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Specific criteria for BCS-type cuprate superconductivity and peculiar isotope effects on the critical superconducting transition temperature

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Abstract. So far, many researchers have been misled to believe that the Bardeen–Cooper–Schrieffer (BCS)-like (*s*- or *d*-wave) pairing theory is adequate for explaining high-T_c superconductivity in doped cuprates from underdoped to overdoped regime. We show that the doped cuprates, depending on the Fermi energy (ε_F) and the energy (ε_A) of the effective attraction between pairing carriers, might be either unconventional (non-BCS-type) superconductors (at intermediate doping) or BCS-type superconductors (at higher doping). We argue that specific criteria for BCS-type superconductivity formulated in terms of two ratios $\varepsilon_A/\varepsilon_F$ and Δ/ε_F (where Δ is the BCS-like gap) must be met in these systems. We demonstrate that these criteria are satisfied only in overdoped cuprates but not in underdoped and optimally doped cuprates, where the origin of high-T_c superconductivity is quite different from the BCS-type (*s*- or *d*-wave) superconductivity. The BCS-like pairing theory is then used to calculate the critical superconducting transition temperature (T_c) and the peculiar oxygen and copper isotope effects on T_c in overdoped cuprates.

Keywords. Cuprate superconductors; doping effects; specific criteria for BCS-type cuprate superconductivity; oxygen and copper isotope effects on T_c.

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1. Introduction

The superconducivity in ordinary metals with large Fermi energies ($\varepsilon_F > 1$ eV) is now well described in terms of the Bardeen–Cooper–Schrieffer (BCS) condensation of Cooper pairs at the critical superconducting transition temperature (T_c). In these weak-coupling superconductors, the formation of Cooper pairs and their BCS condensation into a superfluid Fermi liquid state occur simultaneously at T_c. The occurrence of such a BCS-type Fermi liquid superconductivity in other systems (which might be fermion superconductors [1]) is also of fundamental interest, especially in the physics of doped semiconductors [2] and doped Mott insulators [3–6] (e.g., the undoped copper oxides (cuprates) are typical Mott insulators). The situation, however, is different in doped cuprates in which the Fermi energy (ε_F) is small and comparable with the energy $\hbar\omega_0$ of the high-frequency optical phonons. The undoped cuprates with charge-transfer gaps $\Delta_{CT} \cong 1.5$–2.0 eV [3,5] are Mott insulators and the doped cuprates behave like doped semiconductors (e.g., Si and Ge) [7,8]. Hole doping of the cuprates produces free holes in the oxygen valence band and the large ionicity of these materials $\eta = \varepsilon_\infty/\varepsilon_0 \ll 1$ (where ε_∞ and ε_0 are the high-frequency and static dielectric constants, respectively) enhances the polar electron–phonon interaction. Actually, the hole carriers in doped cuprates are self-trapped just like the self-trapping holes in ionic crystals of alkali halides [9]. These charge carriers in a polar crystal interact with acoustic and optical phonons and they become self-trapped quasiparticles (large polarons) in doped cuprates [10,11]. There is now experimental evidence that polarons are relevant charge carriers in doped cuprates [7,12,13]. In this fundamentally different physical situation, the validity of the BCS-like theory of superconductivity is obscure. However, it is natural to assume that polaronic effects in cuprate superconductors weaken with increasing doping level and disappear at higher doping levels [14–16].

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One can expect that the cuprates might be similar to ordinary metals in the overdoped regime and superconductivity in the cuprates at high doping (corresponding overdoping) level can be described in terms of the BCS-like pairing of quasi-free carriers. However, for doped cuprates, it is not obvious which specific criteria should be satisfied for the existence of BCS-type superconductivity.

It is not yet clarified at what levels of doping and under which specific conditions the cuprate compounds can become BCS-like superconducting systems. In ordinary metals, ε_F is nearly 10^2 times larger than the phonon energy ($\hbar\omega_D$) (where ω_D is the Debye frequency) and the fermionic nature of strongly overlapping Cooper pairs is quite apparent. Therefore, the formation of the energy gap on the Fermi surface at the Cooper pairing of quasi-free electrons just below T_c is a criterion for the occurrence of BCS superconductivity in metals. However, for doped cuprates with low Fermi energies ($\varepsilon_F \ll 1$ eV), it is not obvious in which cases Cooper pairs have fermionic nature and which specific criteria should be satisfied for the occurrence of BCS-type Fermi liquid superconductivity.

