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Lie symmetry analysis for the coupled integrable dispersionless equations

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Abstract. In this paper, we primarily investigate Lie symmetry analysis and exact solutions for the coupled integrable dispersionless equations. First of all, based on the Lie symmetry analysis, an optimal system of one-dimensional subalgebras is constructed. Furthermore, similarity reductions and group invariant solutions are given. Next, exact solutions of the reduced equations have been derived by the method of power series. Finally, by means of Ibragimov's method, conservation laws are obtained.

Keywords. Coupled integrable dispersionless equations; symmetry analysis; optimal system; conservation laws.

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1. Introduction

Nonlinear differential equations play an indispensable role in nature. A host of natural phenomena could be modelled by them. In the past few years, the study of nonlinear differential equations has improved many fields, such as physics, chemistry, mechanics, etc. In this paper, we consider the coupled integrable dispersionless equations of the form

$$\begin{cases} u_{xt} + v_x w + v w_x = 0, \\ v_{xt} - 2v u_x = 0, \\ w_{xt} - 2w u_x = 0, \end{cases} \tag{1}$$

where u, v, w are functions of x, t and subscripts stand for partial derivatives. The equations arise from the coupled system solved by Konno and Oono [1], and describe the current-fed string in the external magnetic field [2]. The coherent structures of eq. (1) have been sought out by the singular manifold method [3], whereas Lie symmetry analysis and power series solutions of eq. (1) remain to be solved.

Multitude of methods have been explored to find exact solutions for nonlinear partial differential equations (PDEs). Some of the most representative methods are Bäcklund method [4], Hirota bilinear method [5], Darboux transformations [6], Painlevé expansion method [7], Lie symmetry analysis [8–12], inverse scattering method [13], etc. Lie symmetry analysis, a quite effective method among these, can get crucially explicit solutions of PDEs. Therefore, we have a clear understanding of some physical phenomena with the aid of exact solutions. This method, originally developed by Sophus Lie, aims to transform the solutions of the PDEs to other solutions. The purpose of this paper is to derive similarity reductions and exact solutions by Lie symmetry analysis method, then use lbragimov's method to construct conservation laws.

This paper is organised as follows. In §2, we focus on obtaining the Lie point symmetry of eq. (1). Section 3 mainly constructs the optimal system of one-dimensional subalgebras. Similarity reductions and group invariant solutions are given in §4. In §5, exact solutions of the reduced equations have been derived by the method of power series. Using lbragimov's method, we set up the conservation laws in §6. Some conclusions are presented in §7.

2. Lie point symmetry

In this section, using Lie symmetry analysis method for eq. (1), we consider a one-parameter Lie group transformation

$$x \to x + \epsilon \xi^1(x, t, u, v, w),$$

$$t \to t + \epsilon \xi^2(x, t, u, v, w),$$

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$$u \to u + \epsilon\eta^1(x, t, u, v, w),$$
$$v \to v + \epsilon\eta^2(x, t, u, v, w),$$
$$w \to w + \epsilon\eta^3(x, t, u, v, w),$$

with a small parameter $\epsilon \ll 1$. And its infinitesimal generator is

$$X = \xi^1(x, t, u, v, w)\frac{\partial}{\partial x} + \xi^2(x, t, u, v, w)\frac{\partial}{\partial t} + \eta^1(x, t, u, v, w)\frac{\partial}{\partial u} + \eta^2(x, t, u, v, w)\frac{\partial}{\partial v} + \eta^3(x, t, u, v, w)\frac{\partial}{\partial w}.$$

The second-order prolongation of the infinitesimal generator is

$$\mathrm{pr}^{(2)}X = X + \eta_x^1\frac{\partial}{\partial u_x} + \eta_x^2\frac{\partial}{\partial v_x} + \eta_x^3\frac{\partial}{\partial w_x} + \eta_{xt}^1\frac{\partial}{\partial u_{xt}} + \eta_{xt}^2\frac{\partial}{\partial v_{xt}} + \eta_{xt}^3\frac{\partial}{\partial w_{xt}},$$

where

$$\eta_x^1 = D_x(\eta^1) - u_x D_x(\xi^1) - u_t D_x(\xi^2),$$
$$\eta_x^2 = D_x(\eta^2) - v_x D_x(\xi^1) - v_t D_x(\xi^2),$$
$$\eta_x^3 = D_x(\eta^3) - w_x D_x(\xi^1) - w_t D_x(\xi^2),$$
$$\eta_{xt}^1 = D_x D_t(\eta^1 - \xi^1 u_x - \xi^2 u_t) + \xi^1 u_{xxt} + \xi^2 u_{xtt},$$
$$\eta_{xt}^2 = D_x D_t(\eta^2 - \xi^1 v_x - \xi^2 v_t) + \xi^1 v_{xxt} + \xi^2 v_{xtt},$$
$$\eta_{xt}^3 = D_x D_t(\eta^3 - \xi^1 w_x - \xi^2 w_t) + \xi^1 w_{xxt} + \xi^2 w_{xtt},$$

and D_x, D_t denote the total derivatives with regard to x and t. According to the invariance conditions, the following equations can be written as

$$\mathrm{pr}^{(2)}X(\Delta_1)|_{\Delta_1=0} = 0,$$
$$\mathrm{pr}^{(2)}X(\Delta_2)|_{\Delta_2=0} = 0,$$
$$\mathrm{pr}^{(2)}X(\Delta_3)|_{\Delta_3=0} = 0,$$

where

$$\Delta_1 = u_{xt} + v_x w + v w_x,$$
$$\Delta_2 = v_{xt} - 2v u_x,$$
$$\Delta_3 = w_{xt} - 2w u_x.$$

Next, the system of overdetermined equations is given as follows:

$$\xi_t^1 = \xi_u^1 = \xi_v^1 = \xi_w^1 = 0,$$
$$\xi_u^2 = \xi_v^2 = \xi_w^2 = \xi_x^2 = \xi_{tt}^2 = 0,$$
$$\eta_u^1 = -\xi_t^2, \quad \eta_v^1 = \eta_w^1 = \eta_x^1 = 0,$$
$$\eta_t^2 = \eta_u^2 = \eta_w^2 = \eta_x^2 = 0,$$
$$\eta_v^2 = \frac{\eta 2}{v}, \quad \eta_3 = -\frac{w}{v}(2\xi_t^2 v + \eta_2).$$

