Pramana – J. Phys. (2019) 93:77
https://doi.org/10.1007/s12043-019-1840-1

The energy fluxes of surface waves propagating along the interface between nonlinear media with different characteristics

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MS received 3 December 2018; revised 16 May 2019; accepted 4 June 2019

Abstract. We describe analytically the nonlinear surface waves at the interface between two nonlinear media with different characteristics. We use one-dimensional nonlinear Schrödinger equation with cubic nonlinearity differing on the opposite sides of the interface. We take into account the interaction of excitations with media interface. We consider the interaction of the wave with the interface using the local potential approximated by Dirac delta function. We derive and analyse three types of dispersion equations determining the surface wave frequencies. We propose two approaches to determine the flux depending on the choice of one of the possible control parameters. We calculate the energy flux of the surface waves and analyse the influence of intensity interaction of excitations with interface and difference of media characteristics on the opposite sides of the interface.

Keywords. Nonlinear Schrödinger equation; media interface; planar defect; localised states; nonlinear surface waves; energy flux.

PACS Nos **05.45.Yv; 42.65.Tg; 03.75.Lm**

1. Introduction

The results of theoretical researches of excitation localisation and existence of surface wave in nonlinear media were published in a number of papers [1–6]. However, the continued interest in such studies is due to the new technological capabilities and the development of new devices using the physical properties of the localised states and surface waves [7]. The intensive development of fibre optics and fabrication of new types of waveguides, filters, switchers, sensors and other devices require the study of localisation features [8].

Many researchers have theoretically considered the localisation of states in nonlinear media or nonlinear surface waves near the defects [9–15]. But the description of the localisation of nonlinear states near the interface between two nonlinear media with different characteristics remains unaffected while taking into account the interaction of the excitations with the boundary.

In this paper, we propose the analytical description of localisation peculiarities in the framework of a simple model based on the one-dimensional nonlinear Schrödinger equation (NSE) [9,10]. We take into account the nonlinearity term in the cubic form (the so-called 'Kerr-type nonlinearity') [2,3]. Also the NSE is known with nonlinear terms in other forms [16–20]. The NSE can describe the excitations of elastic [21], electrical [4,10,11] and magnetic fields [22,23].

As is well known, the stationary NSE with cubic nonlinearity is equivalent to the equation derived from the Ginsburg–Landau functional minimisation (see refs [24–26]). In particular, in ref. [26] in the framework of the Ginsburg–Landau theory it was shown that the redistribution of the charge (and, hence, spin) density near the Fe/Cr interfaces has given rise to the formation of an essentially inhomogeneous spin-density-wave state in the chromium spacer.

It is also known that the stationary NSE with cubic nonlinearity is equivalent to the Gross–Pitaevskii equation (GPE) derived for the Bose–Einstein condensation [3,27,28]. In ref. [29] it is shown that in the case of indirect excitons in an annular trap, the one-dimensional GPE permits an analytical solution and it turns out that there can be no bound state in a one-dimensional symmetric potential well. Matter–wave GPE solitons are self-trapped modes in Bose gases with attractive interactions between atoms, which have been formed in

Published online: 22 August 2019

Bose–Einstein condensates of ^{7}Li in ref. [30] and ^{85}Rb in ref. [31].

In the present paper, we take into account the interaction of excitations with media interface. The excitation (or wave) interaction with defects was considered with local (point) potential approximated by the Dirac delta function in refs [3,9,10,32,33]. In particular, the wave localisation in nonlinear medium with defect and nonlinear localised states near the boundary between linear and nonlinear media based on the NSE with the Dirac delta function term were described in refs [34,35]. Recently, the authors of refs [36–44] considered additional nonlinear response of defect for different cases of the NSE solutions.

We consider localisation peculiarities near the interface between two nonlinear media with different characteristics. As a result, in the framework of the proposed model, the possibility of describing localisation peculiarities near the interface of the media with opposite signs of nonlinearity arises. In this paper, we analyse the influence of two effects on nonlinear excitation localisation: the interaction of excitations with interface and difference of media characteristics on opposite sides of the interface.

