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Quantisation of particle motion in dissipative harmonic environment

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Abstract. In this work, the quantisation of particle propagating in a dissipative harmonic medium will be investigated using the creation and annihilation operator formalism, which is more appropriate in some fields of physics. Modelling the problem as damped harmonic oscillator, the equations of motion are then written in terms of Poisson brackets, and the Heisenberg equations are written in terms of the quantum counterpart of the Poisson bracket, known as commutators. The creation and annihilation operators are introduced and used to obtain the energy and eigenstates. Our results are in exact agreement with different quantisation approaches as in Serhan *et al*, *J. Math. Phys.* **59**, 082105 (2018). The normalisable coherent states are obtained as eigenstates of the annihilation operator, which overcome the non-normalisability of these states that appeared via the dual coordinate method.

Keywords. Dissipative system; Poisson bracket; creation and annihilation operators; damped quantum oscillator; coherent state; energy loss and dissipation.

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1. Introduction

One of the most important problems in the description of microscopic systems is the relation between their classical and quantum descriptions. The transition between classical and quantum is achieved by constructing the Hamiltonian and treating it as an operator. Then, the energy eigenvalues and eigenfunctions can be obtained. The quantisation of damped harmonic oscillator is of prime importance. Most of the physical systems are damping systems. In [1], the Hamiltonian of the damped harmonic oscillator starting from Bateman Lagrangian is constructed. The system is then quantised using the Wentzel–Kramers–Brillouin (WKB) approximation and the canonical quantisation. In [2], the coherent states, which are the eigenstates of the annihilation operator, of a damped harmonic oscillator is constructed with desirable properties. The states obtained are normalisable. In addition, the coherent states of a damped harmonic oscillator employing the method of dual coordinates are obtained by Ghosh and Hasse [3]. Their states are not normalisable. Besides, the time-dependent creation and annihilation operators are proposed in [2]. It is shown that the creation and annihilation operators satisfy the usual commutation relations and the Hamiltonian is written in terms of these operators. The quantum Bateman system for a damped harmonic oscillator has been analysed by many workers. It is considered as a simple dissipative model system (see [4–7]). Suzuki and Majima [8] studied the case of damped harmonic oscillator where the oscillator's mass changes with time. It is shown that the time-dependent mass plays the same role of the control parameter for damping as the damping factor in the damped harmonic oscillator does in the dissipative system. The time-dependent harmonic oscillator and a charged particle in a particular type of time-dependent potential are discussed in [9]. In other words, the theory of explicitly time-dependent invariants for a general quantum system whose Hamiltonian operator is explicitly time-dependent is considered. Ballhausen [10] shows how the eigenvectors can be obtained for the harmonic oscillator with a time-dependent force constant by constructing raising and lowering operators. The quantum dynamics for the generalised Caldirola–Kanai oscillator in the Glauber coherent state is studied in [11] by employing linear invariant operator method. In [12], the Glauber coherent states are used to propose a phase-space

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formulation of quantum mechanics as an appropriate and consistent method, where phase-space wave functions play a central role, and position q and momentum p are treated on the same footing.

And the least action principle for classical mechanical dissipative systems is formulated in [13]. The whole conservative system composed of a damped moving body and its environment receiving the dissipated energy is considered. The correct equation of the damped motion is obtained from the usual variation calculus of the least action.

In this paper, the equation of motion for the damped harmonic oscillator is introduced in terms of Poisson brackets. Besides, the Heisenberg equations are written in terms of the quantum counterpart of the Poisson bracket, known as the commutators. The creation and annihilation operators will be then introduced and used to obtain the energy and eigenstates. The Caldirola–Kanai Hamiltonian is recovered by a given canonical transformation. Then, the creation and annihilation operators are transformed to be explicitly time-dependent.