The purpose of the present paper is to resolve the aforementioned key and challenging issues arising in the physics of doped cuprate superconductors, which is controlled by the polaronic and doping effects, characteristic energy (ε_A) of the effective pairing interaction between charge carriers and low ε_F. We argue that BCS-like regime of superconductivity can exist in certain doped cuprates, where ε_F is much larger than the energy of the effective attraction (ε_A) between fermionic quasiparticles (holes or electrons). We formulate new and specific criteria for the BCS-type Fermi liquid superconductivity and show that the so-called *s*- or *d*-wave superconductivity described by BCS-like pairing theory can exist in overdoped cuprates, where the onset temperature of the Cooper pairing of quasi-free charge carriers just like in ordinary metals coincides with T_c. The Cooper pairing of holes in overdoped cuprates is possible if we consider their effective interaction with high-frequency optical phonons. Actually, various experiments [17–21] revealing the peculiar isotope effects on T_c and other physical quantities show that the polar electron–phonon interactions play an important role in cuprate superconductors. Despite all these well-performed experiments that lead to the consistent conclusion about the key role of electron–phonon interactions in these materials, there is no consensus on the origin of the observed peculiar isotope effects on T_c. Although the isotope effects on T_c in cuprate superconductors have been studied by many researchers [21–26], not very much is known, in particular, of the oxygen and copper isotope effects on T_c in overdoped cuprates. The understanding of the peculiar oxygen and copper isotope effects on T_c in overdoped cuprates is yet to be obtained. Here we also address this issue by studying the peculiar isotope effects on T_c in BCS-type cuprate superconductors.

2. Specific criteria for the BCS-type cuprate superconductivity

There are key differences between the superconducting mechanisms of the fermionic and bosonic Cooper pairs in doped cuprates, because the underlying mechanism of superconductivity in these systems depends on the fermionic or bosonic nature of Cooper pairs. The distinctive feature of the underdoped-to-overdoped cuprates is that they might be in the bosonic limit of Cooper pairs and in the non-BCS-type regime of superconductivity. In these high-T_c superconductors, the relevant charge carriers are rather polaronic quasiparticles than quasi-free electrons or holes and the formation of polaronic Cooper pairs is expected at the characteristic temperature T^* in the normal state (above T_c). However, one expects that polaronic effects disappear in certain overdoped materials, where the pairing of quasi-free carriers (electrons or holes) leads to the formation of large fermionic Cooper pairs and their simultaneous BCS condensation into a superfluid Fermi liquid state at T_c. One can assume that such overdoped cuprate materials become better metals and BCS-type superconductors. The bosonic or fermionic nature of Cooper pairs in doped cuprates can be specified by comparing the size of the Cooper pairs (a_c) with the average distance (R_c) between them. It may be important to identify which of the doped cuprates are the BCS-type or non-BCS-type cuprate superconductors. The size of a Cooper pair in a superconductor can be determined by using the uncertainty principle as [27]

$$a_c \simeq \frac{\hbar}{2\Delta}\sqrt{\frac{\varepsilon_F}{2m^*}}, \qquad (1)$$

where Δ is the BCS-like pairing gap, ε_F is the Fermi energy of the pairing carriers and m^* is the effective mass of these carriers. The mean distance between the Cooper pairs is determined by the expression

$$R_c \simeq \left(\frac{3}{4\pi n_c}\right)^{1/3}, \qquad (2)$$

where n_c is the concentration of Cooper pairs. According to the BCS-like pairing theory, n_c is determined from the equation

$$n_c = \frac{1}{2}\sum_k \left(1 - \frac{\varepsilon}{E}\right)[1 - f(k)] = \frac{\sqrt{2m^{*3}}}{4\pi^2\hbar^3} \times \int_{-\varepsilon_{\rm A}}^{+\varepsilon_{\rm A}} \left[1 - \frac{\varepsilon}{E}\right] \frac{{\rm e}^{E/k_{\rm B}T}}{{\rm e}^{E/k_{\rm B}T}+1} (\varepsilon + \varepsilon_{\rm F})^{1/2}\,{\rm d}\varepsilon, \tag{3}$$

where $f(k) = [{\rm e}^{E/k_{\rm B}T} + 1]^{-1}$ is the Fermi distribution function, $E(k) = \sqrt{\varepsilon^2(k) + \Delta^2}$ is the single-particle excitation energy, Δ is a BCS-like energy gap (or the minimum energy) for the excitation of Cooper pairs, $\varepsilon(k) = \hbar^2 k^2/2m^*$ is the energy of charge carriers measured from $\varepsilon_{\rm F}$ and $\varepsilon_{\rm A}$ is the cut-off parameter for the attractive pairing interaction between charge carriers, which take part in the Cooper pairing.

