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Exact solitary wave solutions for a system of some nonlinear space–time fractional differential equations

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Abstract. We have enumerated new and exact general wave solutions, along with multiple exact soliton solutions of space–time nonlinear fractional differential equations (FDE), namely Zakharov–Kuznetsov–Benjamin–Bona–Mahony (ZKBBM), foam drainage and symmetric regularised long-wave (SRLW) equations, by employing a relatively new technique called $(G'/G, 1/G)$-expansion method. Also, based on fractional complex transformation and the properties of the modified Riemann–Liouville fractional-order operator, the fractional partial differential equations transform into a form of ordinary differential equation (ODE). This method is a recollection of the commutation of the well-appointed (G'/G)-expansion method introduced by Wang *et al*, *Phys. Lett. A* **372**, 417 (2008) In this paper, it is mentioned that the two-variable $(G'/G, 1/G)$-expansion method is more legitimate, modest, sturdy and effective in the sense of theoretical and pragmatical point of view. Lastly, the peculiarities of these analytic solutions are illustrated graphically by utilising the computer symbolic programming Wolfram Mathematica.

Keywords. $(G'/G, 1/G)$-expansion method; Zakharov–Kuznetsov–Benjamin–Bona–Mahony equation; foam drainage equation; symmetric regularised long-wave equation; fractional derivative; solitary wave solution.

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1. Introduction

The first concept of fractional calculus is found in 1695 [1], when Gottfried–Wilhelm–Leibniz suggested the possibility of fractional derivatives for the first time. The foundations of the subject were laid by Liouville in a paper from 1832, and the fractional derivative of a power function obtained by Riemann in 1847 [2]. Fractional calculus is a powerful tool for modelling complex systems, especially for viscoelastic materials. The study of exact travelling and solitary wave solutions for nonlinear fractional differential equations (FDE) play an important role because of their prospective implementations in various scientific and technological fields, especially in solid-state physics, mathematical physics, mechanics, plasma physics, signal processing, bioengineering, optical fibres, geochemistry, stochastic dynamical systems, nonlinear optics, economics and business [3,4]. In the last few decades, many researchers and scholars have been attracted to solve nonlinear FDE by interposing several explicit, effective and powerful approaches. Researchers have introduced different methods such as the first integral method [5,6], modified simple equation method [7–9], auxiliary equation method [10], fractional subequation method [11–13], fractional (G'/G)-expansion method [14–16], ansatz method [17,18], fractional functional variable method [19], generalised (G'/G)-expansion method [20], generalised Kudryashov method [21–25] and so on. Recently, based on the original (G'/G)-expansion method, various modified techniques such as the generalised (G'/G)-expansion method, extended (G'/G)-expansion method, (G'/G^2)-expansion method etc. have been promoted. The main idea of the original single-variable (G'/G)-expansion method is that, the exact travelling wave solution of nonlinear PDEs can be exposed by a polynomial in one-variable (G'/G), in which $G = G(\eta)$ satisfies the second-order linear ordinary differential equation (LODE) $G''(\eta) + \lambda G'(\eta) + \mu G(\eta) = 0$, wherein λ and μ are non-zero constants and G' is the derivative of G. This equation can also be used as an auxiliary equation. However, inspired and motivated by the ongoing research in this arena, we also introduce and make best use of $(G'/G, 1/G)$-expansion method, which can be envisaged as the modification of the original (G'/G)-expansion method [26]. The

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key point of this two-variable $(G'/G, 1/G)$-expansion method is that the exact travelling wave solution of nonlinear PDEs can be revealed by a polynomial in the two-variable (G'/G) and $(1/G)$ in which $G = G(\eta)$ gratifies the second-order LODE $G''(\eta) + \lambda\, G(\eta) = \mu$, where λ and μ are constants. The degree of this polynomial can be discerned by introducing homogeneous balance principle between the highest order derivatives with highest order nonlinear term appearing in the conferred nonlinear PDEs. Also, the coefficient of this polynomial can be ascertained by solving a set of algebraic equations resulted by using this technique. More recently, some scholars like Yasar and Giresunlu [27] obtained exact solutions of space–time fractional Chan–Allen and Klein–Gordon equation by treating $(G'/G, 1/G)$-expansion method. Likewise, utilising similar technique, Topsakal *et al* [28] have accomplished the exact and travelling wave solutions for space–time fractional modified Benjamin–Bona–Mahony (mBBM) and modified nonlinear Kawahara equations. Hereafter, in this manuscript for the first time, we disclose the exact travelling and solitary wave solutions for the system of nonlinear space–time FDE such as: Zakharov–Kuznetsov–Benjamin–Bona–Mahony (ZKBBM), foam drainage and symmetric regularised long-wave (SRLW) equations by applying two-variable $(G'/G, 1/G)$-expansion method. The principal motive for selecting this method is to give solutions in more general form. Moreover, the advantages of our proposed method over the original (G'/G)-expansion method is that the solutions treating the first method recapture the solutions treating the second one. So, we can say that the two-variable $(G'/G, 1/G)$-expansion method is an extension of the (G'/G)-expansion method.

First, we describe the well-known space–time fractional ZKBBM equation, which arises as the gravity water wave phenomena in the unidirectional propagation of long waves in certain nonlinear dispersive systems. The form of the equation is represented by [29]

$$D_t^{\alpha} v + D_x^{\alpha} v - 2av D_x^{\alpha} v - b D_t^{\alpha}\left(D_x^{2\alpha} v\right) = 0,$$
$$t > 0, \quad 0 < \alpha \leq 1. \tag{1}$$

In this equation, a and b are arbitrary constants.

Secondly, we consider the space–time fractional foam drainage equation [30] as

$$D_t^{\alpha} w = \frac{1}{2} w D_x^{\alpha} D_x^{\alpha} w - 2w^2 D_x^{\alpha} w + \left(D_x^{\alpha} w\right)^2,$$
$$0 < \alpha \leq 1. \tag{2}$$

Finally, we designate the space–time fractional nonlinear SRLW equation, which emerges in various physical forms including ion sound waves in plasma. The equation takes the following form [31]:

$$D_t^{2\alpha} v + D_x^{2\alpha} v + v D_t^{\alpha}\left(D_x^{\alpha} v\right)$$
$$+ D_t^{\alpha} v D_x^{\alpha} v + D_t^{2\alpha}\left(D_x^{2\alpha} v\right) = 0, \quad 0 < \alpha \leq 1. \tag{3}$$

This paper is organisd as follows: In §2, basic ideas of the modified Riemann–Liouville fractional-order derivative are given. In §3, we represent the structure of the renewed $(G'/G, 1/G)$-expansion method. In §4, we execute this method to seek new and further exact wave solutions of the space–time FDE written above. The nature of the solutions along with their graphical representation is provided in §5. In §6, we give the results and discussion and lastly, in §7, the concluding remarks of this present paper is given.

2. Basic ideas of the modified Riemann–Liouville fractional-order derivative

The Jumarie's modified Riemann–Liouville derivative of order α with the continuous function $f: R \to R$, $x \to f(x)$ is stated by the following expression [32]:

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x (x-\xi)^{-\alpha-1} (f(\xi) - f(0))\, \mathrm{d}\xi,$$

where $\Gamma(x)$ is the gamma function which is defined as

$$\Gamma(x) = \int_0^x \mathrm{e}^{-t} t^{(x-1)} \mathrm{d}x.$$

If $\alpha < 0$ then

$$D_x^{\alpha} f(x) = \left(f^{(\alpha-1)}(x)\right)',$$
$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \times \int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0)) \mathrm{d}\xi. \tag{4}$$

Also if $0 < \alpha < 1$,

$$D_x^{\alpha} f(x) = \left(f^{(n)}(x)\right)^{(\alpha-n)} \tag{5}$$

if $n \leq \alpha \leq n+1$ and $n \geq 1$.

For fractional-order derivative, some essential postulates which we use further in this paper are selected as follows:

Postulate 1.

$$D_x^{\alpha} x^r = \frac{\Gamma(1+r)}{\Gamma(1+r-\alpha)} x^{r-\alpha}, \tag{6}$$

where r is the real number.

Postulate 2.

$$D_x^{\alpha}(af(x) + bg(x)) = a D_x^{\alpha} f(x) + b D_x^{\alpha} g(x), \tag{7}$$

where a and b are arbitrary constants.

Postulate 3.

$$D_x^\alpha f(\xi) = \frac{\mathrm{d}f}{\mathrm{d}\xi} D_x^\alpha(\xi), \tag{8}$$

where $\xi = g(x)$.

3. Formation of $(G'/G,\ 1/G)$-expansion method

In this section, we explain the major concept of $(G'/G, 1/G)$-expansion method for attaining exact solitary wave solutions of the nonlinear time FDE. As usual, we envisage the second-order LODE in $G = G(\eta)$ as

$$G''(\eta) + \lambda G(\eta) = \mu, \tag{9}$$

and we assume two rational functions ϕ and ψ as

$$\phi = \frac{G'(\eta)}{G(\eta)}, \quad \psi = \frac{1}{G(\eta)}, \tag{10}$$

where λ and μ are two arbitrary constants and G' is the derivative of G.

From eqs (9) and (10) we obtain

$$\phi' = -\phi^2 + \mu\psi - \lambda, \quad \psi' = -\phi\psi. \tag{11}$$

The solution of LODE (9), accomplish the following three distinct annotations:

Annotation 1: If $\lambda < 0$, the general solution of LODE (9) gives

$$G(\eta) = A_1 \sinh\left(\sqrt{-\lambda}\,\eta\right) + A_2 \cosh\left(\sqrt{-\lambda}\,\eta\right) + \frac{\mu}{\lambda}, \tag{12}$$

where A_1 and A_2 are two arbitrary constants. Therefore, from eqs (10)–(12) the following relation can be obtained:

$$\psi^2 = \frac{-\lambda}{\lambda^2\varepsilon + \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right),$$

where

$$\varepsilon = A_1^2 - A_2^2. \tag{13}$$

Annotation 2: If $\lambda > 0$, the general solution of LODE (9) is

$$G(\eta) = A_1 \sin\left(\sqrt{\lambda}\,\eta\right) + A_2 \cos\left(\sqrt{\lambda}\,\eta\right) + \frac{\mu}{\lambda}, \tag{14}$$

and similarly, from eqs (10), (11) and (14) the corresponding relations are

$$\psi^2 = \frac{\lambda}{\lambda^2\varepsilon - \mu^2}\left(\phi^2 - 2\mu\psi + \lambda\right),$$

where

$$\varepsilon = A_1^2 + A_2^2. \tag{15}$$

Annotation 3: Finally, if $\lambda = 0$, the general solution of LODE (9) takes the form

$$G(\eta) = \frac{\mu}{2}\eta^2 + A_1\eta + A_2. \tag{16}$$

Consequently, we have

$$\psi^2 = \frac{1}{A_1^2 - 2\mu A_2}\left(\phi^2 - 2\mu\psi\right). \tag{17}$$

Now suppose we have a nonlinear FDE, say in two independent variables x and t of the form

$$F\left(v, D_t^\alpha v, D_x^\alpha v, D_t^\alpha D_t^\alpha v, D_t^\alpha D_x^\alpha v, D_x^\alpha D_x^\alpha v, \ldots\right) = 0, \tag{18}$$

where F is a polynomial in $v = v(x,t)$ and its various partial fractional derivatives. The main structure of the $\left(G'/G, 1/G\right)$-expansion method are present as follows:

Step 1: Take into account, the travelling wave variable

$$v(x,t) = v(\eta), \quad \eta = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)}, \tag{19}$$

where k and c are non-zero arbitrary constants. Substituting eq. (19) and the values of $v = v(x,t)$ and their various fractional derivatives into eq. (18), it transforms into the following ordinary differential equation (ODE):

$$H\left(v, v', v'', v''', \ldots\right) = 0, \tag{20}$$

where $'$ indicates $\mathrm{d}/\mathrm{d}\eta$ and so on. If needed, we integrate eq. (20) one or more times and assume the constant of integration to be zero.