2. Poisson bracket formulation of the damped harmonic oscillator

Consider the following Lagrangian that is deduced by Bateman [14]:

$$L(q,\dot{q},t)=\frac{m}{2}(\dot{q}^2-\omega^2q^2)\mathrm{e}^{\lambda t}. \tag{1}$$

According to [1], the equations of motion is then

$$\ddot{q}+\lambda\dot{q}+\omega^2q=0 \tag{2}$$

which clearly describes the one-dimensional damped harmonic oscillator. Using the transformation

$$y=q\mathrm{e}^{\lambda t/2} \tag{3}$$

the Lagrangian is then transformed as

$$\begin{aligned}L&=\frac{1}{2}m\left(\dot{y}^2-\frac{\lambda}{2}y\right)^2-\frac{m\omega^2}{2}y^2\\&=\frac{1}{2}m\left(\dot{y}^2-\frac{\lambda}{2}\dot{y}y+\frac{\lambda^2}{4}y^2\right)-\frac{m\omega^2}{2}y^2.\end{aligned} \tag{4}$$

The term $\dot{y}y$ does not influence the equations of motion as it is a total time derivative of a function F: $(\mathrm{d}F/\mathrm{d}t)=\dot{y}y$ implies that $F=y^2/2$. Thus, we have the following equivalent Lagrangian:

$$L=\frac{m}{2}\dot{y}^2-\frac{m}{2}\left(\omega^2-\frac{\lambda^2}{4}\right)y^2. \tag{5}$$

The corresponding equation of motion is then

$$\ddot{y}+\left(\omega^2-\frac{\lambda^2}{4}\right)y=0. \tag{6}$$

It is worth noting that the angular frequency of the damped oscillator is reduced to $\sqrt{\omega^2-(\lambda^2/4)}$, where ω is the angular frequency of the corresponding conservative system. The simple harmonic oscillator is simply recovered by setting $\lambda=0$ in eq. (6).

The Hamiltonian in terms of the new coordinate y is then

$$H=p_y\dot{y}-L, \tag{7}$$

where

$$p_y=\frac{\partial L}{\partial\dot{y}}=m\dot{y}. \tag{8}$$

The Hamiltonian is then

$$H(y,p_y)=\frac{p_y^2}{2m}+\frac{m}{2}\left(\omega^2-\frac{\lambda^2}{4}\right)y^2. \tag{9}$$

Note that this Hamiltonian is independent of time, and it is equivalent to the following Hamiltonian obtained in [1]:

$$H(y,p_y)=\frac{p_y^2}{2m}+\frac{1}{2}m\omega^2y^2+\frac{1}{2}\lambda yp_y. \tag{10}$$

The formulations of Poisson brackets of Hamilton's equations of motion are then calculated as

$$\begin{aligned}\dot{y}&=\{y,H\}\\&=\left\{y,\frac{p_y^2}{2m}+\frac{1}{2}m\omega^2y^2+\frac{\lambda}{2}yp_y\right\}\\&=\{y,p_y\}\frac{p_y}{2m}+\frac{p_y}{2m}\{y,p_y\}+\frac{\lambda}{2}y\{y,p_y\}\\&=\frac{p_y}{m}+\frac{\lambda}{2}y,\end{aligned} \tag{11}$$

if $\{y,p_y\}=1$. Similarly,

$$\begin{aligned}\dot{p_y}&=\{p_y,H\}\\&=\{p_y,y\}\frac{1}{2}m\omega^2y+\frac{1}{2}m\omega^2y\{p_y,y\}+\frac{\lambda}{2}p_y\{p_y,y\}\\&=-m\omega^2y-\frac{\lambda}{2}p_y.\end{aligned} \tag{12}$$

The canonical quantisation can be done by applying the standard quantisation rules: $[y,p_y]=i\hbar$; $[y,y]=[p_y,p_y]=0$ where position and momentum operators are denoted respectively by y and p_y. Then, the Heisenberg equations read as

$$\frac{\mathrm{d}}{\mathrm{d}}\langle y\rangle = \frac{i}{\hbar}\left\langle\left[\frac{p_y^2}{2m} + \frac{\lambda}{2}yp_y, y\right]\right\rangle$$
$$= \left\langle\frac{p_y}{m} + \frac{1}{2}\lambda y\right\rangle$$
$$= \frac{1}{m}\langle p_y\rangle + \frac{1}{2}\lambda\langle y\rangle \tag{13}$$