At low temperature ($T \to 0$), eq. (3) can be written as

$$n_{\rm c} = \frac{\sqrt{2m^{*3}}}{4\pi^2\hbar^3} \int_{-\varepsilon_{\rm A}}^{+\varepsilon_{\rm A}} \left[1 - \frac{\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}}\right] (\varepsilon + \varepsilon_{\rm F})^{1/2}\,{\rm d}\varepsilon = \frac{\sqrt{2(m^*\varepsilon_{\rm F})^3}}{4\pi^2\hbar^3} \times I_{\rm A}, \tag{4}$$

where

$$I_{\rm A} = \int_{-x_{\rm A}}^{+x_{\rm A}} \left[1 - \frac{x}{\sqrt{x^2 + \Delta_{\rm F}^2}}\right] \sqrt{x+1}\,{\rm d}x,$$

$$\Delta_{\rm F} = \Delta/\varepsilon_{\rm F}, \quad x_{\rm A} = \varepsilon_{\rm A}/\varepsilon_{\rm F}.$$

If $a_{\rm c}$ is larger than $R_{\rm c}$ between them, we deal with the overlapping Cooper pairs, which behave like fermions, as argued by Bardeen and Schrieffer (see refs [28,29]). Thus, it is quite clear that we deal with the BCS-type superconductors if the condition $a_{\rm c}(T) > R_{\rm c}$ is satisfied. Using relations (1) and (2), the condition $R_{\rm c} < a_{\rm c}(T)$ can be written as

$$\frac{R_{\rm c}}{a_{\rm c}} = \frac{2\Delta}{\varepsilon_{\rm F}}\left[\frac{3}{4\pi n_{\rm c}}\right]^{1/3} \sqrt{\frac{2m^*\varepsilon_{\rm F}}{\hbar^2}} < 1. \tag{5}$$

By using eq. (4) we can rewrite criterion (5) for the existence of BCS-type Fermi liquid superconductivity in the form

$$\frac{\Delta}{\varepsilon_{\rm F}} < \frac{1}{2}\left(\frac{I_{\rm A}}{6\pi}\right)^{1/3}, \tag{6}$$