Step 2: In accordance with the $(G'/G, 1/G)$-expansion technique the exact solution of eq. (20) can be disclosed by a finite power series in two variables ϕ and ψ as follows:

$$v(\eta) = \sum_{i=0}^{M} a_i\phi^i + \sum_{j=1}^{M} b_j\phi^{j-1}\psi, \tag{21}$$

where a_i $(i=0, 1, 2, \ldots, M)$ and b_j $(j = 1, 2, \ldots, M)$ are constants, which will be determined later, as for instance, evaluating the values of positive integer M by homogeneous balance principle, balancing the highest-order derivative with the highest-order nonlinear term appearing in eq. (20). But, balance number M is not positive all the times: sometimes it is fraction or negative. In this case, we use the following transformation [33]:

(a) If $M = q/p$, where q/p is a fraction in the lowest terms, then

$$v(\eta) = u^{q/p}(\eta), \tag{22}$$

then substitute eq. (22) into eq. (20) to attain an equation in the renewed function $u(\eta)$ with a positive integer balance number.

(b) If M is a negative number, then

$$v(\eta) = u^{M}(\eta), \tag{23}$$

and set eq. (23) into eq. (20) to obtain an equation in the new function $u(\eta)$ with a positive integer balance number.

Step 3. Substituting eq. (21) into eq. (20), also operating eqs (11) and (13) will transform into a polynomial in ϕ and ψ, in which the degree of ψ is not longer than 1. Equalising all the coefficient of this polynomial to zero, yields a system of algebraic equations for a_i $(i = 0, 1, 2, \ldots, M)$, b_j $(j = 1, 2, \ldots, M)$, k, c, μ, λ $(\lambda < 0)$, A_1 and A_2.

Step 4. Solve the system which is attained in Step 3 with the aid of any computer symbolic program, like Mathematica and substituting the values of a_i, b_j, k, c, μ, λ $(\lambda < 0)$, A_1 and A_2 into eq. (21) and finally, we come up with different types of exact wave solutions of eq. (18) represented by the hyperbolic functions.

Step 5. Likewise, pursuing Steps 3 and 4, plugging eq. (21) into eq. (20), treating eqs (11) and (15) (or eqs (11) and (17)), we come up with the solution of eq. (18) in the case of trigonometric functions (or by rational functions) as before.

4. Implementations of the method

In this section, the $\left(G'/G, 1/G\right)$-expansion method has been assigned to visualise different exact wave solutions of the nonlinear space–time fractional ZKBBM equation (1), foam drainage equation (2) and SRLW equation (3).

4.1 *Space–time fractional ZKBBM equation*

In this subsection, we shall make best use of the $\left(G'/G, 1/G\right)$-expansion method to constitute the exact travelling and solitary wave solutions of the space–time fractional ZKBBM equation (1). Take into account, the following wave transformation

$$v(x,t) = v(\xi), \quad \xi = \frac{kx^{\alpha}}{\Gamma(1+\alpha)} - \frac{ct^{\alpha}}{\Gamma(1+\alpha)}, \tag{24}$$

where k and c are non-nzero constants. This wave transformation permits eq. (1) to be reduced to an ODE

$$(k-c)v' - 2akvv' + bck^2v''' = 0, \tag{25}$$

where prime denotes the derivative with respect to ξ. Integrating eq. (25) once with respect to ξ and setting the constant of integration to be zero, we get

$$(k-c)v - akv^2 + bck^2v'' = 0. \tag{26}$$

By using the homogeneous balance principle, balancing the highest-order derivative v'' with the highest-order nonlinear term v^2, balance number $M = 2$ is obtained. By putting the value of $M = 2$ in eq. (21), then the solution formula becomes

$$v(\xi) = a_0 + a_1\phi + a_2\phi^2 + b_1\psi + b_2\phi\psi, \tag{27}$$

where a_0, a_1, a_2, b_1, and b_2 are constants to be decided later. Therefore, the aforementioned three annotations are studied as follows:

Annotation 1: When $\lambda < 0$ (Hyperbolic function solutions)

Plugging the value of $v(\xi)$ from eq. (27) into eq. (26), in addition with eqs. (11) and (13), the left-hand side of eq. (26) becomes a polynomial in ϕ and ψ. Assuming each coefficient of the polynomial to be zero, we arrive at a system of algebraic equations as follows:

$$\phi\colon -ca_1 + ka_1 + 2bck^2\lambda a_1 - 2aka_0a_1 + \frac{6bck^2\lambda^2\mu b_2}{\mu^2+\lambda^2\sigma} + \frac{2ak\lambda^2 b_1b_2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^2\colon -aka_1^2 - ca_2 + ka_2 + 8bck^2\lambda a_2 - 2aka_0a_2 - \frac{2bck^2\lambda\mu^2a_2}{\mu^2+\lambda^2\sigma} + \frac{bck^2\lambda\mu b_1}{\mu^2+\lambda^2\sigma} + \frac{ak\lambda b_1^2}{\mu^2+\lambda^2\sigma} + \frac{ak\lambda^2b_2^2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^3\colon 2bck^2a_1 - 2aka_1a_2 + \frac{6bck^2\lambda\mu b_2}{\mu^2+\lambda^2\sigma} + \frac{2ak\lambda b_1b_2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^4\colon 6bck^2a_2 - aka_2^2 + \frac{ak\lambda b_2^2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\psi\colon -4bck^2\lambda\mu a_2 - cb_1 + kb_1 + bck^2\lambda b_1 - 2aka_0b_1 + \frac{4bck^2\lambda\mu^3a_2}{\mu^2+\lambda^2\sigma} - \frac{2bck^2\lambda\mu^2b_1}{\mu^2+\lambda^2\sigma} - \frac{2ak\lambda\mu b_1^2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi\psi:\ -3bck^2\mu a_1 - 2aka_1b_1 - cb_2 + kb_2 + 5bck^2\lambda b_2 - 2aka_0b_2 - \frac{12bck^2\lambda\mu^2 b_2}{\mu^2+\lambda^2\sigma} - \frac{4ak\lambda\mu b_1 b_2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^2\psi:\ -10bck^2\mu a_2 + 2bck^2 b_1 - 2aka_2b_1 - 2aka_1b_2 - \frac{2ak\lambda\mu b_2^2}{\mu^2+\lambda^2\sigma} = 0,$$

$$\phi^3\psi:\ 6bck^2b_2 - 2aka_2b_2 = 0,$$

$$\text{Const.}:\ -ca_0 + ka_0 - aka_0^2 + 2bck^2\lambda^2 a_2 - \frac{2bck^2\lambda^2\mu^2 a_2}{\mu^2+\lambda^2\sigma} = 0.$$

On solving the above algebraic equations with the aid of a symbolic software program, like Mathematica, we come up with the following different sets of results:
Set 1:

$$a_0 = \frac{3}{2a},\quad a_1 = 0,\quad a_2 = \frac{3}{2a\lambda},\quad b_1 = -\frac{3\mu}{2a\lambda},$$
$$b_2 = \pm\frac{3\sqrt{-\mu^2-\lambda^2\sigma}}{2a\lambda^{3/2}},$$
$$k = \frac{1}{\sqrt{b}\sqrt{\lambda}},\quad c = \frac{1}{2\sqrt{b}\sqrt{\lambda}}. \tag{28}$$

Setting the above values into eq. (27), we attain the exact wave solution of eq. (1) as

$$v(\xi) = \frac{3}{2a} - \frac{3\mu}{2a\lambda\left(A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)} + \frac{3\sqrt{-\lambda(-\mu^2-\lambda^2\sigma)}\left(A_1\cosh(\xi\sqrt{-\lambda}) + A_2\sinh(\xi\sqrt{-\lambda})\right)}{2a\lambda^{3/2}\left(A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)^2} - \frac{3\left(A_1\cosh(\xi\sqrt{-\lambda}) + A_2\sinh(\xi\sqrt{-\lambda})\right)^2}{2a\left(A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)^2} \tag{29}$$

where
$\sigma = A_1^2 - A_2^2$
and
$$\xi = \frac{1}{\sqrt{b\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{t^\alpha}{2\,\Gamma(1+\alpha)}\right).$$

Particularly, if we put $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$ in eq. (29), the soliton solution is

$$v(\xi) = \frac{3}{2a}\left(1 + \sqrt{\sigma}\tanh\left(\xi\sqrt{-\lambda}\right)\operatorname{sech}\left(\xi\sqrt{-\lambda}\right) - \left(\tanh\left(\xi\sqrt{-\lambda}\right)\right)^2\right). \tag{30}$$

But if we take $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$ then the solution becomes

$$v(\xi) = \frac{3}{2a}\left(1 + \sqrt{\sigma}\coth\left(\xi\sqrt{-\lambda}\right)\operatorname{csch}\left(\xi\sqrt{-\lambda}\right) - \left(\coth\left(\xi\sqrt{-\lambda}\right)\right)^2\right). \tag{31}$$

Set 2:

$$a_0 = -\frac{1}{a},\quad a_1 = 0,\quad a_2 = -\frac{3}{2a\lambda},\quad b_1 = \frac{3\mu}{2a\lambda},$$
$$b_2 = \pm\frac{3\sqrt{-\mu^2-\lambda^2\sigma}}{2a\lambda^{3/2}},$$
$$k = -\frac{i}{\sqrt{b}\sqrt{\lambda}},\quad c = -\frac{i}{2\sqrt{b}\sqrt{\lambda}}. \tag{32}$$