which agrees with eq. (11). Similarly,

$$\frac{\mathrm{d}}{\mathrm{d}}\langle p_y\rangle = \frac{i}{\hbar}\left\langle\left[\frac{1}{2}m\omega^2y^2 + \frac{\lambda}{2}yp_y, p_y\right]\right\rangle$$
$$= -\left\langle m\omega^2 y + \frac{1}{2}\lambda p_y\right\rangle$$
$$= -m\omega^2\langle y\rangle - \frac{1}{2}\lambda\langle p_y\rangle \tag{14}$$

which also agrees with eq. (12). Equations (13) and (14) state that the expectation values obey the classical equations of motion. These equations hold for any state ψ of the particle, and they are equations for expectation values. Using eqs (11) and (12), we then have

$$\ddot{y} = -\omega^2 y - \frac{\lambda}{2m}p_y + \lambda\dot{y} \tag{15}$$

which can be rewritten as

$$\ddot{y} + \left(\omega^2 - \frac{\lambda^2}{4}\right)y^2 = 0. \tag{16}$$

This is in full agreement with eq. (5) in [1]. Therefore, $\{y, p_y\} = 1$ implies $[y, p_y] = i\hbar$. With this, we can now introduce the pairs of the creation and annihilation operators $a^\dagger$ and a as

$$a = \frac{\beta}{\sqrt{2}}\left(y + \frac{ip_y}{m\Omega}\right),$$
$$a^\dagger = \frac{\beta}{\sqrt{2}}\left(y - \frac{ip_y}{m\Omega}\right), \tag{17}$$

where $\beta^2 = m\Omega/\hbar$. Then solving for y and p_y, we have

$$y = \frac{a + a^\dagger}{\sqrt{2}\beta},$$
$$p_y = \frac{m\Omega}{i}\frac{(a - a^\dagger)}{\sqrt{2}\beta}. \tag{18}$$

It is an easy task to verify that $[y, p_y] = i\hbar$ implies $[a, a^\dagger] = 1$. Substituting in eq. (10), and after calculations we obtain

$$H = \hbar\Omega\left(a^\dagger a + \frac{1}{2}\right), \tag{19}$$

where

$$\Omega = \sqrt{\omega^2 - \frac{\lambda^2}{4}}.$$

The problem is then reduced to the standard quantum harmonic oscillator known in literature. The energy eigenvalues are given by replacing ω by Ω in the quantum simple harmonic energy eigenvalue equation

$$E_n = \hbar\Omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, 3, \ldots. \tag{20}$$

The ground-state wave function $\psi_0(y)$ obeys $a\psi_0(y) = 0$, from which we obtain

$$\psi_0(y) = \left(\frac{\beta^2}{\pi}\right)^{1/4} \mathrm{e}^{-\beta^2y^2/2}. \tag{21}$$

For any state n, the eigenfunctions are then expressed in terms of the Hermite polynomials

$$H_n(y) = \mathrm{e}^{y^2}\frac{\mathrm{d}^n}{\mathrm{d}y^n}\mathrm{e}^{-y^2}$$

as

$$\psi_n(y) = \frac{1}{\sqrt{n!}}(a^\dagger)^n\psi_0(y) = (-1)^n\mathrm{e}^{\beta^2y^2/2}H_n(\beta y). \tag{22}$$

The loss-energy states are obtained by setting $y = q\mathrm{e}^{\lambda t/2}$ in eq. (22), from which we obtain the loss-energy states

$$\psi_n(q, t) = \frac{(-1)^n}{\sqrt{2^n n!}}\left(\frac{m\Omega}{\pi\hbar}\right)^{1/4}\exp\left(\frac{\beta^2}{2}q^2\mathrm{e}^{\lambda t}\right)$$
$$\times H_n(\beta q\mathrm{e}^{\lambda t/2})\mathrm{e}^{-i(E_nt/\hbar)} \tag{23}$$

which are in agreement (up to a multiplicative phase factor) with the result obtained in [1].