from which it follows that the new and specific criteria for BCS-type superconductivity should be defined by the values of $\varepsilon_{\rm A}/\varepsilon_{\rm F}$ and $\Delta/\varepsilon_{\rm F}$ satisfying condition (6). If $\varepsilon_{\rm A}/\varepsilon_{\rm F}$ and $\Delta/\varepsilon_{\rm F}$ satisfy condition (6), Cooper pairs are in the fermionic limit and we deal with BCS-type superconductors. The Fermi energy ($\varepsilon_{\rm F}$) of the optimally doped cuprates is larger than that of the underdoped cuprates, but smaller than $\varepsilon_{\rm F}$ of the overdoped cuprates. To illustrate this point, we estimate $\varepsilon_{\rm F}$ of the typical cuprate superconductor $La_{2-x}Sr_xCuO_4$ (LSCO) in the optimally doped regime. In so doing, we determine the doping level in the dimensionless form $x = n/n_a$ (where n is the density of doped hole carriers, $n_a = 1/V_a$ is the density of the host lattice atoms, $V_a \simeq 190\,\text{Å}^3$ is the volume per CuO_2 unit in the orthorhombic LSCO) and argue that polaronic carriers with effective masses $m^* = m_p = (2$–$3)m_e$ [7,30] (where m_e is the free electron mass) can exist in cuprates up to some overdoped level above which polaronic effects disappear and quasi-free charge carriers exist in overdoped cuprates. We find that in LSCO, optimally doped level ($x \simeq 0.15$) corresponds to the value of $n \simeq 0.8 \times 10^{21}{\rm cm}^{-3}$. Then the optimally-doped LSCO has $\varepsilon_{\rm F} = \hbar^2(3\pi^2 n_a x)^{2/3}/2m_p \approx 0.16$ eV at $n_a \simeq 5.3 \times 10^{21}{\rm cm}^{-3}$, $m_p = 2m_e$ and $x = 0.15$. Therefore, the values $\varepsilon_{\rm F} \simeq 0.10$–$0.15$ eV correspond to the underdoped and optimally doped cuprates, while the overdoped cuprates have relatively large Fermi energies ($\varepsilon_{\rm F} > 0.16$ eV) at $m^* = m_p = 2m_e$ and $x > 0.16$. Actually, the doped cuprate superconductors are characterised by low Fermi energies ($\varepsilon_{\rm F} \simeq (0.1$–$0.3)$ eV) [31,32] and high-frequency optical phonons $\hbar\omega_0 \simeq (0.04$–$0.08)$ eV [7,31,32]. In the absence of polaronic mass effect, $\varepsilon_{\rm F}$ of quasi-free carriers in overdoped cuprates is larger than 0.3 eV. Experimental results show that the values of Δ in underdoped and optimally doped cuprates vary from 0.025 to 0.042 eV [33], whereas in the overdoped cuprates the BCS-like gap Δ varies from 0.015 to 0.020 eV [34]. One can see that the ratio $\varepsilon_{\rm A}/\varepsilon_{\rm F}$ in underdoped and optimally doped cuprates varies from 0.5 to 1.0. Condition (6) in these high-$T_{\rm c}$ cuprates is not satisfied for $\Delta \simeq 0.025$–0.042 eV and $\varepsilon_{\rm F} < 0.15$ eV. For $\varepsilon_{\rm A}/\varepsilon_{\rm F} > 0.5$, the BCS-type regime of superconductivity is unrealisable in underdoped, optimally doped and slightly overdoped cuprates, which are non-BCS-type superconductors (figure 1). In the absence of any polaronic effect, the value of $\varepsilon_{\rm A}/\varepsilon_{\rm F}$ in overdoped cuprates with $\varepsilon_{\rm F} \gtrsim 0.3$ eV is always less than 0.5 or even much less than 0.5. Therefore, the BCS-type regime of superconductivity is viable only in such overdoped cuprates, where $\Delta/\varepsilon_{\rm F} \ll 0.1$ for $\Delta \lesssim 0.020$ eV and $\varepsilon_{\rm F} \gtrsim 0.3$ eV.

Thus, BCS-type Fermi liquid superconductivity is realised only in the overdoped cuprates with relatively large $\varepsilon_{\rm F}$, but not in underdoped, optimally doped and slightly overdoped cuprates with small $\varepsilon_{\rm F}$ ($\varepsilon_{\rm F} \lesssim 0.15$ eV), where the theory of BCS-type (s- or d-wave) superconductivity, often discussed by many researchers, is not applicable. In non-BCS-type cuprate superconductors, the onset temperature of Cooper pairing does not

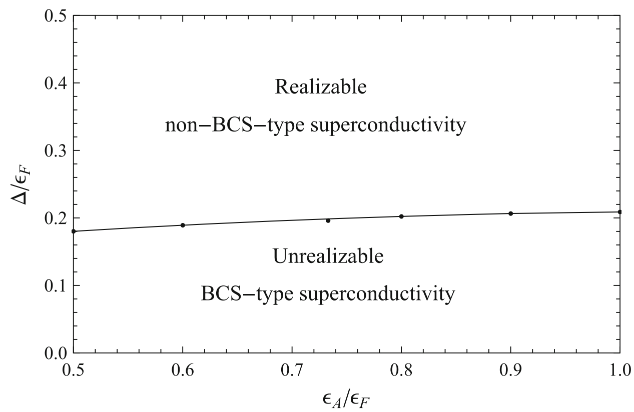

Figure 1. Diagrammatic representation of the two characteristic ratios Δ/ε_F and $\varepsilon_A/\varepsilon_F$ for underdoped, optimally doped and slightly overdoped cuprates, which are beyond the BCS-type regime of superconductivity.

coincide with T_c [1], which is not determined from the BCS-like gap equation. In these unconventional superconductors, the superconducting order parameter Δ_{SC} and T_c are determined from the equation of the mean-field theory of attracting bosonic Cooper pairs [35].

