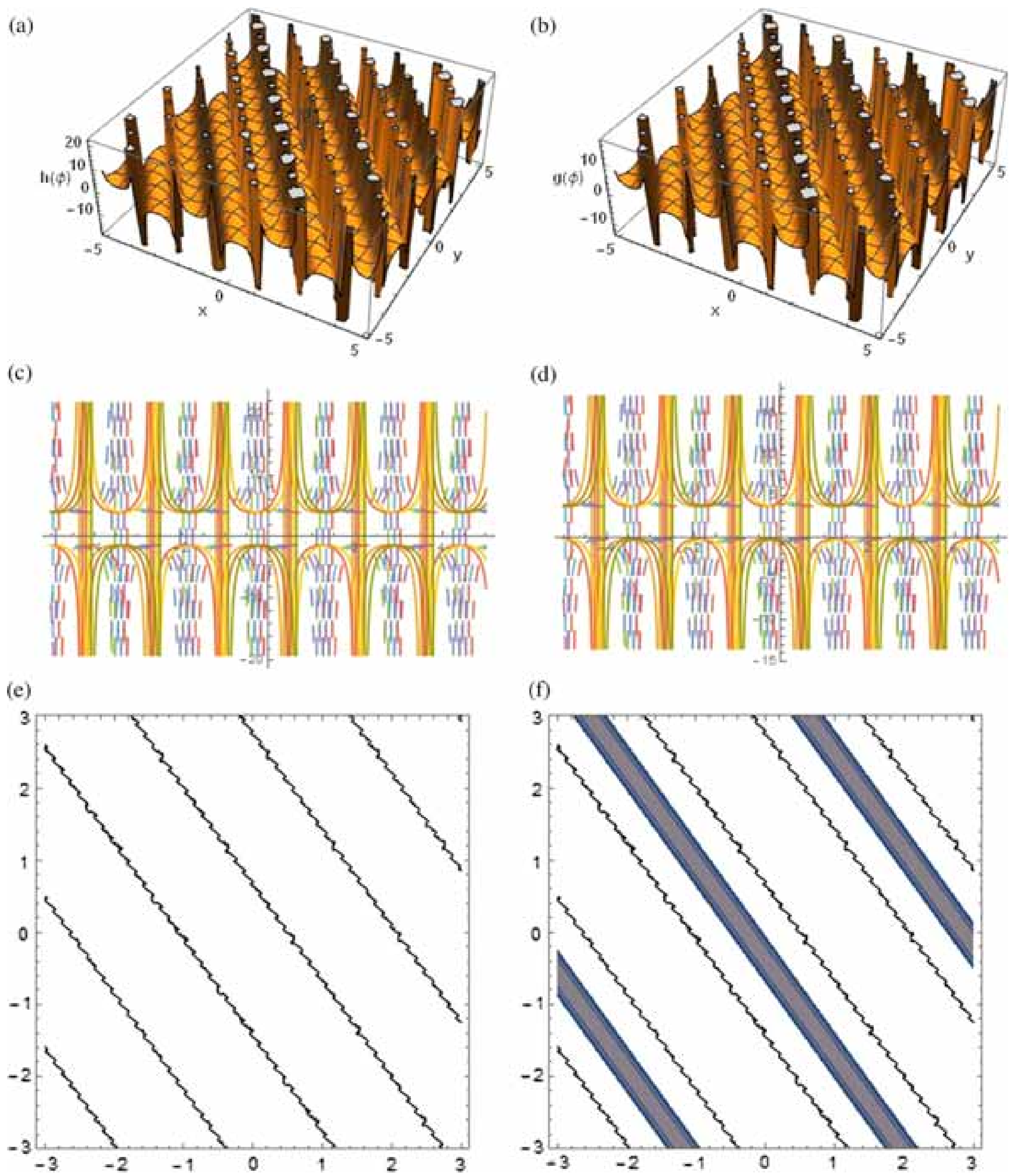

Figure 4. Solutions of eq. (40) (when $m \to 0$ of Case 5i) with different plotted shapes at $\beta = 1.5$, $\gamma = 2, \alpha = 1, \kappa_1 = 2, \kappa_2 = 1.5$, $f_0 = 1$, $f_2 = -(m^2 + 1)$, $f_4 = m^2$: (**a**), (**c**) periodic wave and (**e**) contour plot of h, (**b**), (**d**) periodic wave and (**f**) contour plot of g.

Mathematica software. The physical nature of the results is analysed as follows:

Figure 1 shows the solution of the KD equation denoted by eq. (16) indicating solitary wave at time $\xi = 1$, and at arbitrary constants $\beta = 1, \gamma = 1, \alpha = 1, \kappa_1 = 1.5, \kappa_2 = 2$, $f_2 = 1$, $f_3 = 6$ and $f_4 = 1$ from Mathematica simulations.

Figure 2 shows the evolution shapes of eq. (25) indicating periodic solitary wave at time $\xi = 1$, and at arbitrary constants $\beta = 1.5, \gamma = 2$, $\alpha = 1, \kappa_1 = 1.5, \kappa_2 = 2$, $f_2 = -1.5$ and $f_4 = 2$.

Figure 3 shows the evolution shapes of eq. (29) indicating the periodic solitary wave nature. We tracked the plots at time $\xi = 1$ and at values of the constants from simulation as: $\beta = 1.5, \gamma = 1, \alpha = 1, \kappa_1 = 1.5, \kappa_2 = 2$, $f_0 = 1$, $f_2 = 1$ and $f_1 = 3$.