Substituting these values into eq. (27), the solution of eq. (1) takes the form

$$v(\xi) = -\frac{1}{a} + \frac{3\mu}{2a\lambda(A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda})} + \frac{3\sqrt{-\lambda(-\mu^2-\lambda^2\sigma)}\left(A_1\cosh(\xi\sqrt{-\lambda}) + A_2\sinh(\xi\sqrt{-\lambda})\right)}{2a\lambda^{3/2}\left(A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)^2} + \frac{3\left(A_1\cosh(\xi\sqrt{-\lambda}) + A_2\sinh(\xi\sqrt{-\lambda})\right)^2}{2a\left(A_1\sinh(\xi\sqrt{-\lambda}) + A_2\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)^2} \tag{33}$$

where

$\sigma = A_1^2 - A_2^2$

and

$$\xi = -\frac{i}{\sqrt{b\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{t^\alpha}{2\,\Gamma(1+\alpha)}\right).$$

For favourable values, set $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$ in eq. (33), and the solution yields

$$v(\xi) = \frac{1}{a}\left(\frac{3\sqrt{\lambda^3\sigma}}{2\lambda^{3/2}}\tanh\left(\xi\sqrt{-\lambda}\right)\operatorname{sech}\left(\xi\sqrt{-\lambda}\right) + \frac{3}{2}\left(\tanh\left(\xi\sqrt{-\lambda}\right)\right)^2 - 1\right). \quad (34)$$

Also if we input $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$, we obtain

$$v(\xi) = \frac{1}{a}\left(\frac{3\sqrt{\lambda^3\sigma}}{2\lambda^{3/2}}\coth\left(\xi\sqrt{-\lambda}\right)\operatorname{csch}\left(\xi\sqrt{-\lambda}\right) + \frac{3}{2}\left(\coth\left(\xi\sqrt{-\lambda}\right)\right)^2 - 1\right). \quad (35)$$

Set 3.

$$a_0 = \frac{2bk^2\lambda}{a\left(1-bk^2\lambda\right)}, \quad a_1 = 0, \quad a_2 = \frac{3bk^2}{a\left(1-bk^2\lambda\right)},$$

$$b_1 = -\frac{3bk^2\mu}{a\left(1-bk^2\lambda\right)}, \quad b_2 = \frac{3ibk^2\sqrt{\mu^2+\lambda^2\sigma}}{\sqrt{\lambda}\left(\pm a \mp abk^2\lambda\right)}$$

and

$$c = \frac{k}{1-bk^2\lambda}. \quad (36)$$

By using these values into eq. (27), we derive the exact solution of eq. (1) as

$$v(\xi) = -\frac{2bk^2\lambda}{a\left(bk^2\lambda-1\right)} - \frac{3bk^2\sqrt{\left(\mu^2+\lambda^2\sigma\right)}\left(A_1\cosh\left(\xi\sqrt{-\lambda}\,\right) + A_2\sinh\left(\xi\sqrt{-\lambda}\,\right)\right)}{a\left(1-bk^2\lambda\right)\left(A_1\sinh\left(\xi\sqrt{-\lambda}\,\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\,\right) + \frac{\mu}{\lambda}\right)^2} + \frac{3bk^2}{a\left(bk^2\lambda-1\right)} \times\left(\frac{\mu}{\left(A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}\right)} + \frac{\lambda\left(A_1\cosh\left(\xi\sqrt{-\lambda}\right) + A_2\sinh\left(\xi\sqrt{-\lambda}\right)\right)^2}{\left(A_1\sinh\left(\xi\sqrt{-\lambda}\right) + A_2\cosh\left(\xi\sqrt{-\lambda}\right) + \frac{\mu}{\lambda}\right)^2}\right), \quad (37)$$

where

$\sigma = A_1^2 - A_2^2$

and

$$\xi = k\left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{t^\alpha}{\left(1-bk^2\lambda\right)\Gamma(1+\alpha)}\right).$$

For special values, put $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$,

$$v(\xi) = \frac{bk^2\lambda}{a\left(1-bk^2\lambda\right)}\left(2 - 3\sqrt{\sigma}\,\tanh\left(\xi\sqrt{-\lambda}\right)\times\operatorname{sech}\left(\xi\sqrt{-\lambda}\right) - 3\left(\tanh\left(\xi\sqrt{-\lambda}\right)\right)^2\right), \quad (38)$$

and if $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$ then

$$v(\xi) = \frac{bk^2\lambda}{a\left(1-bk^2\lambda\right)}\left(2 - 3\sqrt{\sigma}\coth\left(\xi\sqrt{-\lambda}\right)\times\operatorname{csch}\left(\xi\sqrt{-\lambda}\right) - 3\left(\coth\left(\xi\sqrt{-\lambda}\right)\right)^2\right). \quad (39)$$

Annotation 2: When $\lambda > 0$ (Trigonometric function solutions)

In a similar manner, as mentioned in Annotation 1, putting the values of $v(\xi)$ from eq. (27) into eq. (26) along with eqs. (11) and (15), the left-hand side of eq. (26) becomes a polynomial in ϕ and ψ. Taking each coefficient of the polynomial equal to zero, we obtain a system of algebraic equations (for the sake of simplicity, the equations are not presented here) for $a_0, a_1, a_2, b_1, b_2, \mu, \lambda$ and σ. Solving this system of equations we obtain the following sets of results:

Set 1.

$$a_0 = \frac{3}{2a}, \quad a_1 = 0, \quad a_2 = \frac{3}{2a\lambda},$$

$$b_1 = -\frac{3\mu}{2a\lambda}, \quad b_2 = \pm\frac{3\sqrt{-\mu^2+\lambda^2\sigma}}{2a\lambda^{\frac{3}{2}}},$$

$$k = \frac{1}{\sqrt{b}\sqrt{\lambda}}, \quad c = \frac{1}{2\sqrt{b}\sqrt{\lambda}}. \quad (40)$$

Therefore, in this result the exact solution of eq. (1) transforms as

$$v(\xi)=\frac{3}{2a}-\frac{3\mu}{2a\lambda\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)}+\frac{3\sqrt{\left(-\mu^2+\lambda^2\sigma\right)}\left(A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)\right)}{2a\lambda\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)^2}+\frac{3\left(A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)\right)^2}{2a\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)^2}, \quad (41)$$

where

$$\sigma=A_1^2+A_2^2$$

and

$$\xi=\frac{1}{\sqrt{b\lambda}}\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{t^\alpha}{2\,\Gamma(1+\alpha)}\right).$$

By setting $A_1=0$, $\mu=0$ and $A_2\neq 0$, the periodic solution becomes

$$v(\xi)=\frac{3}{2a}\left(\left(\tan\left(\xi\sqrt{\lambda}\right)\right)^2-\sqrt{\sigma}\tan\left(\xi\sqrt{\lambda}\right)\sec\left(\xi\sqrt{\lambda}\right)+1\right). \quad (42)$$

Besides, for $A_2=0$, $\mu=0$ and $A_1\neq 0$, we have

$$v(\xi)=\frac{3}{2a}\left(\left(\cot\left(\xi\sqrt{\lambda}\right)\right)^2+\sqrt{\sigma}\cot\left(\xi\sqrt{\lambda}\right)\csc\left(\xi\sqrt{\lambda}\right)+1\right). \quad (43)$$

Set 2.

$$a_0=-\frac{2bk^2\lambda}{a\left(bk^2\lambda-1\right)},\quad a_1=0,\quad a_2=-\frac{3bk^2}{a\left(bk^2\lambda-1\right)},$$

$$b_1=\frac{3bk^2\mu}{a\left(bk^2\lambda-1\right)},\ b_2=\frac{3ibk^2\sqrt{\mu^2-\lambda^2\sigma}}{\sqrt{\lambda}(\pm a\mp abk^2\lambda)}$$

and

$$c=-\frac{k}{bk^2\lambda-1}. \quad (44)$$

Hence, the exact travelling wave solution of eq. (1) is

$$v(\xi)=-\frac{2bk^2\lambda}{a\left(bk^2\lambda-1\right)}+\frac{3bk^2\,\mu}{a\left(bk^2\lambda-1\right)\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)}+\frac{3ibk^2\sqrt{\left(\mu^2-\lambda^2\sigma\right)}\left(A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)\right)}{a\left(1-bk^2\lambda\right)\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)^2}-\frac{3bk^2\lambda\left(A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)\right)^2}{a\left(bk^2\lambda-1\right)\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)^2} \quad (45)$$

where

$$\sigma=A_1^2+A_2^2$$

and

$$\xi=k\left(\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{t^\alpha}{\left(bk^2\lambda-1\right)\Gamma(1+\alpha)}\right).$$

In particular, if we input $A_1=0$, $\mu=0$ and $A_2\neq 0$, we achieve the solution of the following form:

$$v(\xi)=\frac{bk^2\lambda}{a\left(1-bk^2\lambda\right)}\left(3\left(\tan\left(\xi\sqrt{\lambda}\right)\right)^2+3\sqrt{\sigma}\tan\left(\xi\sqrt{\lambda}\right)\sec\left(\xi\sqrt{\lambda}\right)+2\right). \quad (46)$$

Also, for $A_2=0$, $\mu=0$ and $A_1\neq 0$,

$$v(\xi)=\frac{bk^2\lambda}{a\left(1-bk^2\lambda\right)}\left(3\left(\cot\left(\xi\sqrt{\lambda}\right)\right)^2-3\sqrt{\sigma}\cot\left(\xi\sqrt{\lambda}\right)\csc\left(\xi\sqrt{\lambda}\right)+2\right). \quad (47)$$

Annotation 3. When $\lambda=0$ (Rational function solution)

Using similar process, as mentioned in Annotation 1, substituting the values of $v(\xi)$ from eq. (27), into eq. (26), along with eqs. (11) and (17), the left-hand side of eq. (26) becomes a polynomial in ϕ and ψ. Taking each coefficient of the polynomial equal to zero, a system of algebraic equations is obtained (for the sake of simplicity, the equations are not represent here) for a_0, a_1, a_2, b_1, b_2, μ, λ and σ. Solving this system of equations, the following results are obtained:

Set 1.