3. BCS-type superconductivity in overdoped cuprates

From the aforementioned considerations, it follows that the validity criterion (i.e., condition (6)) for BCS-type superconductivity is well satisfied in overdoped cuprates with relatively large ε_F ($\varepsilon_F \gtrsim 0.3$ eV), where the quasi-free charge carriers (with $m^* = m_e$) exist in the absence of polaronic effects. In the case of such overdoped cuprates, the attractive interaction mechanism (mainly due to the exchange of optical phonons) between the hole carriers operating in the energy range $\{-\hbar\omega_0, \hbar\omega_0\}$ is more effective than in the simple BCS picture in the narrow energy range $\{-\hbar\omega_D, \hbar\omega_D\}$. We now present the BCS-like pairing theory which describes the superconductivity of overlapping large fermionic Cooper pairs in overdoped cuprates with $\varepsilon_F \gg \hbar\omega_0$. The BCS-like Hamiltonian of the system of interacting quasi-free carriers is diagonalised by using the Bogoliubov transformations of Fermi operators. Then, the BCS-like gap equation has the form

$$\Delta(k) = -\sum_{\vec{k}'} V(k, k') \frac{\Delta(k')}{2E(k')} \tanh \frac{E(k')}{2k_B T}, \quad (7)$$

where $V(k, k')$ is the pairing interaction potential (which has both an attractive electron–phonon interaction part and a repulsive Coulomb interaction part) between the charge carriers, k is their wave vector. Further, we use the model potential which we may write as

$$V(k, k') = \begin{cases} V_{ph} - V_c & \text{for } |\varepsilon(k)|, \ |\varepsilon(k')| \leq \varepsilon_A = \hbar\omega_0, \\ V_c & \text{for } \varepsilon_A \leq |\varepsilon(k)|, \ |\varepsilon(k')| < \varepsilon_c, \\ 0 & \text{otherwise}, \end{cases} \quad (8)$$

where V_{ph} is the phonon-mediated interaction potential between two carriers, V_c is the bare repulsive Coulomb potential and ε_c is the cut-off energy for the Coulomb interaction. Using model potential (8) and replacing the sum over k by an integral over ε in eq. (7), we obtain the following BCS-like equation for determining the energy gap ($\Delta(T)$) and the mean-field pairing temperature (T^*) which coincides with T_c:

$$\frac{1}{\lambda_{BCS}} = \int_0^{\varepsilon_A} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2(T)}} \tanh \frac{\sqrt{\varepsilon^2 + \Delta^2(T)}}{2k_B T}, \quad (9)$$

where $\lambda_{BCS} = N(\varepsilon_F)\tilde{V}$ is the BCS-like coupling constant, $N(\varepsilon_F)$ is the density of states at the Fermi level, $\tilde{V} = V_{ph} - \tilde{V}_c$ is the effective pairing interaction potential, $\tilde{V}_c = V_c/[1 + N(\varepsilon_F)V_c \ln(\varepsilon_c/\varepsilon_A)]$ is the screened Coulomb interaction between two carriers, $\varepsilon_c = \varepsilon_F$. According to criterion (6), the overdoped cuprates are in the BCS limit and the order parameter Δ serves as the superconducting order parameter. The onset temperature of the Cooper pairing of quasi-free carriers in the BCS-type cuprate superconductors coincides with T_c, so that at $T = T_c$, eq. (9) becomes

$$\frac{1}{\lambda_{BCS}} = \int_0^{\varepsilon_A} \frac{d\varepsilon}{\varepsilon} \tanh \frac{\varepsilon}{2k_B T_c}. \quad (10)$$

At $\varepsilon_A = \hbar\omega_0 \gtrsim 7k_B T_c$, the solution of eq. (10) gives the following result:

$$k_B T_c \simeq 1.134\hbar\omega_0 \exp\left[-\frac{1}{\lambda_{BCS}}\right]. \quad (11)$$

One can see that both the BCS-like mean-field pairing temperature T_c (which represents the superconducting transition temperature) and the coupling constant (λ_{BCS}) depend on phonon energy ($\hbar\omega_0$). The BCS-like pairing theory extending to the case of overdoped cuprate superconductors allows us to calculate T_c and the peculiar oxygen and copper isotope effects on T_c.