Solving this system, one obtains

$$\xi^1 = F_1(x), \quad \xi^2 = c_1 t + c_2, \quad \eta^1 = -c_1 u + F_2(t),$$
$$\eta^2 = c_3 v, \quad \eta^3 = -w(2c_1 + c_3),$$

where c_1, c_2 and c_3 are arbitrary constants, $F_1(x)$ and $F_2(t)$ are arbitrary functions. In order to obtain physically significant solutions, we take $F_1(x) = c_4$, $F_2(t) = c_5$ and substitute above, getting

$$\xi^1 = c_4, \quad \xi^2 = c_1 t + c_2, \quad \eta^1 = -c_1 u + c_5,$$
$$\eta^2 = c_3 v, \quad \eta^3 = -w(2c_1 + c_3),$$

where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants. Thus, Lie algebra L_5 of infinitesimal symmetries is spanned by the following generators:

$$X_1 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2w\frac{\partial}{\partial w},$$
$$X_2 = \frac{\partial}{\partial t},$$
$$X_3 = v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w},$$
$$X_4 = \frac{\partial}{\partial x},$$
$$X_5 = \frac{\partial}{\partial u}. \tag{2}$$

3. The optimal system of one-dimensional subalgebras

To get the optimal system, one divides one-dimensional subalgebras into equivalence classes [14] or computes by algebraic adjoint representations [15]. However, in this section, we apply this method only depending on the commutator table to obtain the optimal system [16]. The commutation relations about X_1, X_2, X_3, X_4 and X_5 are listed in table 1.

An arbitrary operator $X \in L_5$ is of the following form:

$$X = l_1 X_1 + l_2 X_2 + l_3 X_3 + l_4 X_4 + l_5 X_5.$$

In order to find the linear transformations of the vector $l = (l_1, l_2, l_3, l_4, l_5)$, the following generator is given as

$$E_i = c_{ij}^k l_j \frac{\partial}{\partial l_k}, \quad i = 1, 2, 3, 4, 5, \tag{3}$$

where c_{ij}^k is determined by $[X_i, X_j] = c_{ij}^k X_k$. On the basis of eq. (3) and table 1, E_1, E_2, E_3, E_4, E_5 can be given as

$$E_1 = -l_2 \frac{\partial}{\partial l_2} + l_5 \frac{\partial}{\partial l_5},$$

$$E_2 = l_1 \frac{\partial}{\partial l_2},$$

$$E_3 = 0,$$

$$E_4 = 0,$$

$$E_5 = -l_1 \frac{\partial}{\partial l_5}.$$

For the generators E_1, E_2, E_3, E_4, E_5, the Lie equations which have parameters a_1, a_2, a_3, a_4, a_5 with the initial condition $\tilde{l}|_{a_i=0} = l,\ i = 1-5$ are expressed as

$$\frac{d\tilde{l}_1}{da_1} = 0, \quad \frac{d\tilde{l}_2}{da_1} = -\tilde{l}_2, \quad \frac{d\tilde{l}_3}{da_1} = 0,$$

$$\frac{d\tilde{l}_4}{da_1} = 0, \quad \frac{d\tilde{l}_5}{da_1} = \tilde{l}_5,$$

$$\frac{d\tilde{l}_1}{da_2} = 0, \quad \frac{d\tilde{l}_2}{da_2} = \tilde{l}_1, \quad \frac{d\tilde{l}_3}{da_2} = 0,$$

$$\frac{d\tilde{l}_4}{da_2} = 0, \quad \frac{d\tilde{l}_5}{da_2} = 0,$$

$$\frac{d\tilde{l}_1}{da_3} = 0, \quad \frac{d\tilde{l}_2}{da_3} = 0, \quad \frac{d\tilde{l}_3}{da_3} = 0,$$

$$\frac{d\tilde{l}_4}{da_3} = 0, \quad \frac{d\tilde{l}_5}{da_3} = 0,$$

$$\frac{d\tilde{l}_1}{da_4} = 0, \quad \frac{d\tilde{l}_2}{da_4} = 0, \quad \frac{d\tilde{l}_3}{da_4} = 0,$$

$$\frac{d\tilde{l}_4}{da_4} = 0, \quad \frac{d\tilde{l}_5}{da_4} = 0,$$

Table 1. Table of Lie brackets.

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5
X_1	0	$-X_2$	0	0	X_5
X_2	X_2	0	0	0	0
X_3	0	0	0	0	0
X_4	0	0	0	0	0
X_5	$-X_5$	0	0	0	0

$$\frac{d\tilde{l}_1}{da_5} = 0, \quad \frac{d\tilde{l}_2}{da_5} = 0, \quad \frac{d\tilde{l}_3}{da_5} = 0,$$

$$\frac{d\tilde{l}_4}{da_5} = 0, \quad \frac{d\tilde{l}_5}{da_5} = -\tilde{l}_1.$$

The transformations of the solutions of these equations are given as follows:

$$T_1: \ \tilde{l}_1 = l_1, \quad \tilde{l}_2 = e^{-a_1} l_2, \quad \tilde{l}_3 = l_3,$$

$$\tilde{l}_4 = l_4, \quad \tilde{l}_5 = e^{a_1} l_5,$$

$$T_2: \ \tilde{l}_1 = l_1, \quad \tilde{l}_2 = a_2 l_1 + l_2, \quad \tilde{l}_3 = l_3,$$

$$\tilde{l}_4 = l_4, \quad \tilde{l}_5 = l_5,$$

$$T_3: \ \tilde{l}_1 = l_1, \quad \tilde{l}_2 = l_2, \quad \tilde{l}_3 = l_3, \quad \tilde{l}_4 = l_4, \quad \tilde{l}_5 = l_5,$$

$$T_4: \ \tilde{l}_1 = l_1, \quad \tilde{l}_2 = l_2, \quad \tilde{l}_3 = l_3, \quad \tilde{l}_4 = l_4, \quad \tilde{l}_5 = l_5,$$

$$T_5: \ \tilde{l}_1 = l_1, \quad \tilde{l}_2 = l_2, \quad \tilde{l}_3 = l_3,$$

$$\tilde{l}_4 = l_4, \quad \tilde{l}_5 = -a_5 l_1 + l_5.$$

The structure of the optimal system needs a simplification of the vector

$$l = (l_1, l_2, l_3, l_4, l_5), \tag{4}$$

through the transformations $T_1 - T_5$. We mainly concentrate on finding the simplest representative of vector (4). The structure is classified into two cases.