$$a_0=0,\quad a_1=0,\quad a_2=\frac{3bk^2}{a},\quad b_1=-\frac{3bk^2\mu}{a},$$

$$b_2=\mp\frac{3bk^2\sqrt{A_1^2-2\mu A_2}}{a}$$

and

$$c=k. \quad (48)$$

Now, for the above values the new exact wave solution of eq. (1) is

$$v(\xi)=\frac{6bk^2\left(2A_1^2+2A_2\left(\mu\xi-\sqrt{A_1^2-2\mu A_2}\right)+\mu\left(\xi\left(\mu\xi-2\sqrt{A_1^2-2\mu A_2}\right)-2A_2\right)\right)}{a\left(\mu\xi^2+2\xi A_1+2A_2\right)^2}, \tag{49}$$

where

$$\xi=k\left(\frac{x^\alpha}{\Gamma(1+\alpha)}-\frac{t^\alpha}{\Gamma(1+\alpha)}\right).$$

Specifically, if we take $A_1=0$, $\mu=-1$ and $A_2\neq 0$, the solution is as follows:

$$v(\xi)=\frac{3bk^2}{a^2\left(1-\frac{\xi^2}{2}\right)^2}\left(\xi^2+\xi\sqrt{2}-a\left(1-\frac{\xi^2}{2}\right)\right). \tag{50}$$

On the other hand, $A_2=0$, $\mu=0$ and $A_1\neq 0$, then the kink-type solution is

$$v(\xi)=-\frac{3bk^2}{a\xi}. \tag{51}$$

4.2 *Space–time fractional foam drainage equation*

The space–time fractional foam drainage equation is a model of the flow of liquid through the channels and nodes between the bubbles, driven by gravity and capillarity. This equation is also stated as the model of waves on shallow water surface, ion-acoustic waves in plasma and the waves on foam may be illustrated in the field of a nonlinear PDE for the foam density as a function of time and vertical position [30]. For solving eq. (2), we envisage the travelling wave transformation

$$\xi=k\left(x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\quad\text{and}\quad\eta=mx-\frac{wt^\alpha}{\Gamma(1+\alpha)}. \tag{52}$$

By means of transformation (52), eq. (2) is converted into the following form:

$$2kw'-m^2ww''+4mw^2w'-2m^2\left(w'\right)^2=0. \tag{53}$$

Again using the condition $w=v^{-1}$, then eq. (53) reduces to the following nonlinear ODE:

$$2kv^2v'+4m^2\left(v'\right)^2-m^2vv''+4mv'=0. \tag{54}$$

By the homogeneous balance principle, balancing between the highest-order derivative with highest-order nonlinear term in eq. (54), we achieve the value of integer $M=1$. Hereafter, eq. (21) becomes

$$v(\xi)=a_0+a_1\phi+b_1\psi, \tag{55}$$

where a_0, a_1 and b_1 are constants to be determined later.

Annotation 1: When $\lambda<0$ (Hyperbolic function solutions)

Substituting the value of $v(\xi)$ and its derivatives from eq. (53) into eq. (54), in cohesion with eqs (11) and (13), the left-hand side of eq. (54) turns out to be a polynomial in ϕ and ψ. Equating each coefficient of different powers to zero, we obtain a system of algebraic equations (for the sake of brevity, the equations are not given here) for a_0, a_1, b_1, μ, λ and σ. Solving these systems by any computer program like Mathematica, we have several types of results as follows:

$$a_0=0,\quad a_1=\frac{2}{m\lambda},$$

$$b_1=\pm\frac{2\sqrt{-\mu^2-\lambda^2\sigma}}{m\lambda^{3/2}}\quad\text{and}\quad k=\frac{m^3\lambda}{4}. \tag{56}$$

Substituting these results into eq. (55), we get the exact solution of eq. (2) in the following form:

$$v(\xi)=\frac{2\sqrt{-(\mu^2+\lambda^2\sigma)}}{m\lambda^{3/2}\left(A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)+\frac{\mu}{\lambda}\right)}+\frac{2\sqrt{-\lambda}\left(A_1\cosh\left(\xi\sqrt{-\lambda}\right)+A_2\sinh\left(\xi\sqrt{-\lambda}\right)\right)}{m\lambda\left(A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)+\frac{\mu}{\lambda}\right)}. \tag{57}$$

According to the condition, $w=v^{-1}$ and simplifying solution (57), we get

$$w(\xi)=\frac{m\lambda\left(A_1\sinh\left(\xi\sqrt{-\lambda}\right)+A_2\cosh\left(\xi\sqrt{-\lambda}\right)+\frac{\mu}{\lambda}\right)}{2\left(\sqrt{-\lambda}\left(A_1\cosh\left(\xi\sqrt{-\lambda}\right)+A_2\sinh\left(\xi\sqrt{-\lambda}\right)\right)+\frac{\sqrt{-(\mu^2+\lambda^2\sigma)}}{\sqrt{\lambda}}\right)}, \tag{58}$$

where

$\sigma = A_1^2 - A_2^2$

and

$\xi = \frac{m^3\lambda}{4}\left(x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right).$

When $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$, the solitary wave solution is

$$w(\xi) = \frac{m\sqrt{\lambda}}{2i\left(\tanh(\xi\sqrt{-\lambda}) + \sqrt{\sigma}\operatorname{sech}(\xi\sqrt{-\lambda})\right)}. \quad (59)$$

and when $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$, we obtain

$$w(\xi) = \frac{m\sqrt{\lambda}}{2i\left(\coth(\xi\sqrt{-\lambda}) + \sqrt{\sigma}\operatorname{csch}(\xi\sqrt{-\lambda})\right)}. \quad (60)$$

Annotation 2. When $\lambda > 0$ (Trigonometric function solutions)

In a similar procedure, putting the value of $v(\xi)$ and its derivatives from eq. (55) into eq. (54), along with eqs (11) and (15) and then solving the sets of algebraic equations, the following solutions will be obtained:

$$a_0 = 0, \quad a_1 = \frac{2}{m\lambda}, \quad b_1 = \mp\frac{2\sqrt{-\mu^2 + \lambda^2\sigma}}{m\lambda^{3/2}}$$

and

$$k = \frac{m^3\lambda}{4}. \quad (61)$$

Hence, the solution of eq. (2) is as follows:

$$v(\xi) = -\frac{2\sqrt{-\mu^2+\lambda^2\sigma}}{m\lambda^{3/2}\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)} + \frac{2\left(A_1\cos\left(\xi\sqrt{\lambda}\right) - A_2\sin\left(\xi\sqrt{\lambda}\right)\right)}{m\sqrt{\lambda}\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)}. \quad (62)$$

Using the condition $w = v^{-1}$, eq. (62) takes the following form:

$$w(\xi) = \frac{m\sqrt{\lambda}\left(A_1\sin\left(\xi\sqrt{\lambda}\right) + A_2\cos\left(\xi\sqrt{\lambda}\right) + \frac{\mu}{\lambda}\right)}{2\left(\left(A_1\cos\left(\xi\sqrt{\lambda}\right) - A_2\sin\left(\xi\sqrt{\lambda}\right)\right) - \frac{\sqrt{-\mu^2+\lambda^2\sigma}}{\lambda}\right)}, \quad (63)$$

where

$\sigma = A_1^2 + A_2^2$

and

$\xi = \frac{m^3\lambda}{4}\left(x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right).$

Particularly, if we select $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$, the periodic solution is

$$w(\xi) = -\frac{m\sqrt{\lambda}}{2\left(\tan\left(\xi\sqrt{\lambda}\right) + \sqrt{\sigma}\sec\left(\xi\sqrt{\lambda}\right)\right)}. \quad (64)$$

For $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$,

$$w(\xi) = \frac{m\sqrt{\lambda}}{2\left(\cot\left(\xi\sqrt{\lambda}\right) - \sqrt{\sigma}\csc\left(\xi\sqrt{\lambda}\right)\right)}. \quad (65)$$

4.3 *Space–time nonlinear fractional SRLW equation*

In this subsection, the space–time fractional SRLW equation will be solved by treating $(G'/G, 1/G)$-expansion method and various types of periodic and solitary wave solutions are acquired. For this purpose, we conceive the following transformation:

$$v(x,t) = v(\xi), \;\; \xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{ct^\alpha}{\Gamma(1+\alpha)} + \xi_0, \quad (66)$$

where k, c and ξ_0 are constants along with $k, c \neq 0$. Using the above transformation, the nonlinear ODE form of eq. (3) is

$$2k^2c^2v'' + 2\left(k^2 + c^2\right)v + kcv^2 = 0. \quad (67)$$

By using the homogeneous balance principle, balancing the highest-order derivative v'' with the highest-order nonlinear term v^2, balance number $M = 2$ is obtained. Hence the solution formula is

$$v(\xi) = a_0 + a_1\phi + a_2\phi^2 + b_1\psi + b_2\phi\psi, \quad (68)$$

where a_0, a_1, a_2, b_1 and b_2 are constants to be discussed later.

Annotation 1. When $\lambda < 0$ (Hyperbolic function solutions)

Proceeding in a similar way as stated in §4.1 (Annotation 1), the following results will be arrived:

Result 1.

$$a_0 = \frac{4k^2\lambda\sqrt{-(1+k^2\lambda)}}{1+k^2\lambda}, \quad a_1 = 0,$$

$$a_2 = -\frac{6k^2}{\sqrt{-(1+k^2\lambda)}},$$

$$b_1 = \frac{6k^2\mu}{\sqrt{-(1+k^2\lambda)}}, \quad b_2 = \pm\frac{6\sqrt{k^4\mu^2+k^4\lambda^2\sigma}}{\sqrt{\lambda+k^2\lambda^2}},$$

$$c = \frac{k}{\sqrt{-(1+k^2\lambda)}}. \tag{69}$$

Plugging these results into eq. (68), we achieve the exact travelling wave solution of eq. (3) as

$$v(\xi) = \frac{4ik^2\lambda}{\sqrt{(1+k^2\lambda)}} + \frac{6k^2\mu}{\sqrt{-(1+k^2\lambda)}\left(A_1\,\sinh(\xi\sqrt{-\lambda}) + A_2\,\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)}$$
$$+\frac{6\sqrt{-\lambda(k^4\mu^2+k^4\lambda^2\sigma)}\,\left(A_1\,\cosh(\xi\sqrt{-\lambda}) + A_2\,\sinh(\xi\sqrt{-\lambda})\right)}{\sqrt{\lambda(1+k^2\lambda)}\left(A_1\,\sinh(\xi\sqrt{-\lambda}) + A_2\,\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)^2}$$
$$+\frac{6k^2\lambda\left(A_1\,\cosh(\xi\sqrt{-\lambda}) + A_2\,\sinh(\xi\sqrt{-\lambda})\right)^2}{\sqrt{-(1+k^2\lambda)}\left(A_1\,\sinh(\xi\sqrt{-\lambda}) + A_2\,\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)^2}, \tag{70}$$

where

$$\sigma = A_1^2 - A_2^2$$

and

$$\xi = \frac{kx^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\sqrt{-(1+k^2\lambda)}\Gamma(1+\alpha)} + \xi_0.$$