4. Peculiar oxygen and copper isotope effects on T_c in BCS-type cuprate superconductors

We now determine the values of T_c and the possible isotope effects on T_c, which are characteristics of overdoped cuprates, using the BCS-like equation

(11), where the prefactor and the effective attractive interaction potential $\tilde{V}_c$ depend on ω_0 which in turn depends on M_O and M_{Cu}, the masses of the oxygen (O) and copper (Cu) atoms. Thus, the peculiar isotope effects on T_c in these cuprate superconductors depend on the optical phonon frequency determined from the relation

$$\omega_0 \simeq \left[2\beta\left(\frac{1}{M}+\frac{1}{M'}\right)\right]^{1/2},$$

where M (=M_O or M_{Cu}), M' (=M_{Cu} or M_O) and β is a force constant of the lattice. According to eq. (11), the prefactor (ω_0) and the BCS-like coupling constant (λ_{BCS}) depend on the reduced mass $\mu = MM'/(M+M')$ of ions. The density of states at the Fermi level (ε_F) can be approximated in a simple form

$$N(\varepsilon_F) = \begin{cases} 1/\varepsilon_F & \text{for } 0 < \varepsilon \le \varepsilon_F = \hbar^2(3\pi^2 n)^{2/3}/2m_e, \\ 0 & \text{otherwise,} \end{cases} \tag{12}$$

where n is the density of hole carriers. The exponent of the isotope effect on T_c is defined as

$$\alpha_{T_c} = -\frac{\mathrm{d}\ln T_c}{\mathrm{d}\ln M}. \tag{13}$$

In order to study the isotope effects on T_c in heavily overdoped cuprates, the expression for T_c can be written as

$$k_B T_c = 1.134\hbar\omega_0(\mu)\exp\left[-\frac{1}{\lambda_{BCS}(\mu)}\right], \tag{14}$$

where

$$\omega_0(\mu) = (2\beta/\mu)^{1/2}, \quad \lambda_{BCS}(\mu) = \lambda_{ph} - \tilde{\lambda}_c,$$

$$\tilde{\lambda}_c = \lambda_c/[1+\lambda_c \ln(\varepsilon_F/\hbar\omega_0(\mu))],$$

$$\lambda_{ph} = [2m_e/\hbar^2(3\pi^2 n)^{2/3}]V_{ph},$$

$$\lambda_c = [2m_e/\hbar^2(3\pi^2 n)^{2/3}]V_c$$

and approximation (12) is taken into account. Using eqs (13) and (14), we obtain the following exponents of the oxygen and copper isotope effects on T_c:

$$\alpha^{O}_{T_c} = \frac{1}{2(1+M_O/M_{Cu})}\left[1-\left(\frac{\Lambda}{1-\Lambda}\right)^2\right] \tag{15}$$

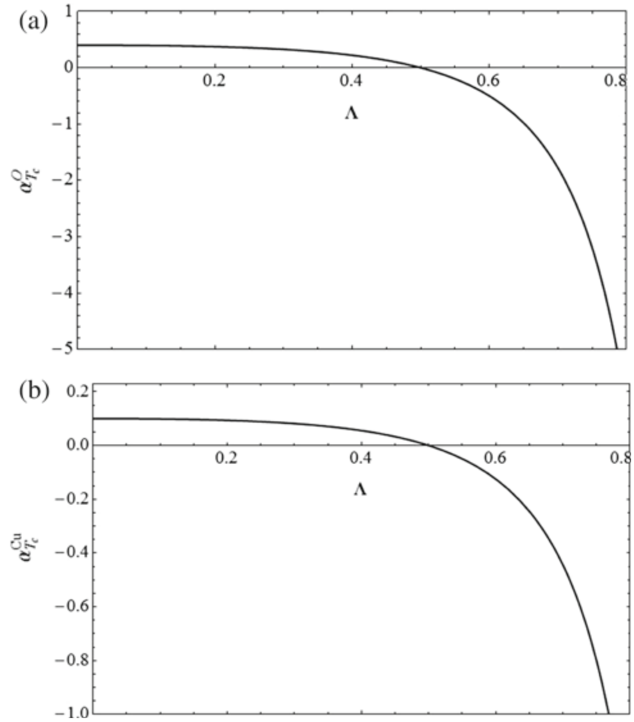

Figure 2. Variation of the oxygen isotope exponent $\alpha^{O}_{T_c}$ (**a**) and the copper isotope exponent $\alpha^{Cu}_{T_c}$ (**b**) as a function of Λ.

and

$$\alpha^{Cu}_{T_c} = \frac{1}{2(1+M_{Cu}/M_O)}\left[1-\left(\frac{\Lambda}{1-\Lambda}\right)^2\right], \tag{16}$$

where $\Lambda = \tilde{\lambda}_c(\mu)/\lambda_{ph}$ (figure 2).