Case 3.1. $l_1 \neq 0$

By taking $a_2 = -l_2/l_1$ in the transformation T_2, we can enable $\tilde{l}_2 = 0$. Consequently, vector (4) is reduced as

$$l = (l_1, 0, l_3, l_4, l_5). \tag{5}$$

Then we take $a_5 = l_5/l_1$ in T_5, reducing vector (5) as

$$l = (l_1, 0, l_3, l_4, 0).$$

As a result, we obtain the following representatives:

$$X_1, \ X_1 \pm X_3, \ X_1 \pm X_4, \ X_1 \pm X_3 \pm X_4. \tag{6}$$

Case 3.2. $l_1 = 0$

We consider vector (4) of the form

$$l = (0, l_2, l_3, l_4, l_5).$$

3.2.1. $l_2 \neq 0$

Given all the possible combinations, we have the following representatives:

$$X_2, \ X_2 \pm X_3, \ X_2 \pm X_4,$$

$$X_2 \pm X_5, \ X_2 \pm X_3 \pm X_4, \ X_2 \pm X_3 \pm X_5,$$

$$X_2 \pm X_4 \pm X_5, \ X_2 \pm X_3 \pm X_4 \pm X_5. \tag{7}$$

3.2.2. $l_2 = 0$

We consider vector (4) of the form

$$l = (0, 0, l_3, l_4, l_5).$$

Then, we obtain the following representatives:

$$X_3,\ X_4,\ X_5,\ X_3 \pm X_4,\ X_3 \pm X_5,$$

$$X_4 \pm X_5,\ X_3 \pm X_4 \pm X_5. \tag{8}$$

Hence, by gathering the operators (6)–(8), we get the following theorem:

Theorem 3.1. *The following operators constitute an optimal system of eq.* (1):

$$X_1,\ X_2,\ X_3,\ X_4,\ X_5,\ X_1 \pm X_3,$$

$$X_1 \pm X_4,\ X_1 \pm X_3 \pm X_4,\ X_2 \pm X_3,$$

$$X_2 \pm X_4,\ X_2 \pm X_5,\ X_2 \pm X_3 \pm X_4,$$

$$X_2 \pm X_3 \pm X_5,\ X_2 \pm X_4 \pm X_5,$$

$$X_2 \pm X_3 \pm X_4 \pm X_5,\ X_3 \pm X_4,$$

$$X_3 \pm X_5,\ X_4 \pm X_5,\ X_3 \pm X_4 \pm X_5.$$

4. Similarity reductions and exact solutions

In this section, we cope with the similarity reductions and derive group invariant solutions.

Case 4.1. For the generator $X_1 + X_3$, we have similarity variables

$$u = \frac{f(z)}{t}, \quad v = tg(z), \quad w = \frac{h(z)}{t^3},$$

where $z = x$. Thus, we obtain

$$\begin{cases} f' - g'h - gh' = 0, \\ g' - 2gf' = 0, \\ 3h' + 2hf' = 0, \end{cases} \tag{9}$$

where

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}z}, \quad h' = \frac{\mathrm{d}h}{\mathrm{d}z}.$$

Solving the reduced equations, we obtain

$$u(x,t) = \frac{c_2}{t}, \quad v(x,t) = tc_1, \quad w(x,t) = \frac{3}{4t^3 g(x)},$$

where c_1 and c_2 are arbitrary constants, $g(x) \neq 0$.

Case 4.2. For the generator $X_2 + X_3$, we have $z = x$, $u = f(z)$, $v = \mathrm{e}^t g(z)$, $w = \mathrm{e}^{-t} h(z)$ and the reduced equations are

$$\begin{cases} g'h + gh' = 0, \\ g' - 2gf' = 0, \\ h' + 2hf' = 0, \end{cases} \tag{10}$$

where

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}z}, \quad h' = \frac{\mathrm{d}h}{\mathrm{d}z}.$$

Solving the reduced equations, we obtain $u(x,t) = f(x)$, $v(x,t) = c_2 \mathrm{e}^{t+2f(x)}$, $w(x,t) = c_1 \mathrm{e}^{-t-2f(x)}$, where c_1 and c_2 are arbitrary constants, $f(x)$ is an arbitrary function.

Case 4.3. For the generator $X_3 + X_4 + X_5$, we have $z = t$, $u = f(z) + x$, $v = \mathrm{e}^x g(z)$, $w = \mathrm{e}^{-x} h(z)$. The reduced equations are

$$\begin{cases} g' - 2g = 0, \\ h' + 2h = 0, \end{cases} \tag{11}$$

where

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}z}, \quad h' = \frac{\mathrm{d}h}{\mathrm{d}z}.$$

Solving the reduced equations, we obtain $u(x,t) = f(t) + x$, $v(x,t) = c_2 \mathrm{e}^{x+2t}$, $w(x,t) = c_1 \mathrm{e}^{-2t-x}$, where c_1 and c_2 are arbitrary constants, $f(t)$ is an arbitrary function.

Case 4.4. For the generator $X_1 + X_4$, we have $u = \mathrm{e}^{-x} f(z)$, $v = g(z)$, $w = \mathrm{e}^{-2x} h(z)$ where $z = t\mathrm{e}^{-x}$. Thus, we obtain

$$\begin{cases} 2f' + zf'' + zgh' + zg'h + 2gh = 0, \\ zg'' + g' - 2fg - 2zf'g = 0, \\ 3h' + h''z - 2hf - 2zf'h = 0, \end{cases} \tag{12}$$

where

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}z}, \quad h' = \frac{\mathrm{d}h}{\mathrm{d}z}.$$

Case 4.5. For the generator $X_2 + X_4$, we have $u = f(z)$, $v = g(z)$, $w = h(z)$ in which $z = -x + t$. The form of the reduced equations is

$$\begin{cases} f'' + g'h + gh' = 0, \\ g'' - 2gf' = 0, \\ h'' - 2hf' = 0, \end{cases} \tag{13}$$

where

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}z}, \quad h' = \frac{\mathrm{d}h}{\mathrm{d}z}.$$

Case 4.6. For the generator $X_2 + X_3 + X_4 + X_5$, we have $z = -x + t$, $u = f(z) + x$, $v = \mathrm{e}^x g(z)$, $w = \mathrm{e}^{-x} h(z)$. The form of the reduced equations is

$$\begin{cases} f'' - g'h - gh' = 0, \\ g' - g'' + 2gf' - 2g = 0, \\ h' + h'' - 2hf' + 2h = 0, \end{cases} \tag{14}$$

where

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}z}, \quad h' = \frac{\mathrm{d}h}{\mathrm{d}z}.$$

Case 4.7. For the generator $X_2 + X_4 + X_5$, we have $z = -x + t$, $u = f(z) + x$, $v = g(z)$, $w = h(z)$. The form of the reduced equations is

$$\begin{cases} f'' + g'h + gh' = 0, \\ g'' - 2gf' + 2g = 0, \\ h'' - 2hf' + 2h = 0, \end{cases} \tag{15}$$

where

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}z}, \quad h' = \frac{\mathrm{d}h}{\mathrm{d}z}.$$

Case 4.8. For the generator $X_1 + X_3 + X_4$, we obtain $z = t\mathrm{e}^{-x}$, $u = \mathrm{e}^{-x} f(z)$, $v = \mathrm{e}^{x} g(z)$, $w = \mathrm{e}^{-3x} h(z)$. The corresponding reduced equations are

$$\begin{cases} 2f' + f''z + 2gh + g'hz + gh'z = 0, \\ g''z - 2fg - 2gf'z = 0, \\ 4h' + h'z - 2hf - 2hf'z = 0, \end{cases} \tag{16}$$

where

$$f' = \frac{\mathrm{d}f}{\mathrm{d}z}, \quad g' = \frac{\mathrm{d}g}{\mathrm{d}z}, \quad h' = \frac{\mathrm{d}h}{\mathrm{d}z}.$$

As is depicted in the contents above, we have obtained the solutions of Cases 4.1–4.3. For Cases 4.4–4.8, we apply power series method to cope with their solutions.