If we take $A_1 = 0$, $\mu = 0$ and $A_2 \neq 0$, we come up with the periodic solution

$$v(\xi) = \frac{ik^2\lambda}{\sqrt{(1+k^2\lambda)}}\Big(4+6\sqrt{\sigma}\tanh\big(\xi\sqrt{-\lambda}\big)$$
$$\times\operatorname{sech}\big(\xi\sqrt{-\lambda}\big) - 6\big(\tanh\big(\xi\sqrt{-\lambda}\big)\big)^2\Big). \tag{71}$$

Consequently, for $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$, we obtain

$$v(\xi) = \frac{ik^2\lambda}{\sqrt{(1+k^2\lambda)}}\Big(4+6\sqrt{\sigma}\coth\big(\xi\sqrt{-\lambda}\big)$$
$$\times\operatorname{csch}\big(\xi\sqrt{-\lambda}\big) - 6\big(\coth\big(\xi\sqrt{-\lambda}\big)\big)^2\Big). \tag{72}$$

Result 2.

$$a_0 = 2i\sqrt{2}, \quad a_1 = 0, \quad a_2 = \frac{3i\sqrt{2}}{\lambda},$$

$$b_1 = -\frac{3i\sqrt{2}\mu}{\lambda}, \quad b_2 = \pm\frac{3\sqrt{2}\sqrt{\mu^2+\lambda^2\sigma}}{\lambda^{3/2}},$$

$$k = -\frac{1}{\sqrt{\lambda}}, \quad c = \frac{i}{\sqrt{2}\sqrt{\lambda}}. \tag{73}$$

Therefore, the exact solution of eq. (3) reduces to

$$v(\xi) = 2i\sqrt{2} - \frac{3i\sqrt{2}\mu}{\lambda\left(A_1\,\sinh(\xi\sqrt{-\lambda}) + A_2\,\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)}$$
$$+\frac{3\sqrt{2}\sqrt{-\lambda(\mu^2+\lambda^2\sigma)}\left(A_1\,\cosh(\xi\sqrt{-\lambda}) + A_2\,\sinh(\xi\sqrt{-\lambda})\right)}{\lambda^{\frac{3}{2}}\left(A_1\,\sinh(\xi\sqrt{-\lambda}) + A_2\,\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)^2}$$
$$-\frac{3i\sqrt{2}\left(A_1\,\cosh(\xi\sqrt{-\lambda}) + A_2\,\sinh(\xi\sqrt{-\lambda})\right)^2}{\left(A_1\,\sinh(\xi\sqrt{-\lambda}) + A_2\,\cosh(\xi\sqrt{-\lambda}) + \frac{\mu}{\lambda}\right)^2}, \tag{74}$$

where

$$\sigma = A_1^2 - A_2^2$$

and

$$\xi = -\frac{x^\alpha}{\sqrt{\lambda}\,\Gamma(1+\alpha)} + \frac{it^\alpha}{\sqrt{2\lambda}\,\Gamma(1+\alpha)} + \xi_0.$$

Specifically, for $A_1=0$, $\mu=0$ and $A_2\neq 0$, we obtain the solitary wave solution as

$$v(\xi)=\sqrt{2}i\left(2+3\sqrt{\sigma}\tanh\left(\xi\sqrt{-\lambda}\right)\operatorname{sech}\left(\xi\sqrt{-\lambda}\right)\right.$$
$$\left.-3\left(\tanh\left(\xi\sqrt{-\lambda}\right)\right)^2\right). \tag{75}$$

Likewise, if we set $A_2=0$, $\mu=0$ and $A_1\neq 0$, we obtain

$$v(\xi)=\sqrt{2}i\left(2+3\sqrt{\sigma}\coth\left(\xi\sqrt{-\lambda}\right)\operatorname{csch}\left(\xi\sqrt{-\lambda}\right)\right.$$
$$\left.-3\left(\coth\left(\xi\sqrt{-\lambda}\right)\right)^2\right). \tag{76}$$

Annotation 2. When $\lambda>0$ (Trigonometric function solutions)

Following the uniform procedure which is described in §4.1 (Annotation 2), the following results will be obtained:

Result 1.

$$a_0=\frac{4k^2\lambda}{\sqrt{-(1+k^2\lambda)}},\quad a_1=0,\quad a_2=\frac{6k^2}{\sqrt{-(1+k^2\lambda)}},$$
$$b_1=-\frac{6k^2\mu}{\sqrt{-(1+k^2\lambda)}},\quad b_2=\mp\frac{6\sqrt{k^4\mu^2-k^4\lambda^2\sigma}}{\sqrt{\lambda+k^2\lambda^2}},$$
$$c=-\frac{k}{\sqrt{-(1+k^2\lambda)}}. \tag{77}$$

Hence, in this result the trigonometric function solution of eq. (3) is

$$v(\xi)=\frac{4k^2\lambda}{\sqrt{-(1+k^2\lambda)}}$$
$$-\frac{6k^2\mu}{\sqrt{-(1+k^2\lambda)}\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)}$$
$$-\frac{6\sqrt{\lambda\left(k^4\mu^2-k^4\lambda^2\sigma\right)}\left(A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)\right)}{\sqrt{\lambda(1+k^2\lambda)}\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)^2}$$
$$+\frac{6k^2\lambda\left(A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sinh\left(\xi\sqrt{\lambda}\right)\right)^2}{\sqrt{-(1+k^2\lambda)}\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)^2}, \tag{78}$$

where

$\sigma=A_1^2+A_2^2$

and

$$\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}-\frac{kt^\alpha}{\sqrt{-(1+k^2\lambda)}\Gamma(1+\alpha)}+\xi_0.$$

In particular, if we put $A_1=0$, $\mu=0$ and $A_2\neq 0$, the obtained periodic solution is

$$v(\xi)=\frac{k^2\lambda}{\sqrt{-(1+k^2\lambda)}}\left(4-6\sqrt{\sigma}\tan\left(\xi\sqrt{\lambda}\right)\sec\left(\xi\sqrt{\lambda}\right)\right.$$
$$\left.+6\left(\tan\left(\xi\sqrt{\lambda}\right)\right)^2\right). \tag{79}$$

If $A_2=0$, $\mu=0$ and $A_1\neq 0$ in eq. (78),

$$v(\xi)=\frac{k^2\lambda}{\sqrt{-(1+k^2\lambda)}}\left(4+6\sqrt{\sigma}\cot\left(\xi\sqrt{\lambda}\right)\right.$$
$$\left.\times\csc\left(\xi\sqrt{\lambda}\right)+6\left(\cot\left(\xi\sqrt{\lambda}\right)\right)^2\right). \tag{80}$$

Result 2.

$$a_0=3i\sqrt{2},\quad a_1=0,\quad a_2=\frac{3i\sqrt{2}}{\lambda},\quad b_1=-\frac{3i\sqrt{2}\mu}{\lambda},$$
$$b_2=\pm\frac{3\sqrt{2}\sqrt{\mu^2-\lambda^2\sigma}}{\lambda^{3/2}},\quad k=-\frac{i}{\sqrt{\lambda}},\quad c=\frac{1}{\sqrt{2}\sqrt{\lambda}}. \tag{81}$$

Now, the exact solution of eq. (3) takes the following form:

$$v(\xi)=3i\sqrt{2}-\frac{3i\sqrt{2}\mu}{\lambda\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)}$$
$$+\frac{3\sqrt{2}\sqrt{(\mu^2-\lambda^2\sigma)}\left(A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)\right)}{\lambda\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)^2}$$
$$+\frac{3i\sqrt{2}\left(A_1\cos\left(\xi\sqrt{\lambda}\right)-A_2\sin\left(\xi\sqrt{\lambda}\right)\right)^2}{\left(A_1\sin\left(\xi\sqrt{\lambda}\right)+A_2\cos\left(\xi\sqrt{\lambda}\right)+\frac{\mu}{\lambda}\right)^2} \tag{82}$$

where

$\sigma=A_1^2+A_2^2$

and

$$\xi=-\frac{ix^\alpha}{\sqrt{\lambda}\Gamma(1+\alpha)}+\frac{t^\alpha}{\sqrt{2\lambda}\Gamma(1+\alpha)}+\xi_0.$$

More preciously, for $A_1=0$, $\mu=0$ and $A_2\neq 0$, the above solution transforms into the form

$$v(\xi)=i3\sqrt{2}\left(1-\sqrt{\sigma}\tan\left(\xi\sqrt{\lambda}\right)\sec\left(\xi\sqrt{\lambda}\right)\right.$$
$$\left.+\left(\tan\left(\xi\sqrt{\lambda}\right)\right)^2\right). \tag{83}$$

Also, if the $A_2=0$, $\mu=0$ and $A_1\neq 0$, we obtain

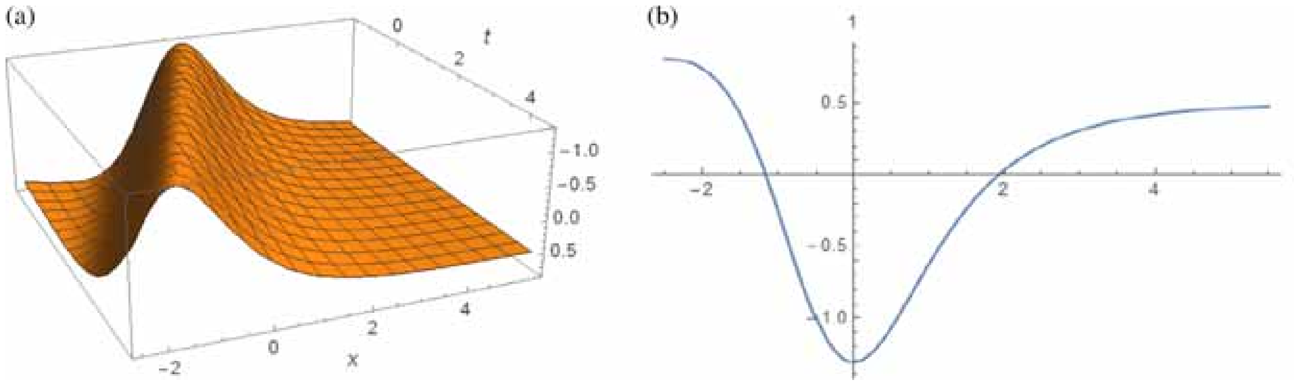

Figure 1. Shape of eq. (34) corresponding to the values (**a**) $b = 1.5$, $\lambda = -1$ and its projection at (**b**) $t = 1$.