According to eq. (11), the value of T_c in overdoped cuprates just like in conventional superconductors becomes rather small and equal to 9.5 K for $\hbar\omega_0 = 0.05$ eV and $\lambda_{BCS} = 0.236$. This is consistent with the experimental value of $T_c = 9.4$ K observed in the overdoped system $Bi_{1.74}Sr_{1.88}Pb_{0.38}CuO_{6+\delta}$ [36]. The overdoped cuprate superconductors $La_{2-x}Sr_xCuO_4$ (LSCO) have a lower T_c than the optimally doped LSCO system with $x = 0.15$, where T_c is the maximum and becomes equal to 37 K [17,37]. Therefore, one can expect that the heavily overdoped LSCO systems have $T_c \lesssim 30$ K and they have larger oxygen isotope-effect exponents $\alpha^{O}_{T_c}$ than the optimally doped LSCO system where $\alpha^{O}_{T_c} \simeq 0.36$ [37]. Using eq. (11), we obtain $T_c \lesssim 30$ K at $\hbar\omega_0 = 0.047$ eV [7] and $\lambda_{BCS} \simeq 0.333$ for these systems. It follows from eqs (13) and (14) that the oxygen and copper isotope effects on T_c in overdoped LSCO systems are positive for $2\Lambda < 1$ and both $\alpha^{O}_{T_c}$ and $\alpha^{Cu}_{T_c}$ are less than 0.5. By taking $\Lambda = 0.2$ for these systems, we find

$$\alpha_{T_c}^{O} \simeq \frac{1}{2(1+M_O/M_{Cu})}[1-0.062] \simeq 0.38 \qquad (17)$$

and

$$\alpha_{T_c}^{Cu} \simeq \frac{1}{2(1+M_{Cu}/M_O)}[1-0.062] \simeq 0.094. \qquad (18)$$

In overdoped cuprate superconductors, the oxygen and copper isotope effects on T_c are positive for $\Lambda < 0.5$ and become negative for $\Lambda > 0.5$ (see figure 2) in accordance with the experimental observations [17,37,38].

Thus, the isotope effects on T_c in overdoped cuprates, which are BCS-type superconductors just like in conventional metals [39] are positive or may become even negative.

5. Conclusions

We have addressed the key issues of superconductivity in doped cuprates, which might be BCS-type or non-BCS-type superconductors under certain conditions. We argue that the new physics of these complex systems is controlled by the polaronic and doping effects, low ε_F ($\varepsilon_F \ll 1$ eV) and characteristic energy ε_A of the effective attraction between pairing charge carriers. We have found that the polaronic effects and the non-BCS-like superconducting behaviours are characteristics of underdoped-to-overdoped cuprates that have low ε_F ($\varepsilon_F \lesssim 0.15$ eV) and are beyond the BCS-type regime of superconductivity. We have formulated new and specific criteria for the existence of BCS-type superconductivity in doped cuprates in terms of characteristic ratios $\varepsilon_A/\varepsilon_F$ and Δ/ε_F. We demonstrated that the overdoped cuprates with relatively large ε_F ($\varepsilon_F \gtrsim 0.3$ eV) are in the fermionic limit of Cooper pairs and the BCS-type Fermi liquid superconductivity is realisable in these systems but not in underdoped, optimally doped and slightly overdoped cuprates, which are in the bosonic limit of Cooper pairs. In so doing, we proved that the interpretations of high-T_c superconductivity in underdoped, optimally doped and slightly overdoped cuprates as the BCS-type (*s*- or *d*-wave) superconductivity are misleading. In overdoped cuprates with $\varepsilon_F \gtrsim 0.3$ eV, $(\varepsilon_A/\varepsilon_F) < 0.5$ or even much less than 0.5, while $\Delta/\varepsilon_F \ll 0.1$ for $\Delta \lesssim 0.02$ eV. The BCS-like pairing theory is used to calculate T_c and peculiar isotope effects on T_c in such overdoped systems. The appropriate values of T_c as well as the possible oxygen and copper isotope effects on T_c in overdoped cuprates are determined and compared with the available experimental results.

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