5. The explicit power series solutions

In this section, we use power series approach to solve Cases 4.4–4.8. It is effective for us to obtain solutions of differential equations.

5.1 *Power series solutions to eq.* (12)

Firstly, the form of power series of eq. (12) is

$$f(z) = \sum_{n=0}^{\infty} p_n z^n, \quad g(z) = \sum_{n=0}^{\infty} q_n z^n,$$

$$h(z) = \sum_{n=0}^{\infty} r_n z^n, \tag{17}$$

where the coefficients p_n, q_n and r_n are constants to be determined. Substituting (17) into (12), we obtain

$$\begin{aligned} &2\sum_{n=0}^{\infty}(n+1)p_{n+1}z^n + z\sum_{n=0}^{\infty}(n+1)(n+2)p_{n+2}z^n \\ &\quad + z\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)q_{n+1-k}r_k z^n \\ &\quad + z\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)r_{n+1-k}q_k z^n \\ &\quad + 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}r_{n-k}q_k z^n = 0, \\ &z\sum_{n=0}^{\infty}(n+1)(n+2)q_{n+2}z^n + \sum_{n=0}^{\infty}(n+1)q_{n+1}z^n \\ &\quad - 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}p_{n-k}q_k z^n \\ &\quad - 2z\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)p_{n+1-k}q_k z^n = 0, \\ &3\sum_{n=0}^{\infty}(n+1)r_{n+1}z^n + z\sum_{n=0}^{\infty}(n+1)(n+2)r_{n+2}z^n \\ &\quad - 2\sum_{n=0}^{\infty}\sum_{k=0}^{n}p_{n-k}r_k z^n \\ &\quad - 2z\sum_{n=0}^{\infty}\sum_{k=0}^{n}(n+1-k)p_{n+1-k}r_k z^n = 0. \end{aligned} \tag{18}$$

Comparing coefficients for (18), when $n = 0$, we obtain

$$p_1 = -r_0 q_0, \quad q_1 = 2p_0 q_0, \quad r_1 = \frac{2}{3}p_0 r_0. \tag{19}$$

Universally, when $n \geq 0$, we have the following recurrence formula:

$$\begin{aligned} p_{n+2} = -\frac{1}{(n+2)(n+3)}\Bigg[&\sum_{k=0}^{n}(n+1-k)q_{n+1-k}r_k \\ &+ \sum_{k=0}^{n}(n-k+1)r_{n+1-k}q_k + 2\sum_{k=0}^{n+1}r_{n+1-k}q_k\Bigg], \end{aligned}$$

$$q_{n+2} = \frac{2}{(n+2)^2}\Bigg[\sum_{k=0}^{n+1}p_{n-k+1}q_k$$

$$+\sum_{k=0}^{n}(n+1-k)p_{n+1-k}q_k\Bigg],$$

$$r_{n+2}=-\frac{2}{(n+2)(n+4)}\left[\sum_{k=0}^{n+1}p_{n-k+1}r_k+\sum_{k=0}^{n}(n+1-k)p_{n+1-k}r_k\right]. \tag{20}$$

On the basis of (20), all the coefficients $p_i, q_i, r_i (i \geq 2)$ of the power series (17) can be obtained, e.g.,

$$p_2=-\frac{1}{2}(q_1r_0+r_1q_0),\quad q_2=\frac{1}{2}(2p_1q_0+p_0q_1),$$

$$r_2=-\frac{1}{4}(2p_1r_0+p_0r_1),$$

$$p_3=-\frac{1}{3}(q_2r_0+q_1r_1+r_2q_0),$$

$$q_3=\frac{2}{9}(3p_2q_0+2p_1q_1+p_0q_2),$$

$$r_3=-\frac{2}{15}(3p_2r_0+2p_1r_1+p_0r_2).$$

Therefore, as p_0, q_0 and r_0 are taken as arbitrary constants, the rest of the sequences $\{p_n\}_{n=0}^{\infty}$, $\{q_n\}_{n=0}^{\infty}$ and $\{r_n\}_{n=0}^{\infty}$, according to (20), can be determined. This is to say, there is a power series solution (17) and its coefficients are formed by (19) and (20). Moreover, for eq. (12), we prove the convergence of the power series solution (17). Actually, with regard to (20), we have

$$|p_{n+2}|\leq 2\left(\sum_{k=0}^{n}|q_{n+1-k}||r_k|+\sum_{k=0}^{n}|r_{n+1-k}||q_k|+\sum_{k=0}^{n+1}|r_{n-k+1}||q_k|\right),\quad n=0,1,\ldots,$$

$$|q_{n+2}|\leq 2\left(\sum_{k=0}^{n+1}|p_{n-k+1}||q_k|+\sum_{k=0}^{n}|p_{n+1-k}||q_k|\right),\quad n=0,1,\ldots,$$

$$|r_{n+2}|\leq 2\left(\sum_{k=0}^{n+1}|p_{n-k+1}||r_k|+\sum_{k=0}^{n}|p_{n+1-k}||r_k|\right),\quad n=0,1,\ldots.$$

Now, we define three power series

$$E=E(z)=\sum_{n=0}^{\infty}e_nz^n,$$

$$S=S(z)=\sum_{n=0}^{\infty}s_nz^n$$

and

$$T=T(z)=\sum_{n=0}^{\infty}t_nz^n,$$

by

$$e_i=|p_i|,\quad s_i=|q_i|,\quad t_i=|r_i|,\quad i=0,1$$

and

$$e_{n+2}=2\left(\sum_{k=0}^{n}s_{n+1-k}t_k+\sum_{k=0}^{n}s_kt_{n+1-k}+\sum_{k=0}^{n+1}t_{n-k+1}s_k\right),$$

$$s_{n+2}=2\left(\sum_{k=0}^{n+1}e_{n-k+1}s_k+\sum_{k=0}^{n}e_{n+1-k}s_k\right),$$

$$t_{n+2}=2\left(\sum_{k=0}^{n+1}e_{n-k+1}t_k+\sum_{k=0}^{n}e_{n+1-k}t_k\right), \tag{21}$$