3. The canonical transformation

Consider the Hamiltonian of the form given by eq. (10). Then, we would like to mention that the damped harmonic oscillator can also be quantised by transforming this Hamiltonian using the following canonical transformation:

$$P = p_y + \frac{\lambda}{2m}y$$
$$Q = y \tag{24}$$

with $\{Q, P\} = 1$ and $\{Q, Q\} = \{P, P\} = 0$. Then the Hamiltonian of eq. (10) can be transformed to the following Hamiltonian:

$$H(Q, P) = \frac{P^2}{2m} + \frac{m}{2}\left(\omega^2 - \frac{\lambda^2}{4}\right)Q^2. \quad (25)$$

In other words, the energy of the damped harmonic oscillator can be treated as the energy of the undamped oscillator but with reduced frequency $\sqrt{(\omega^2 - (\lambda^2/4))}$. This shows that a damped harmonic oscillator is equivalent to an undamped harmonic oscillator whose new coordinate and momentum are related. The generating function $F = F_2(P, y)$ can be calculated from

$$p_y = \frac{\partial F_2}{\partial y} = P - \frac{\lambda}{2m}y$$

and

$$Q = \frac{\partial F_2}{\partial P}$$

from which we obtain

$$F_2 = yP - \frac{\lambda}{4}my^2 + C. \quad (26)$$

4. The Caldirola–Kanai Hamiltonian recovered

Define the canonical transformation

$$p_y = P\mathrm{e}^{\lambda t/2},$$
$$y = Q\mathrm{e}^{-\lambda t/2}. \quad (27)$$

The Hamiltonian equation (9) straightforwardly becomes

$$H_{\mathrm{CK}} = \frac{P^2}{2m}\mathrm{e}^{\lambda t} + \frac{1}{2}m\Omega^2\mathrm{e}^{-\lambda t}Q^2, \quad (28)$$

which is the Caldirola–Kanai Hamiltonian [15,16]. Thus, we derived it without needing Bateman's dual Hamiltonian. This transformation just used is clearly canonical in the sense that

$$[y, p_y] = [Q\mathrm{e}^{-\lambda t/2}, P\mathrm{e}^{\frac{\lambda t}{2}}]$$
$$= [Q, P] = i\hbar. \quad (29)$$

The creation and annihilation operators, A and $A^\dagger$, for this Hamiltonian can now be naturally defined using eq. (17),

$$A = \frac{\beta}{\sqrt{2}}\left(Q\mathrm{e}^{-\lambda t/2} + \frac{iP}{m\Omega}\mathrm{e}^{\lambda t/2}\right)$$
$$A^\dagger = \frac{\beta}{\sqrt{2}}\left(Q\mathrm{e}^{-\lambda t/2} - \frac{iP}{m\Omega}\mathrm{e}^{\lambda t/2}\right), \quad (30)$$

where $\beta^2 = m\Omega/\hbar$. Note that $[A, A^\dagger] = 1$. The Hamiltonian equation (28) then becomes

$$H_{\mathrm{CK}} = \hbar\Omega\left(A^\dagger A + \frac{1}{2}\right). \quad (31)$$

It is now clear that

$$E_n = \hbar\Omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, 3, \ldots \quad (32)$$

$$\psi_n(Q, t) = (-1)^n\mathrm{e}^{\gamma y^2/2}H_n(\sqrt{\alpha}y). \quad (33)$$

The coherent states, that is the eigenstates of the annihilation operator, can be constructed using $\hat{A}\psi_\alpha = \alpha\psi_\alpha$:

$$\frac{\beta}{\sqrt{2}}\left(Q\mathrm{e}^{-\lambda t/2} + \frac{iP}{m\Omega}\mathrm{e}^{\lambda t/2}\right)\psi_\alpha = \alpha\psi_\alpha \quad (34)$$

from which we obtain

$$\psi_\alpha(q, t) = N_0\exp\left(-\frac{\beta^2}{4}\mathrm{e}^{-\lambda t}q^2 - \alpha\mathrm{e}^{-\lambda t/2}q\right). \quad (35)$$

These states are obviously normalisable in the sense

$$\langle\psi_\alpha(q, t)|\psi_\alpha(q, t)\rangle = \frac{\sqrt{2\pi}}{\beta}\mathrm{e}^{(2\alpha^2/\beta^2)+(\lambda t/2)}.$$

Thus, we overcome the difficulty of non-normalisability of the coherent states that appeared using the dual coordinate method discussed in [3].