We derive and analyse the dispersion equations of the localised states of three types and calculate the energy flux conserved along the interface between the media. The use of the model based on the NSE with the Dirac delta function allows us to obtain all results in an explicitly analytical form.

2. The problem formulation

2.1 *Equations*

We consider the contact of two nonlinear media with a cubic Kerr nonlinearity. The media are separated by an ultrathin layer. The thickness of the layer is much smaller than the characteristic scale of localisation of medium parameter perturbations. We can consider the ultrathin layer as a planar interface between the media in the limit of its infinitely small thickness. The media interface plays the role of a waveguide in this context. We choose a coordinate system so that the media interface lies in the yz plane and passes through the origin perpendicular to the x-axis.

Let the field distribution $\Psi(x,t)$ obey the NSE in the standard form:

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \Omega(x)\psi - \gamma(x)|\psi|^2\psi + U(x)\psi. \tag{1}$$

The function $U(x)$ in the NSE (1) is determined by the parameters of the interaction of the field with the boundary between the layers and represents the potential that simulates the interaction of the field with the boundaries between the layers [45].

A monochromatic polarised wave with frequency ω is described by the function $\psi(x,t) = u(x)\exp(-\mathrm{i}\omega t)$. The function $u(x)$ obeys the stationary NSE represented in the standardised form

$$\omega u = -u''/2 + \Omega(x) - \gamma(x)|u|^2 + U(x)u. \tag{2}$$

We approximate the function $\Omega(x)$ by the piece-constant function:

$$\Omega(x) = \begin{cases} \Omega_1, & x < 0, \\ \Omega_2, & x > 0. \end{cases}$$

Here $\Omega_{1,2}$ are constants.

Also, we approximate the Kerr nonlinearity function $\gamma(x)$ by the piece-constant function:

$$\gamma(x) = \begin{cases} \gamma_1, & x < 0, \\ \gamma_2, & x > 0. \end{cases}$$

Here $\gamma_{1,2}$ are constants having any signs.

Because the thickness of the boundary is significantly smaller than the characteristic localisation length of the field perturbations, we represent the local potential in the short-range ('point') approximation in the form

$$U(x) = U_0\delta(x), \tag{3}$$

where $\delta(x)$ is the Dirac delta function, U_0 is the interaction intensity of field excitations with interface in linear approximation (often called the 'power' of a defect or interface): when $U_0 > 0$ excitations are repelled from the interface and when $U_0 < 0$ excitations are attracted by the interface.

In the present paper, we analyse the energy (power) flux conserved along the interface between the media. The energy flux is the first NSE (2) integral:

$$N = \int_{-\infty}^{+\infty} |u(x)|^2 \,\mathrm{d}x. \tag{4}$$

The solution of the NSE (2) with the potential (3) is reduced to the solution of two stationary NSE without potential on the semiaxes:

$$u'' + 2(\omega - \Omega_1 + \gamma_1|u|^2)u = 0, \quad x < 0, \tag{5}$$

$$u'' + 2(\omega - \Omega_2 + \gamma_2|u|^2)u = 0, \quad x > 0 \tag{6}$$

with the following boundary conditions:

$$u(-0) = u(+0) = u_0, \tag{7}$$

$$u'(+0) - u'(-0) = 2U_0u_0. \tag{8}$$

Thus, the formulation of the model is reduced to a contact boundary-value problem for nonlinear equations (5) and (6) with boundary conditions (7) and (8).

2.2 *Physical interpretation*

The physical applicability of models using the NSE (2) has been repeatedly discussed earlier in refs [22–29,39–41,45–49]. For example, in optically layered media, we indicate the following interpretation of eq. (2) coefficients in refs [46,48]. In particular, wave function u has the sense of the component of the electric field strength E_y in transverse electric (TE)-polarised monochromatic electromagnetic waves with diffraction coefficient $D = 1/2$. The frequency ω in eq. (2) is the propagation constant. The function $\Omega(x)$ is directly proportional to the linear refractive index n_{L}. The nonlinearity term γ of eq. (2) is proportional to the coefficient of Kerr nonlinearity α of nonlinear refractive index $n_{\mathrm{N}}(x, |E_y|^2) = \alpha(x)|E_y|^2$ [32]. Parameter U_0 of potential (2) is directly proportional to the linear refractive index inside a thin-film interface.