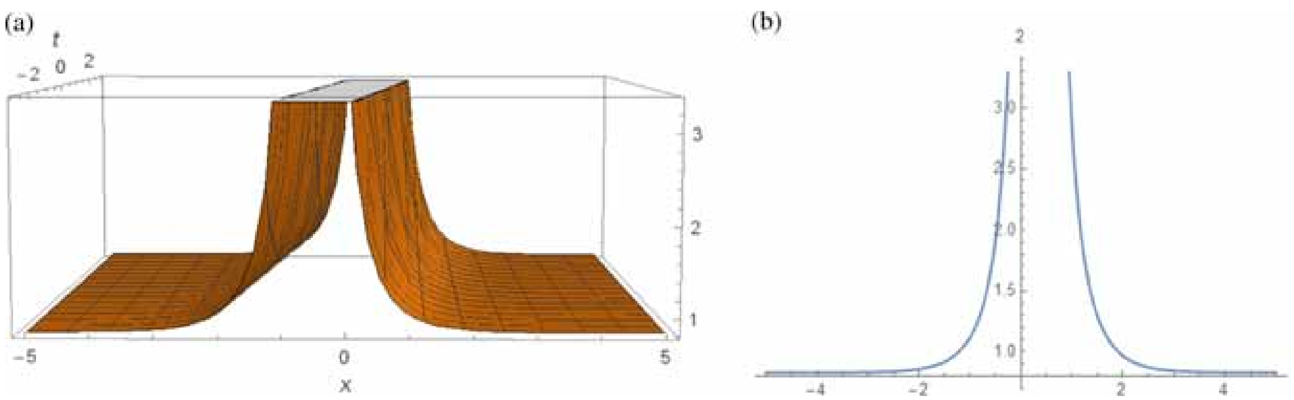

Figure 2. Shape of eq. (39) corresponding to the values (**a**) $b = 1$, $k = 2$, $a = 1$, $\lambda = -1.2$ and its projection at (**b**) $t = 2$.

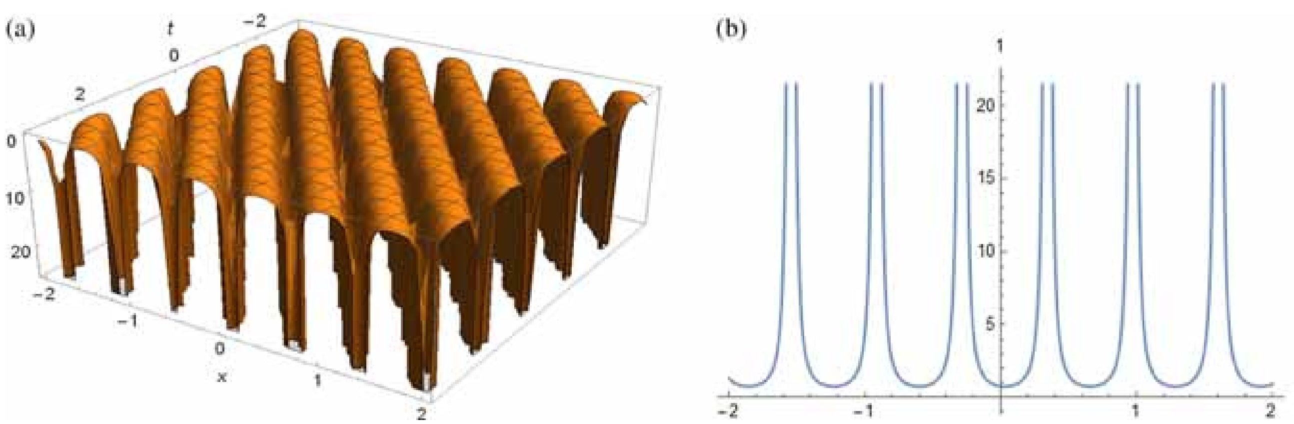

Figure 3. Shape of eq. (42) corresponding to the values (**a**) $b = 0.01$, $\lambda = 1$ and its projection at (**b**) $t = 1$.

$$v(\xi) = i3\sqrt{2}\left(1 + \sqrt{\sigma}\cot\left(\xi\sqrt{\lambda}\right)\csc\left(\xi\sqrt{\lambda}\right) + \left(\cot\left(\xi\sqrt{\lambda}\right)\right)^2\right). \tag{84}$$

Annotation 3. When $\lambda = 0$ (Rational function solution)

Taking the similar procedure described in §4.1 (Annotation 3), the following results will be obtained:

$$a_0 = 0, \quad a_1 = 0, \quad b_1 = -\mu a_2,$$

$$b_2 = \mp a_2\sqrt{A_1^2 - 2\mu A_2},$$

$$k = \pm\frac{(-1)^{1/4}\sqrt{a_2}}{\sqrt{6}},$$

$$c = \pm\frac{(-1)^{3/4}\sqrt{a_2}}{\sqrt{6}} \quad \text{and} \quad a_2 \text{ is a free parameter.} \tag{85}$$

Substituting these results into eq. (68) and simplifying, we come up with the solution of eq. (3) as

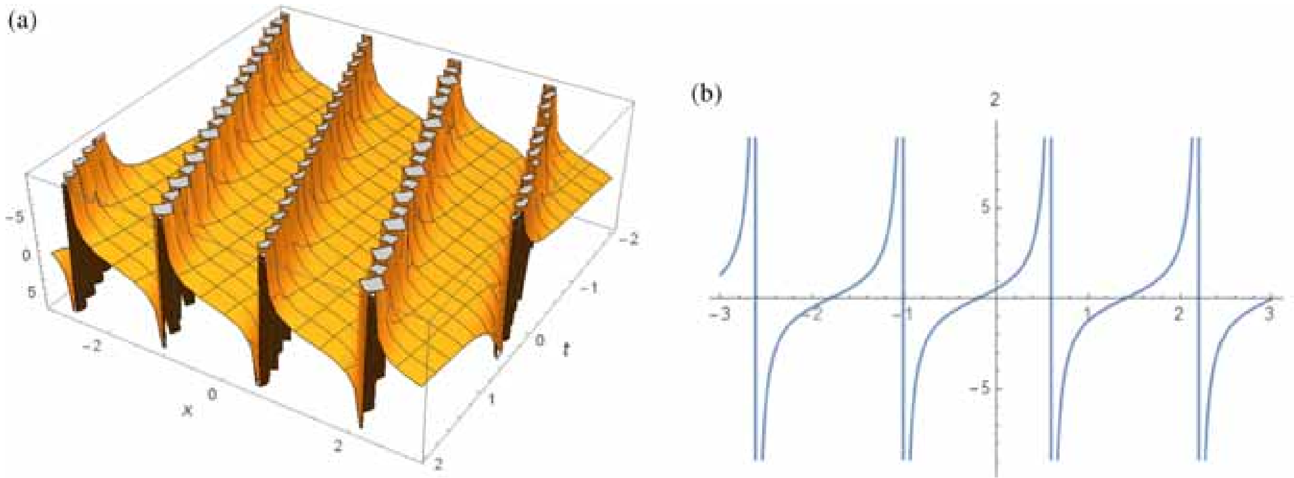

Figure 4. Shape of eq. (64) corresponding to the values (**a**) $m = 2.5$, $c = 0.5$, $\lambda = 1$ and its projection at (**b**) $t = 2$.

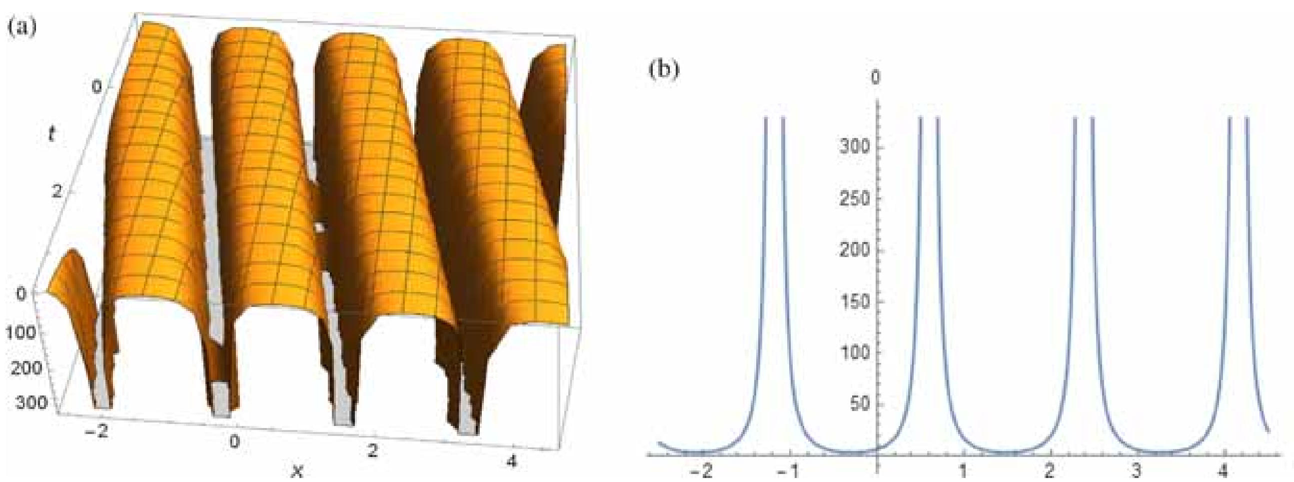

Figure 5. Shape of eq. (80) corresponding to the values (**a**) $k = -2.5$, $\xi_0 = 1.5$, $\lambda = 2$ and its projection at (**b**) $t = 0$.

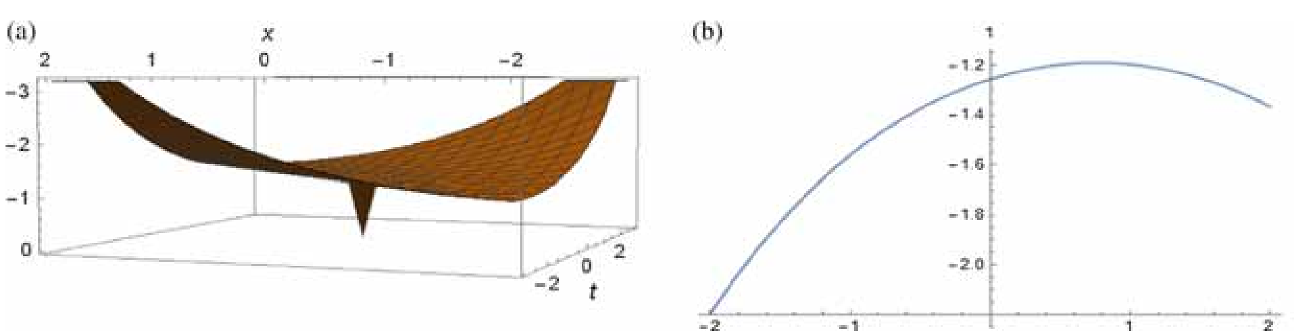

Figure 6. Shape of eq. (47) corresponding to the values (**a**) $b = -2.5$, $k = 0.5$, $a = 0.1$, $\lambda = 0.5$ and its projection at (**b**) $t = 1$.