where $n=0,1,\ldots$. Next, it is obviously known that

$$|p_n|\leq e_n,\quad |q_n|\leq s_n,\quad |r_n|\leq t_n,\quad n=0,1,2,\ldots.$$

In consequence, three series

$$E=E(z)=\sum_{n=0}^{\infty}e_nz^n,$$

$$S=S(z)=\sum_{n=0}^{\infty}s_nz^n$$

and

$$T=T(z)=\sum_{n=0}^{\infty}t_nz^n$$

are majorant series of (17). Then, it is demonstrated that the series $E=E(z)$, $S=S(z)$ and $T=T(z)$ have positive radius of convergence

$$E(z)=e_0+e_1z+\sum_{n=0}^{\infty}e_{n+2}z^{n+2}$$

$$=e_0+e_1z+2\left[\sum_{n=0}^{\infty}\sum_{k=0}^{n}(s_{n+1-k}t_k+t_{n+1-k}s_k)+\sum_{n=0}^{\infty}\sum_{k=0}^{n+1}t_{n+1-k}s_k\right]z^{n+2}$$

$$
\begin{aligned}
&= e_0 + e_1 z + 2[z(ST - s_0 T) \\
&\quad + z(ST - t_0 S) + z(TS - t_0 s_0)] \\
&= e_0 + e_1 z + 2z[T(S - s_0) \\
&\quad + S(T - t_0) + (TS - t_0 s_0)],
\end{aligned}
$$

$$
\begin{aligned}
S(z) &= s_0 + s_1 z + \sum_{n=0}^{\infty} s_{n+2} z^{n+2} = s_0 + s_1 z \\
&\quad + 2\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n+1} e_{n+1-k} s_k + \sum_{n=0}^{\infty}\sum_{k=0}^{n} e_{n+1-k} s_k\right) z^{n+2} \\
&= s_0 + s_1 z + 2z(ES - e_0 s_0 + ES - e_0 S) \\
&= s_0 + s_1 z + 2z[S(E - e_0) + ES - e_0 s_0]
\end{aligned}
$$

and

$$
\begin{aligned}
T(z) &= t_0 + t_1 z + \sum_{n=0}^{\infty} t_{n+2} z^{n+2} = t_0 + t_1 z \\
&\quad + 2z\left(\sum_{n=0}^{\infty}\sum_{k=0}^{n+1} e_{n+1-k} t_k + \sum_{n=0}^{\infty}\sum_{k=0}^{n} e_{n+1-k} t_k\right) z^{n+1} \\
&= t_0 + t_1 z + 2z(TE - e_0 t_0 + TE - e_0 T) \\
&= t_0 + t_1 z + 2z[T(E - e_0) + TE - e_0 t_0].
\end{aligned}
$$

Given the implicit functional equations about the independent variable z,

$$
\begin{aligned}
F(z, E, S, T) &= E - e_0 - e_1 z - 2z[T(S - s_0) \\
&\quad + S(T - t_0) + (TS - s_0 t_0)], \\
G(z, E, S, T) &= S - s_0 - s_1 z - 2z[(E - e_0)S \\
&\quad + ES - s_0 e_0], \\
H(z, E, S, T) &= T - t_0 - t_1 z - 2z[(E - e_0)T \\
&\quad + ET - t_0 e_0].
\end{aligned}
$$

Because F, G, H are analytic in the neighbourhood of $(0, e_0, s_0, t_0)$ and $F(0, e_0, s_0, t_0) = 0$, $G(0, e_0, s_0, t_0) = 0$, and $H(0, e_0, s_0, t_0) = 0$.

Besides, if we select the appropriate parameters $e_0 = |p_0|$, $s_0 = |q_0|$ and $t_0 = |r_0|$, the Jacobian determinant

$$
\left.\frac{\partial(F, G, H)}{\partial(E, S, T)}\right|_{(0, e_0, s_0, t_0)} = 1 \neq 0.
$$

Applying the implicit function theorem [17] to the above statements, we observe that, in a neighbourhood of the point $(0, e_0, s_0, t_0)$, $E = E(z)$, $S = S(z)$ and $T = T(z)$ are analytic and have positive radius. This means that, in a neighbourhood of the point $(0, e_0, s_0, t_0)$, $E = E(z)$, $S = S(z)$ and $T = T(z)$ converge. The proof is complete.

Consequently, the power series solution (17) of eq. (12) is analytic and has the following form:

$$
\begin{aligned}
f(z) &= p_0 + p_1 z + \sum_{n=0}^{\infty} p_{n+2} z^{n+2} \\
&= p_0 - r_0 q_0 z - \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)} \\
&\quad \times \left[\sum_{k=0}^{n} (n+1-k) q_{n+1-k} r_k + \sum_{k=0}^{n} (n+1-k)\right. \\
&\quad \left. \times r_{n+1-k} q_k + 2\sum_{k=0}^{n+1} r_{n-k+1} q_k\right] z^{n+2},
\end{aligned}
$$

$$
\begin{aligned}
g(z) &= q_0 + q_1 z + \sum_{n=0}^{\infty} q_{n+2} z^{n+2} \\
&= q_0 + 2 p_0 q_0 z + \sum_{n=0}^{\infty} \frac{2}{(n+2)^2} \\
&\quad \times \left[\sum_{k=0}^{n+1} p_{n+1-k} q_k + \sum_{k=0}^{n} (n-k+1) p_{n-k+1} q_k\right] z^{n+2},
\end{aligned}
$$

$$
\begin{aligned}
h(z) &= r_0 + r_1 z + \sum_{n=0}^{\infty} r_{n+2} z^{n+2} \\
&= r_0 + \frac{2}{3} p_0 r_0 z - \sum_{n=0}^{\infty} \frac{2}{(n+2)(n+4)} \\
&\quad \times \left[\sum_{k=0}^{n+1} p_{n+1-k} r_k + \sum_{k=0}^{n} (n-k+1) p_{n-k+1} r_k\right] z^{n+2}.
\end{aligned}
$$

Then, the explicit power series solution of eq. (1) is

$$
\begin{aligned}
u(x, t) &= p_0 \mathrm{e}^{-x} - r_0 q_0 t \mathrm{e}^{-2x} - \sum_{n=0}^{\infty} \frac{1}{(n+2)(n+3)} \\
&\quad \times \left[\sum_{k=0}^{n} (n-k+1) q_{n+1-k} r_k \right. \\
&\quad + \sum_{k=0}^{n} (n-k+1) q_k r_{n-k+1}) \\
&\quad \left. + 2\sum_{k=0}^{n+1} r_{n-k+1} q_k\right] \mathrm{e}^{-xn-3x} t^{n+2},
\end{aligned}
$$