In ref. [50] the distribution of the light field in a nonlinear surface wave was experimentally studied, depending on the intensity of the light flux with a dependent dielectric film along the self-focussing medium. The implementation of dependence of the field profile on the energy flux is reflected in the design of waveguide devices. Nonlinear waveguide devices are based on the fact that the profile of the light field and the propagation constant in a nonlinear medium can depend on the energy flux of the incident beam. In this case, in contacting media, the refractive index depends on the intensity of the incident beam. This property is used in the construction of two types of optical devices. The first type includes devices in which a nonlinear change in the refractive index is insignificant compared to the difference in the refractive indices of the contacting media. In this case, the distribution of the light field is described using linear waveguide modes. The devices operating in this mode include nonlinear prism and lattice couplers [51,52] and nonlinear coherent coupler [53,54]. The second type of optical devices is those in which the optically induced change in the refractive index is comparable to or exceeds the difference in refractive indices between the contacting media. In this case, the distribution of the light field depends on the energy flux of the incident beam. The devices operating in this mode include optical power limiters, devices with low critical threshold energy flow and optical switches [55,56].

As we have noted, an equation in the form of eq. (2) is formally equivalent to the GPE arising in the Bose–Einstein condensation theory [27–31]. Within the framework of this theory the flux defined by eq. (4) is interpreted as the total number of particles in the system and, in fact, plays the role of the normalisation condition. A model of the interferometer which uses a local (δ-functional) potential, embedded into a harmonic-oscillator trapping potential, as the splitter for the incident soliton was elaborated in ref. [27]. An estimate demonstrates that this setting may be implemented by means of the localised Feshbach resonance controlled by a focussed laser beam. The same system may be realised as a nonlinear waveguide in optics.

Another physical interpretation of the results obtained in this paper to describe the localised electron phase transition under the framework of the Ginsburg–Landau theory [24,25] can be suggested. Specific examples of such systems are diluted alloys of chromium. In these systems, the generalised order parameter describes the distribution of linearly polarised spin-density waves. We note that eq. (2) is applicable not only for optical systems. In particular, in refs [24,25], an equation of this form was used to describe the occurrence of localised states in the model of antiferromagnetics with congruent sections of the Fermi surface. In this interpretation, the function u is a generalised order parameter describing the distribution of linearly polarised spin-density waves.

The Ginsburg–Landau theory can be used near the Lifshitz point for the case of small order parameter $|u| \ll T, \mu$ and its slow variation $|u'/u| \ll 1/\zeta_0$, where T is the temperature, μ is an arbitrary value of the non-congruence parameter and ζ_0 is a correlation length when $T = 0$. The parameter μ describes the deviation of band occupation from one half in the Peierls-transition model and the spin splitting of the electronic states in an exchange field in superconductors. In ref. [24] the dimensionless order parameter φ and variable x were introduced by $u \to \varphi u_0$ and $x \to x u_0/v_{\mathrm{F}}$, where u_0 is an order parameter when $T = 0$ and ideal congruent parameter $\mu = 0$, v_{F} is the velocity on the Fermi surface. The Ginsburg–Landau functional near the Lifshitz point has the form

$$F[\varphi] = \int \{C_1\varphi^2 + C_2(\varphi^4 + \varphi'^2) - \Gamma[\varphi]\delta(x)\}\,\mathrm{d}x.$$

Here Γ is a source function (see ref. [24]): $\Gamma \sim \varphi^2$ for a model of local phase transition. The coordinate axis x is perpendicular to the plane of the planar spin-polarised defect. The coefficients of Ginsburg–Landau functional F have been derived in ref. [24]. Variation of the Ginsburg–Landau functional F leads to the equation

$$\varphi'' - (C_1/C_2 + 2|\varphi|^2)\varphi = -\delta(x)D(\varphi)/2C_2,$$

where $D(\varphi) = \partial\Gamma[\varphi]/\partial\varphi \sim \varphi$. The periodic solution of similar equation was analysed in ref. [26].