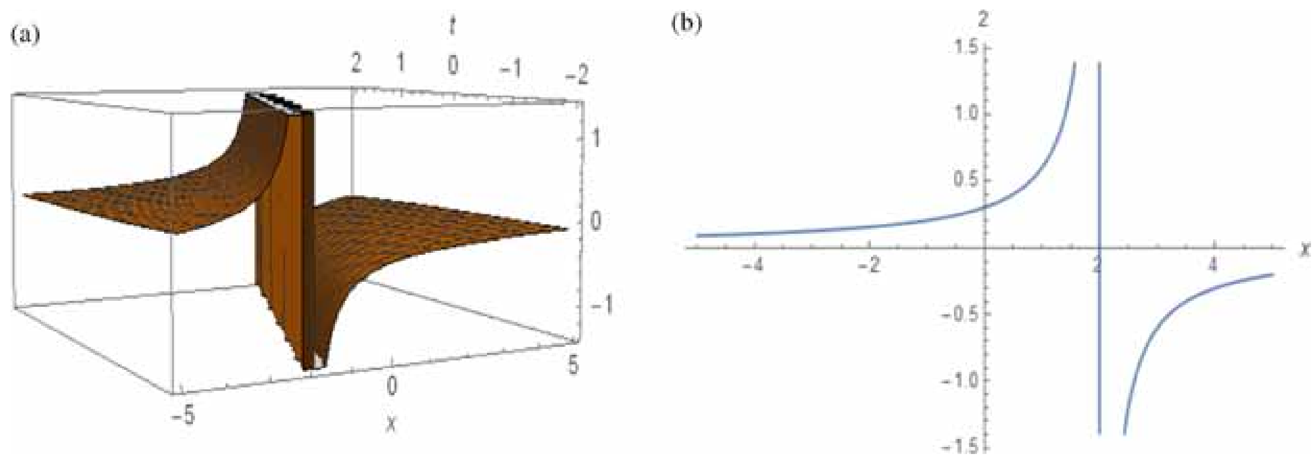

Figure 7. Shape of eq. (51) corresponding to the values (**a**) $b = -0.5$, $k = 0.02$, a $= -0.05$, $\lambda = 0.5$ and its projection at (**b**) $t = 2$.

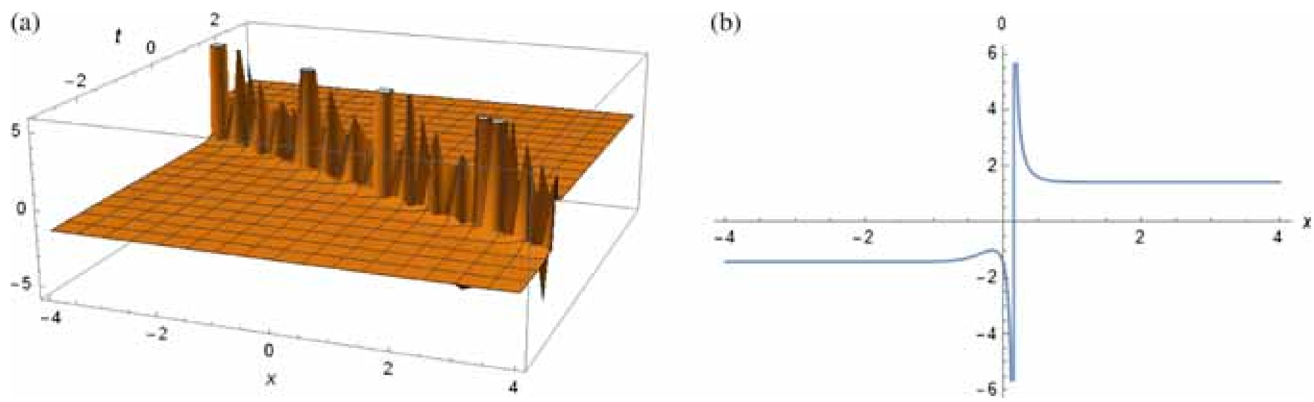

Figure 8. Shape of eq. (59) corresponding to the values (**a**) $m = -4$, $c = -2.5$, $\lambda = 0.5$ and its projection at (**b**) $t = 0$.

$$v(\xi) = \frac{2a_2\left(2A_1^2 + 2A_2\left(\mu\xi - \sqrt{A_1^2 - 2\mu A_2}\right) + \mu\left(\xi\left(\mu\xi - 2\sqrt{A_1^2 - 2\mu A_2}\right) - 2A_2\right)\right)}{\left(\mu\xi^2 + 2\xi A_1 + 2A_2\right)^2}, \tag{86}$$

where

$$\xi = \pm \frac{(-1)^{1/4}\sqrt{a_2}}{\sqrt{6}} \frac{x^\alpha}{\Gamma(1+\alpha)} \pm \frac{(-1)^{3/4}\sqrt{a_2}}{\sqrt{6}} \frac{t^\alpha}{\Gamma(1+\alpha)} + \xi_0.$$

Specifically, if $A_1 = 0$, $\mu = -1$ and $A_2 \neq 0$, the travelling wave solution is

$$v(\xi) = \frac{4a_2\left(1 + (\xi^2/2) - \xi\sqrt{2}\right)}{\left(\xi^2 - 2\right)^2}, \tag{87}$$

while $A_2 = 0$, $\mu = 0$ and $A_1 \neq 0$, yields

$$v(\xi) = \frac{2a_2\xi^2}{\left(\xi^2 + 2\xi\right)^2}. \tag{88}$$

Remark 1. All the above-mentioned solutions have been verified by substituting them back into the main equation via the symbolic computer software package program Mathematica and these solutions are found to be correct.

5. Graphical illustration of some solutions

In this section, the physical properties are explained and various shapes of the three-dimensional corresponding with two-dimensional graphical patterns are displayed for the acquired solutions of space–time fractional ZKBBM, foam drainage and SRLW equations. These solutions are of hyperbolic, trigonometric and rational

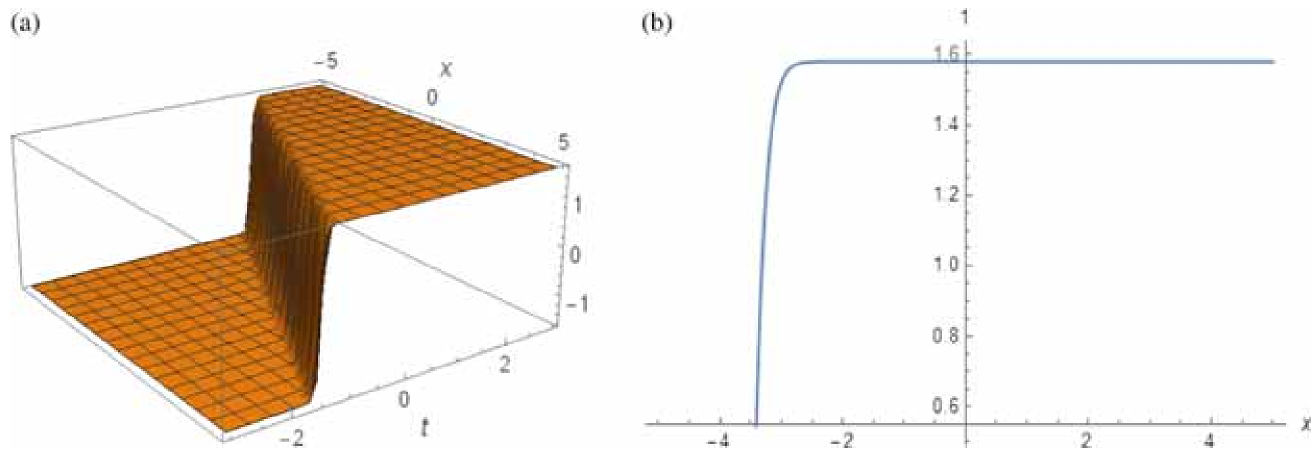

Figure 9. Shape of eq. (60) corresponding to the values (**a**) $m = -2$, $c = -3.5$, $\lambda = 2.5$ and its projection at (**b**) $t = 1$.

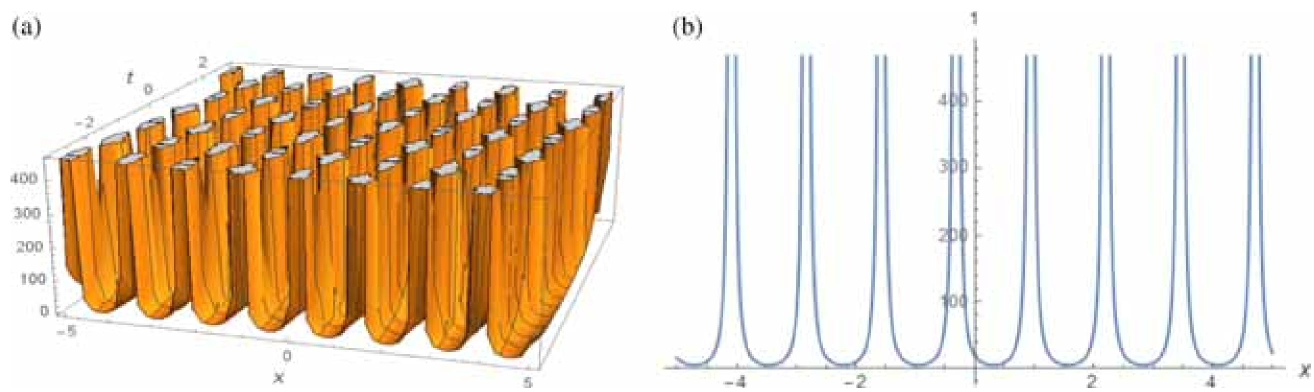

Figure 10. Shape of eq. (79) corresponding to the values (**a**) $k = 5$, $\xi_0 = 1$, $\lambda = 1$ and its projection at (**b**) $t = 1$.

Table 1. Comparison of our solution with the space–time fractional ZKBBM equation.