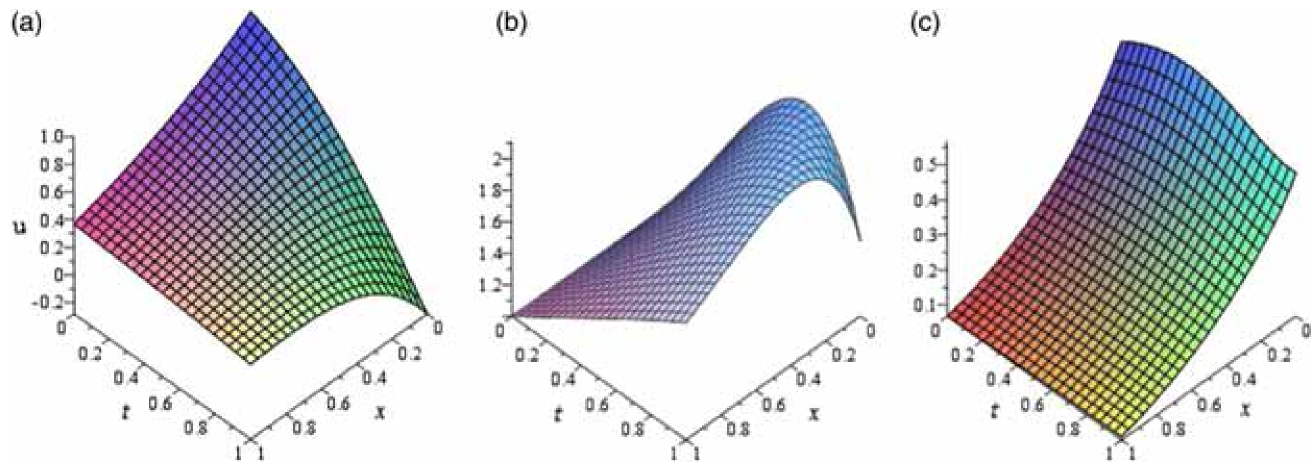

Figure 1. The power series solution of eq. (12). (**a**) $u(x,t)$ for $p_0 = 1, q_0 = 1, r_0 = \frac{1}{2}$, (**b**) $v(x,t)$ for $p_0 = 1, q_0 = 1, r_0 = \frac{1}{2}$ and (**c**) $w(x,t)$ for $p_0 = 1, q_0 = 1, r_0 = \frac{1}{2}$.

$$v(x,t) = q_0 + 2p_0q_0e^{-x}t + \sum_{n=0}^{\infty}\frac{2}{(n+2)^2}\times\left[\sum_{k=0}^{n+1}p_{n-k+1}q_k + \sum_{k=0}^{n}(n-k+1)q_k p_{n-k+1}\right]\times e^{-xn-2x}t^{n+2},$$

$$w(x,t) = e^{-2x}r_0 + \frac{2}{3}p_0r_0e^{-3x}t - \sum_{n=0}^{\infty}\frac{2}{(n+2)(n+4)}\times\left[\sum_{k=0}^{n+1}(n-k+1)q_{n+1-k}r_k + \sum_{k=0}^{n}(n-k+1)q_kr_{n-k+1})\right]e^{-xn-4x}t^{n+2},$$

where p_0, q_0 and r_0 are arbitrary constants and the other coefficients $p_n(n \geq 1)$, $q_n(n \geq 1)$ and $r_n(n \geq 1)$ can be obtained according to (19) and (20).

When we take $p_0 = 1, q_0 = 1, r_0 = 1/2$ and substitute them into (19) and (20), we get figure 1 for the values of u, v and w. Proofs of convergence of the power series solutions of eqs (13)–(16) are the same as that of eq. (12). So the details are omitted here. Besides, we list their solutions in table 2, where p_i, q_i and r_i $(i = 0, 1)$ are arbitrary constants.

6. Construction of conservation laws

In this section, based on the obtained multipliers [18, 19], we prove that eq. (1) is nonlinearly self-adjoint. Moreover, using Ibragimov's method, we construct conservation laws.

6.1 *Proof of nonlinear self-adjointness*

For eq. (1), conservation laws multipliers have the following form:

$$\Lambda_1 = \Lambda_1(x,t,u,v,w), \quad \Lambda_2 = \Lambda_2(x,t,u,v,w),$$
$$\Lambda_3 = \Lambda_3(x,t,u,v,w).$$

Besides,

$$E_u(\Lambda_1(u_{xt} + v_xw + vw_x) + \Lambda_2(v_{xt} - 2vu_x) + \Lambda_3(w_{xt} - 2wu_x)) = 0,$$
$$E_v(\Lambda_1(u_{xt} + v_xw + vw_x) + \Lambda_2(v_{xt} - 2vu_x) + \Lambda_3(w_{xt} - 2wu_x)) = 0,$$
$$E_w(\Lambda_1(u_{xt} + v_xw + vw_x) + \Lambda_2(v_{xt} - 2vu_x) + \Lambda_3(w_{xt} - 2wu_x)) = 0, \tag{22}$$

where the Euler operators E_u, E_v and E_w are written as

$$E_u = \frac{\partial}{\partial u} - D_t\frac{\partial}{\partial u_t} - D_x\frac{\partial}{\partial u_x} + D_x^2\frac{\partial}{\partial u_{xx}}\cdots,$$
$$E_v = \frac{\partial}{\partial v} - D_t\frac{\partial}{\partial v_t} - D_x\frac{\partial}{\partial v_x} + D_x^2\frac{\partial}{\partial v_{xx}}\cdots,$$
$$E_w = \frac{\partial}{\partial w} - D_t\frac{\partial}{\partial w_t} - D_x\frac{\partial}{\partial w_x} + D_x^2\frac{\partial}{\partial w_{xx}}\cdots. \tag{23}$$

Substituting (23) into (22), we obtain the system generated by unknown variables Λ_1, Λ_2 and Λ_3. Solving this system, we can get