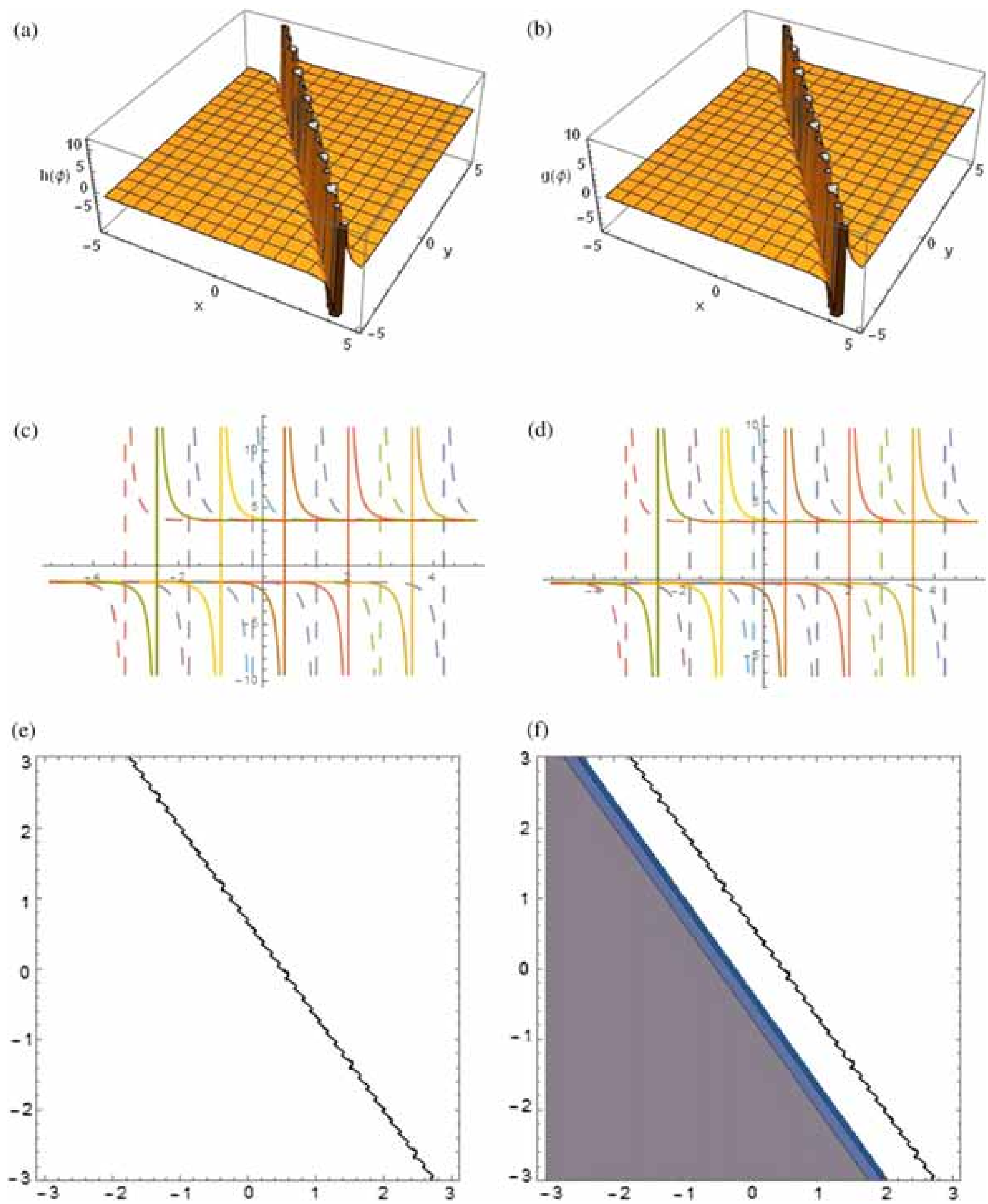

Figure 5. Solutions of eq. (40) (when $m \to 1$ of Case 5i) with different plotted shapes at $\beta = 1.5$, $\gamma = 2, \alpha = 1, \kappa_1 = 2, \kappa_2 = 1.5$, $f_0 = 1$, $f_2 = -(m^2 + 1)$, $f_4 = m^2$: (**a**), (**c**) dark singular combo solitons and (**e**) contour plot of h, (**b**), (**d**) dark singular combo solitons and (**f**) contour plot of g.

Figures 4 and 5 show the results denoted by eq. (40) indicating periodic wave (when $m \to 0$) and singular combo solitons (when $m \to 1$) nature respectively. We tracked the figures at time $\xi = 1$ and at values of the constants from simulation as: $\beta = 1.5, \gamma = 2, \alpha = 1, \kappa_1 = 2, \kappa_2 = 1.5$, $f_0 = 1$, $f_2 = -(m^2 + 1)$ and $f_4 = m^2$.

4. Conclusion

We effectively applied the modified extended direct algebraic method to find the exact travelling wave solutions of the KD equation. Using this method, we reduce the KD equation to the ODE, which is easy to solve. Using the modified extended direct algebraic method and the solutions of the auxiliary first-order nonlinear ODE, we attain novel diversity of the explicit and exact travelling wave solutions of KD equation, including solitary wave, periodic, new elliptic, Jacobi elliptic function and Weierstrass elliptic function solutions. The solutions we obtain in this work will help to better understand the occurrences described by the KD equation.

We obtain more and novel exact solutions with periodic and soliton natures by comparing our solutions in this work and that in [39,43–45]. We believe that this is the first work that shows most of the Jacobi elliptic function solutions to the KD equation. The method applied in this work permits us to perform complex and cumbersome algebraic calculations and can be extended to several other NEEs.

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