Mohyud-Din and Bibi [29] solution	Our solution
For $a_0 = -\frac{2bk^2\lambda\mu}{a(4bk^2\lambda\mu-1)}$, $a_1 = 0$, $a_2 = 0$, $b_1 = 0$, $b_2 = -\frac{6bk^2\lambda\mu}{a(4bk^2\lambda\mu-1)}$, $k = k$ and $c = -\frac{k}{4bk^2\lambda\mu-1}$, the hyperbolic solution of eq. (3.8) is $u_{12}(\xi) = -\frac{2bk^2\lambda\mu}{a(4bk^2\lambda\mu-1)} - \frac{6bk^2\lambda\mu}{a(4bk^2\lambda\mu-1)} \times \left(-\frac{\sqrt{\lvert\mu\lambda\rvert}}{\lambda}\left(\frac{C\sinh(2\sqrt{\lvert\mu\lambda\rvert}\xi)+C\cosh(2\sqrt{\lvert\mu\lambda\rvert}\xi)+D}{C\sinh(2\sqrt{\lvert\mu\lambda\rvert}\xi)+C\cosh(2\sqrt{\lvert\mu\lambda\rvert}\xi)-D}\right)\right)^{-2}.$	For $a_0 = \frac{3}{2a}$, $a_1 = 0$, $a_2 = \frac{3}{2a\lambda}$, $b_1 = -\frac{3\mu}{2a\lambda}$, $b_2 = \pm\frac{3\sqrt{-\mu^2-\lambda^2\sigma}}{2a\lambda^{3/2}}$, $k = \frac{1}{\sqrt{b}\sqrt{\lambda}}$, $c = \frac{1}{2\sqrt{b}\sqrt{\lambda}}$ the hyperbolic solution is $v(\xi) = \frac{3}{2a}\left(1 + \sqrt{\sigma}\tanh\left(\xi\sqrt{-\lambda}\right)\mathrm{sech}\left(\xi\sqrt{-\lambda}\right) - \left(\tanh\left(\xi\sqrt{-\lambda}\right)\right)^2\right)$ and $v(\xi) = \frac{3}{2a}\left(1 + \sqrt{\sigma}\coth\left(\xi\sqrt{-\lambda}\right)\mathrm{csch}\left(\xi\sqrt{-\lambda}\right) - \left(\coth\left(\xi\sqrt{-\lambda}\right)\right)^2\right).$

function solutions. Moreover, by assigning several free parameters, the exact solutions of these equations are converted into various wave shapes of kink, singular kink, periodic, soliton and compacton. Hereafter, the solution of eqs (34) and (39) shown in figures 1 and 2, depicts the soliton shape wave solution with $b = 1.5$, $\lambda = -1$ and $b = 1$, $k = 2$, $a = 1$ and $\lambda = -1.2$, within the interval $-2.5 \le x \le 5.5$, $-1.5 \le t \le 4.5$ and $-5 \le x \le 5$, $-3 \le t \le 3$ respectively. Taking different values of free parameters, the solutions of eqs (42), (64) and (80) are represented in figures 3–5, which describe the periodic solution within the interval

Table 2. Comparison of our solution with the space–time fractional foam drainage equation.

Islam and Akbar [30] solution	Our solution
For $a_0=0,\quad a_1=-\frac{A\psi}{m(4E\psi+B^2)},\quad b_1=-\frac{A}{4m\psi},\quad d=-\frac{B}{2\psi},$ $k=-\frac{m^3(4E\psi+B^2)}{4A^2},$ then the travelling wave solution of hyperbolic form is $w_1(x,t)=\frac{\left(\frac{\sqrt{\Omega}}{2\psi}r_2\ \coth\left(\frac{\sqrt{\Omega}}{2A}k\left(x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)}{-\frac{A\psi}{m(4E\psi+B^2)}\left(\frac{\sqrt{\Omega}}{2\psi}r_2\ \coth\left(\frac{\sqrt{\Omega}}{2A}k\left(x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^2-\frac{A}{4m\psi}}.$ Also the trigonometric solution is $w_3(x,t)=\frac{\left(\frac{\sqrt{-\Omega}}{2\psi}r_2\cot\left(\frac{\sqrt{-\Omega}}{2A}k\left(x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)}{-\frac{A\psi}{m(4E\psi+B^2)}\left(\frac{\sqrt{-\Omega}}{2\psi}r_2\ \cot\left(\frac{\sqrt{-\Omega}}{2A}k\left(x-\frac{ct^\alpha}{\Gamma(1+\alpha)}\right)\right)\right)^2-\frac{A}{4m\psi}}.$	For $a_0=0,\quad a_1=\frac{2}{m\lambda},\quad b_1=\pm\frac{2\sqrt{-\mu^2-\lambda^2\sigma}}{m\lambda^{\frac{3}{2}}}$ and $k=\frac{m^3\lambda}{4},$ the soliton wave solution is $w(\xi)=\frac{m\sqrt{\lambda}}{2i(\tanh(\xi\sqrt{-\lambda})+\sqrt{\sigma}\operatorname{sech}(\xi\sqrt{-\lambda}))}$ and $w(\xi)=\frac{m\sqrt{\lambda}}{2i\ (\coth(\xi\sqrt{-\lambda})+\sqrt{\sigma}\operatorname{csch}(\xi\sqrt{-\lambda}))}.$ Also the periodic wave solution is $w(\xi)=-\frac{m\sqrt{\lambda}}{2\left(\tan\left(\xi\sqrt{\lambda}\right)+\sqrt{\sigma}\sec\left(\xi\sqrt{\lambda}\right)\right)}$ and $w(\xi)=\frac{m\sqrt{\lambda}}{2\left(\cot\left(\xi\sqrt{\lambda}\right)-\sqrt{\sigma}\csc\left(\xi\sqrt{\lambda}\right)\right)}.$

Table 3. Comparison of our solution with the space–time nonlinear fractional SRLW equation.

Sonmezoglu [31] solution	Our solution
For $a_1=b_1=b_2=0,\quad a_2=-12ckm^2,$ $a_0=4ck\left(1+m^2+\sqrt{1-m^2+m^4}\right),$ where $c=i\sqrt{\frac{k^2}{1+4k^2\sqrt{1-m^2+m^4}}},$ the solution of eq. (42) takes the form $u_1(x,t)=4ck\left(1+m^2+\sqrt{1-m^2+m^4}\right)-12ck\mathrm{m}^2\mathrm{sn}^2 \times\left(\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{i}{\Gamma(1+\alpha)}\sqrt{\frac{k^2}{1+4k^2\sqrt{1-m^2+m^4}}}t^\alpha+\xi_0\right),$ where ξ_0 is an arbitrary constant.	For $a_0=0,\quad a_1=0,\quad b_1=-\mu a_2,$ $b_2=\mp a_2\sqrt{A_1^2-2\mu A_2},\quad k=\pm\frac{(-1)^{1/4}\sqrt{a_2}}{\sqrt{6}},$ $c=\pm\frac{(-1)^{3/4}\sqrt{a_2}}{\sqrt{6}},$ the rational function solution gives $v(\xi)=\frac{4a_2\left(1+\xi^2/2-\xi\sqrt{2}\right)}{(\xi^2-2)^2}$ and $v(\xi)=\frac{2a_2\xi^2}{(\xi^2+2\xi)^2},$

$-2\le x\le 2$, $-3\le t\le 3$; $-3\le x\le 3$, $-2\le t\le 2$ and $-2.5\le x\le 4.5$, $-1.5\le t\le 3.5$ respectively. Equation (47) designates the compacton shape solution for $b=-2.5$, $k=0.5$, $a=0.1$ and $\lambda=0.5$ within the interval $-2\le x\le 2$ and $-3\le t\le 3$, which is shown in figure 6. The solution of eq. (51) sketches the singular kink-type wave solution for $b=-0.5$, $k=0.02$, $a=-0.05$, $\lambda=0.5$, $-5\le x\le 5$ and $-2\le t\le 2$, in figure 7. On the other hand, figure 8 describes the multiple soliton solution with $m=-4$, $c=-2.5$, $\lambda=0.5$, $-4\le x\le 4$ and $-3\le t\le 3$. Also, the solution of eq. (60) depicted in figure 9, gives the kink-type wave solution with $m=-2$, $c=-3.5$, $\lambda=2.5$, $-5\le x\le 5$ and $-3\le t\le 3$. Figure 10 shows the shape of eq. (79) corresponding to $k=5$, $\xi_0=1$, $\lambda=1$ and $-4\le t\le 4$.

6. Results and discussion

The fundamental keystone of the advanced method called the two-variable $(G'/G, 1/G)$-expansion method is to improve new general and various existing wave solutions of eqs (1)–(3). In our obtained solutions, as the two parameters A_1 and A_2 receive various special values, the travelling wave solution converts to the solitary wave solution. When $\mu=0$ and $b_j=0$ in eqs (9) and

(21), the two-variable $(G'/G, 1/G)$-expansion method transforms into the original (G'/G)-expansion method. In ref. [29], the solution of ZKBBM equation has been studied and only 12 solutions are obtained, which is in the forms of trigonometric and hyperbolic function solutions. But utilising $\left(G'/G, 1/G\right)$-expansion method in this paper, we obtain 18 solutions and these are represented in the forms of hyperbolic, trigonometric and rational function solutions. Also, setting various particular values of the parameters, soliton, kink, periodic and compacton solutions of ZKBBM equation are found. Islam and Akbar [30] investigated the generalised (G'/G)-expansion method and obtained different travelling wave solutions of fractional foam drainage equation. But treating $(G'/G, 1/G)$-expansion method, we have gained more general solitary wave solutions, which have not been listed in previous literatures. Sonmezoglu [31] obtained only 8 solutions of fractional SRLW equation. On the other hand, applying advanced two-variable $\left(G'/G, 1/G\right)$-expansion method, we establish 15 solutions including periodic, soliton and singular kink solutions, which also have not been reported in previous literatures. We also compare some of our obtained solutions with [29–31] solutions, which will be furnished as follows (see tables 1–3):

Remark 2. From the above representation, it can be seen that all our obtained solutions are completely novel compared to the results obtained by other scholars.

7. Conclusion

The main reason for this work has been to successfully exploit two-variable $(G'/G, 1/G)$-expansion method for analysing new and more exact solitary wave solutions for a class of nonlinear space–time FDE, namely ZKBBM, foam drainage and SRLW equations. Also, by imposing various arbitrary values of free parameters, the solutions convert to different shapes of wave solutions such as soliton, periodic, compacton etc. which are illustrated graphically. We have also compared our obtained solutions with other well-known solutions existing in the literature. To the best our knowledge, we are assured that the solutions retain their uniformity upon interacting with others. The results show that the current method provides a powerful mathematical instrument that shorten the computational complication, and is efficient. In all respects, it is unavoidable to notice that this method can be more frequently applicable to solve different nonlinear FDE along with NLEEs which regularly emerge in mathematical physics, engineering and other technological arenas.

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