5. Conclusion and final remark

The Poisson bracket and commutation relations are introduced for the damped harmonic oscillator. The equations of motion are written in terms of Poisson brackets, and Heisenberg equations are written in terms of commutators. The creation and annihilation operators are introduced and used to obtain the energy and eigenstates. We have seen that the Poisson bracket relations lead to the same results as obtained in [1]. Moreover, we found that the eigenenergies and eigenstates obtained using creation and annihilation operators are in exact agreement with those obtained in [1]. This agreement is thus in support of the canonical quantisation method used there. We then concluded that our formulation leads to the Caldirola–Kanai Hamiltonian straightforwardly, and the coherent states are normalisable. Also, it is worth mentioning that the current treatment overcomes the Hermiticity condition that should be satisfied before quantisation takes place, and that the $\mathcal{PT}$-invariant alternative condition is sometimes sufficient to support real energy eigenvalues [17–19].

As a final remark, it should be mentioned that in [1], it is demonstrated that the distribution $|\psi_n(y, t)|^2$, which is in terms of y, is independent of time, and thus

$$\int_{-\infty}^{\infty} |\psi_n(y,t)|^2 \, \mathrm{d}y = 1. \tag{36}$$

The translation from the y-coordinate to the q-coordinates shows that $\psi_n^*(y,t)\psi_n(y,t)\mathrm{e}^{-\lambda t/2}$ is equivalent to $\psi_n^*(q,t)\psi_n(q,t)$ at any instant of time, from which one may define the probability distribution, for damping systems in terms of the original coordinate q, as

$$\begin{aligned} P_d(t) &= \int_{-\infty}^{\infty} \psi_n^*(q,t)\psi_n(q,t)\,\mathrm{d}q \\ &= \int_{-\infty}^{\infty} \psi_n^*(y,t)\psi_n(y,t)\mathrm{e}^{-\lambda t/2}\,\mathrm{d}y. \end{aligned} \tag{37}$$

Thus, the probability density is conserved when expressed in terms of the coordinates used to quantise the system, namely y, while dissipation is clear when expressed in terms of the original coordinate of the classical problem, namely q. The balance equation (or continuity equation) derived in [1] clarifies the present interpretation.

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References

[1] M Serhan, M Abusini, A Al-Jamel, H El-Nasser and E M Rabei, *J. Math. Phys.* **59**, 082105 (2018)
[2] S K Bose, U B Dubey and V N Tewari, *Pramana – J. Phys.* **24**, 59 (1985)
[3] G Ghosh and R W Hasse, *Phys. Rev. A* **24**, 1621 (1981)
[4] H Dekker, *Phys. Rev. A* **16**, 2126 (1977)
[5] H Dekker, *Phys. Rep*. **80**(**1**), 1 (1981)
[6] C-I Um, K-H Yeon and F T George, *Phys. Rep.* **362**(**2–3**), 63 (2002)
[7] H Majima and A Suzuki, *Ann. Phys*. **326**, 3000 (2011)
[8] A Suzuki and H Majima, *J. Mod. Phys*. **7**, 2329 (2016)
[9] H R Lewis and W B Riesenfeld, *J. Math. Phys*. **10**, 1458 (1969)
[10] C J Ballhausen, *Chem. Phys. Lett.* **192**(**1**), 49 (1992)
[11] J R Choi, *Results Phys*. **3**, 115 (2013)
[12] D Campos, *Pramana – J. Phys*. **87**: 27 (2016)
[13] Q A Wang and R Wang, *J. Phys.: Conf. Ser.* **1113**, 012003 (2018)
[14] H Bateman, *Phys. Rev.* **38**, 815 (1931)
[15] P Caldirola, *Il Nuovo Cimento* **18**, 393 (1941)
[16] E Kanai, *Prog. Theor. Phys*. **3**, 440 (1950)
[17] M Serhan, M Abusini, A Al-Jamel, H El-Nasser and E M Rabei, *J. Math. Phys.* **60**, 094101 (2019), *J. Math. Phys.* **60**, 094102 (2019)
[18] A Singh and S R Jain, *Pramana – J. Phys.* **92**: 47 (2019)
[19] M Abusini, M Serhan, M F Al-Jamal, A Al-Jamel and E M Rabei, *Pramana – J. Phys.* **93**: 93 (2019), https://doi.org/10.1007/s12043-019-1860-x