Table 2. Power series solutions of Cases 4.5–4.8

Case	Power series solution
X_2+X_4	$u(x,t)=p_0+p_1(t-x)-\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}[\sum_{k=0}^{n}(n+1-k)q_{n-k+1}r_k+\sum_{k=0}^{n}(n-k+1)r_{n-k+1}q_k](t-x)^{n+2}$,
	$v(x,t)=q_0+q_1(t-x)+\sum_{n=0}^{\infty}\frac{2}{(n+1)(n+2)}[\sum_{k=0}^{n}(n-k+1)p_{n-k+1}q_k](t-x)^{n+2}$,
	$w(x,t)=r_0+(t-x)r_1+\sum_{n=0}^{\infty}\frac{2}{(n+1)(n+2)}[\sum_{k=0}^{n}(n+1-k)p_{n-k+1}r_k](t-x)^{n+2}$.
$X_2+X_3+X_4+X_5$	$u(x,t)=p_0+p_1(t-x)+\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}[\sum_{k=0}^{n}(n+1-k)q_{n-k+1}r_k+\sum_{k=0}^{n}(n-k+1)r_{n-k+1}q_k](t-x)^{n+2}+x$,
	$v(x,t)=q_0\mathrm{e}^x+q_1\mathrm{e}^x(t-x)+\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}[2\sum_{k=0}^{n}(n-k+1)p_{n-k+1}q_k+(n+1)q_{n+1}-2q_n](t-x)^{n+2}\mathrm{e}^x$,
	$w(x,t)=r_0\mathrm{e}^{-x}+r_1\mathrm{e}^{-x}(t-x)-\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}[-2\sum_{k=0}^{n}(n-k+1)p_{n-k+1}r_k+(n+1)r_{n+1}+2r_n](t-x)^{n+2}\mathrm{e}^{-x}$.
$X_2+X_4+X_5$	$u(x,t)=p_0+p_1(t-x)-\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}[\sum_{k=0}^{n}(n+1-k)q_{n-k+1}r_k+\sum_{k=0}^{n}(n-k+1)r_{n-k+1}q_k](t-x)^{n+2}+x$,
	$v(x,t)=q_0+q_1(t-x)+\sum_{n=0}^{\infty}\frac{2}{(n+1)(n+2)}[\sum_{k=0}^{n}(n-k+1)p_{n-k+1}q_k-q_n](t-x)^{n+2}$,
	$w(x,t)=r_0+(t-x)r_1+\sum_{n=0}^{\infty}\frac{2}{(n+1)(n+2)}[\sum_{k=0}^{n}(n+1-k)p_{n-k+1}r_k-r_n](t-x)^{n+2}$.
$X_1+X_3+X_4$	$u(x,t)=\mathrm{e}^{-x}p_0+p_1t\mathrm{e}^{-2x}-\sum_{n=0}^{\infty}\frac{1}{(n+3)(n+2)}[2\sum_{k=0}^{n+1}r_{n+1-k}q_k+\sum_{k=0}^{n}(n+1-k)r_kq_{n-k+1}+\sum_{k=0}^{n}(n-k+1)q_kr_{n-k+1}]\mathrm{e}^{-nx-3x}t^{n+2}$,
	$v(x,t)=\mathrm{e}^{x}q_0+q_1t+\sum_{n=0}^{\infty}\frac{2}{(n+1)(n+2)}[\sum_{k=0}^{n+1}p_{n+1-k}q_k+\sum_{k=0}^{n}(n+1-k)q_kp_{n-k+1}]\mathrm{e}^{-nx-x}t^{n+2}$,
	$w(x,t)=\mathrm{e}^{-3x}r_0+\mathrm{e}^{-4x}r_1t-\sum_{n=0}^{\infty}\frac{1}{4(n+2)}[(n+1)r_{n+1}-2\sum_{k=0}^{n+1}p_{n+1-k}r_k-2\sum_{k=0}^{n}(n+1-k)r_kp_{n-k+1}]\mathrm{e}^{-nx-5x}t^{n+2}$.

$$\Lambda_1(x,t,u,v,w)=F_1(t),\quad \Lambda_2(x,t,u,v,w)=0,$$
$$\Lambda_3(x,t,u,v,w)=0,$$

where $F_1(t)$ is an arbitrary function.

Given a PDE system of order m

$$\mathscr{R}^\alpha(x,u,\ldots,u_{(k)})=0,\quad \alpha=1,\ldots,m, \tag{24}$$

where $x=(x^1,x^2,\ldots,x^n)$, $u=(u^1,u^2,\ldots,u^m)$ and $u_{(1)},u_{(2)},\ldots,u_{(k)}$ denote the set of all first, second,…, kth-order derivatives of u with regard to x.

The adjoint equations of eq. (24) are defined as

$$(\mathscr{R}^\alpha)^*(x,u,v,\ldots,u_{(k)},v_{(k)})=0,$$
$$\alpha=1,\ldots,m,\quad v=v(x).$$

Moreover,

$$(\mathscr{R}^\alpha)^*(x,u,v,\ldots,u_{(k)},v_{(k)})=\frac{\delta\mathscr{L}}{\delta u^\alpha},$$

where the formal Lagrangian $\mathscr{L}$ is written as

$$\mathscr{L}=v^\beta\mathscr{R}^\beta(x,u,\ldots,u_{(k)}),\quad \beta=1,\ldots,m$$

and the Euler–Lagrange operator has the following form:

$$\frac{\delta}{\delta u^\alpha}=\frac{\partial}{\partial u^\alpha}+\sum_{j=1}^{\infty}(-1)^j D_{i_1}\cdots D_{i_j}\frac{\partial}{\partial u^\alpha_{i_1\cdots i_j}},\quad \alpha=1,2,\ldots,m.$$

DEFINITION 6.1 [20]

System (24) is nonlinearly self-adjoint if the adjoint system is satisfied for all the solutions u of system (24) upon a substitution $v=\varphi(x,u)$ such that $\varphi(x,u)\neq 0$. This implies that

$$(\mathscr{R}^\alpha)^*(x,u,\varphi,\ldots,u_{(k)},\quad \varphi_{(k)})|_{v=\varphi(x,u)}$$
$$=\lambda^\beta_\alpha\mathscr{R}^\beta(x,u,u,\ldots,u_{(k)},u_{(k)}),\quad \alpha,\beta=1,\ldots,m,$$

where λ^β_α is a particular function.

Theorem 6.1 [21]. *The determining system of the multiplier $\Lambda(x,u)$ is the same as the nonlinearly self-adjoint substitution system.*

If the formal Lagrangian of eq. (1) is of the following form

$$\mathscr{L}=\varphi_1(x,t,u,v,w)[u_{xt}+v_xw+vw_x]$$
$$+\varphi_2(x,t,u,v,w)[v_{xt}-2vu_x]$$
$$+\varphi_3(x,t,u,v,w)[w_{xt}-2wu_x],$$

based on Theorem 6.1, we can obtain

$$\varphi_1(x,t,u,v,w)=\Lambda_1(x,t,u,v,w)=F_1(t),$$
$$\varphi_2(x,t,u,v,w)=\Lambda_2(x,t,u,v,w)=0,$$
$$\varphi_3(x,t,u,v,w)=\Lambda_3(x,t,u,v,w)=0. \tag{25}$$

Hence, eq. (1) is nonlinearly self-adjoint with substitution (25).

6.2 *Construction of conservation laws*

Theorem 6.2 [20]. *For eq. (24), every Lie point, Lie–Bäcklund, nonlocal symmetry*

$$X=\xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i}+\eta^\alpha(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u^\alpha}, \tag{26}$$

brings about a conservation law, where $\mathscr{C}^i$ is

$$\mathscr{C}^i=W^\alpha\left[\frac{\partial\mathscr{L}}{\partial u^\alpha_i}-D_j\left(\frac{\partial\mathscr{L}}{\partial u^\alpha_{ij}}\right)+D_jD_k\left(\frac{\partial\mathscr{L}}{\partial u^\alpha_{ijk}}\right)-\cdots\right]+D_j(W^\alpha)\left[\frac{\partial\mathscr{L}}{\partial u^\alpha_{ij}}-D_k\left(\frac{\partial\mathscr{L}}{\partial u^\alpha_{ijk}}\right)+\cdots\right]+D_jD_k(W^\alpha)\left[\frac{\partial\mathscr{L}}{\partial u^\alpha_{ijk}}-\cdots\right],$$

$W^\alpha=\eta^\alpha-\xi^ju^\alpha_j$, and the Lagrangian $\mathscr{L}$ is

$$\mathscr{L}=\Lambda_1[u_{xt}+v_xw+vw_x]+\Lambda_2[v_{xt}-2vu_x]+\Lambda_3[w_{xt}-2wu_x]. \tag{27}$$

For the generator

$$X=\xi^1\frac{\partial}{\partial x}+\xi^2\frac{\partial}{\partial t}+\eta^1\frac{\partial}{\partial u}+\eta^2\frac{\partial}{\partial v}+\eta^3\frac{\partial}{\partial w},$$

according to Theorem 6.2, we can obtain

$$W^1=\eta^1-\xi^1u_x-\xi^2u_t,\ W^2=\eta^2-\xi^1v_x-\xi^2v_t,$$
$$W^3=\eta^3-\xi^1w_x-\xi^2w_t,$$

and so the conserved vectors $\mathscr{C}^t,\mathscr{C}^x$ are

$$\mathscr{C}^t=W^1\left(\frac{\partial\mathscr{L}}{\partial u_t}\right)+W^2\left(\frac{\partial\mathscr{L}}{\partial v_t}\right)+W^3\left(\frac{\partial\mathscr{L}}{\partial w_t}\right), \tag{28}$$

Table 3. Conservation laws of the coupled integrable dispersionless equations.

Generators	C^t	C^x
X_2	0	$u_t F_1'(t) - v_t w F_1(t) - v w_t F_1(t) - u_{tt} F_1(t)$
X_3	0	0
X_4	0	$u_x F_1'(t) - v_x w F_1(t) - w_x v F_1(t) - u_{xt} F_1(t)$
X_5	0	$-F_1'(t)$

$$\mathscr{C}^x = W^1\left(\frac{\partial \mathscr{L}}{\partial u_x} - D_t\frac{\partial \mathscr{L}}{\partial u_{xt}}\right) + W^2\left(\frac{\partial \mathscr{L}}{\partial v_x} - D_t\frac{\partial \mathscr{L}}{\partial v_{xt}}\right) + W^3\left(\frac{\partial \mathscr{L}}{\partial w_x} - D_t\frac{\partial \mathscr{L}}{\partial w_{xt}}\right) + D_t(W^1)\frac{\partial \mathscr{L}}{\partial u_{xt}} + D_t(W^2)\frac{\partial \mathscr{L}}{\partial v_{xt}} + D_t(W^3)\frac{\partial \mathscr{L}}{\partial w_{xt}}. \quad (29)$$

For the generator

$$X_1 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2w\frac{\partial}{\partial w},$$

we have $W^1 = -u - tu_t$, $W^2 = -tv_t$, $W^3 = -2w - tw_t$. On the basis of formulas (28) and (29), we obtain

$$\mathscr{C}^t = 0,$$
$$\mathscr{C}^x = (u + tu_t)F_1'(t) - tv_t w F_1(t) - (2w + tw_t)vF_1(t) - (2u_t + tu_{tt})F_1(t).$$

With regard to other generators of eq. (1), we list their conservation laws in table 3.

7. Conclusions

In this paper, we apply Lie symmetry analysis method to the coupled integrable dispersionless equations. Based on this method, the optimal system of one-dimensional subalgebras and similarity reductions are derived. Furthermore, exact solutions of the reduced equations, by the method of power series, are given. Eventually, it is shown that eq. (1) is nonlinearly self-adjoint. At the same time, we obtain conservation laws. Lie symmetry analysis method is helpful to address nonlinear partial differential equations, which is of great significance in physics, mechanics and other areas.

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References

[1] K Konno and H Oono, *J. Phys. Soc. Jpn.* **63**, 377 (1994)
[2] Z W Xu, *Commun. Nonlinear Sci. Numer. Simul.* **28**, 40 (2016)
[3] Z T Fu and J Y Mao, *Commun. Nonlinear Sci. Numer. Simul.* **17**, 2362 (2012)
[4] C H Gu, *Soliton theory and its application* (Springer, Berlin, 1995)
[5] R Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971)
[6] V B Matveev and M A Salle, *Darboux transformations and solitons* (Springer, Berlin, 1991)
[7] F Cariello and M Tabor, *Physica D* **39**, 77 (1989)
[8] S Lie, *Theories der Tranformationgruppen* (Dritter and Letzter Abschnitt, Teubner, Leipzig, Germany, Berlin, 1888)
[9] G W Bluman and J D Cole, *J. Math. Mech.* **18**, 1025 (1969)
[10] S Sahoo and S S Ray, *Comput. Math. Appl.* **73**, 253 (2017)
[11] A A Afify and M Abd El-Aziz, *Pramana – J. Phys.* **88**: 31 (2017)
[12] G W Wang and M S Hashemi, *Pramana – J. Phys.* **88**: 7 (2017)
[13] C S Gardner, J M Greene, M D Kruskal and R M Miura, *Phys. Rev. Lett.* **19**, 1095 (1967)
[14] R O Popovych, V M Boyko, M O Nesterenko and M W Lutfullin, *J. Phys. A* **36**, 7337 (2003)
[15] P J Olver, *Applications of Lie groups to differential equations* (Springer, Berlin, 1986)
[16] Y N Grigoriev, N H Ibragimov, V F Kovalev and S V Meleshko, *Symmetry of integro-differential equations: With applications in mechanics and plasma physics* (Springer, Berlin, 2010)
[17] W Rudin, *Principles of mathematical analysis*, 3rd edn (China Machine Press, Beijing, 2004)
[18] J F Ganghoffer and I Mladenov, *Similarity and symmetry methods* (Springer, Berlin, 2014)
[19] N H Ibragimov, *J. Math. Anal. Appl.* **333**, 311 (2007)
[20] N H Ibragimov, *J. Phys. A* **44**, 432002 (2011)
[21] Z Y Zhang, *Commun. Nonlinear Sci. Numer. Simul.* **20**, 338 (2015)