One more physical interpretation of the coefficients of eq. (2) is present in refs [22,23,45,48]. The small-amplitude spin waves in multilayer magnetic materials described by NSE have been described in refs [22,23,45,48]. For a system with one narrow magnetic layer that differs from a matrix material in the single-ion anisotropy constant, the existence of a spin wave state localised near this layer was demonstrated. The wave function u plays the role of a complex-valued function related to the components of the magnetisation vector. The defect power U_0 is related to the magnetic anisotropy β_b, which remains constant inside the thin-film interface. The nonlinearity parameter γ of the NSE (2) is related to the magnetic anisotropy of the contacting media. The coefficient Ω of the NSE (2) has a sense of ferromagnetic resonance frequency (see ref. [48]).

3. Nonlinear surface waves

In this paper, we consider only localised in-space solutions of the formulated boundary value problem (5)–(8) satisfying the vanishing condition at infinity: $|u(x)| \rightarrow 0$, $x \rightarrow \infty$. Such solutions describe the distribution of the field travelling along the interface wave quickly decaying with the distance from the interface. In this paper, we obtain and analyse the analytical solutions of the boundary-value problems (5)–(8) and the surface wave dispersion equations. Equations (5) and (6), with the constant coefficients, can be solved analytically and there is no need to use numerical methods to solve it. Numerical analysis can be applied to solve the non-stationary (time-dependent) NSE (1). However, its consideration is beyond the scope of this article.

As is well known, surface waves propagating along the interfaces between two nonlinear media or linear and nonlinear media have been considered several times [4,5,13–15,33,34,57–63]. However, we deem it unnecessary to carry out a detailed analysis of the influence of interface characteristic and the difference of the media characteristics on the field localisation within the framework of different approaches. Approaches to analysis are determined by a set of parameters that physically allow one to control such processes in specific experiments.

We consider the following three cases of nonlinearity signs:

(1) The interface between two self-focussing (attractive) media corresponds to the positive value of Kerr nonlinearity function $\gamma(x)$: $\gamma_{1,2} > 0$.
(2) The interface between two defocussing (repulsive) media corresponds to the negative value of Kerr nonlinearity function $\gamma(x)$: $\gamma_{1,2} < 0$.
(3) The interface between self-focussing (attractive) and defocussing (repulsive) media corresponds to the opposite signs of Kerr nonlinearity function $\gamma(x)$ from different sides of interface: $\gamma_1 < 0$ and $\gamma_2 > 0$.

3.1 *Nonlinear surface waves near the interface of self-focussing media*

In the case of the nonlinear self-focussing media contact with $\gamma_{1,2} > 0$ NSE, (5) and (6) have the solution in the range $\omega < \omega_{\max} = \min\{\Omega_{1,2}\}$:

$$u(x) = \begin{cases} \dfrac{q_1}{\gamma_1^{1/2}\cosh(q_1(x+x_1))}, & x < 0, \\ \dfrac{q_2}{\gamma_2^{1/2}\cosh(q_2(x-x_2))}, & x > 0. \end{cases} \tag{9}$$

Solution (9) describes the nonlinear surface wave with the spatial damping factors that are determined by the expressions

$$q_j^2 = 2(\Omega_j - \omega). \tag{10}$$

Hereinafter, the index value $j = 1$ refers to the characteristics of the environment when $x < 0$ and the index value $j = 2$ refers to the characteristics of the environment when $x > 0$.

From eq. (10) follows the relationship of the spatial damping factors:

$$q_2^2 = 2\Delta\Omega + q_1^2, \quad \Delta\Omega = \Omega_2 - \Omega_1.$$

Substituting solution (9) into the boundary condition (7) leads to the relation:

$$q_1\gamma_2^{1/2}\cosh(q_2x_2) = q_2\gamma_1^{1/2}\cosh(q_1x_1). \tag{11}$$

Substituting solution (9) into the boundary condition (8) leads to the dispersion equation:

$$q_1\tanh(q_1x_1) + q_2\tanh(q_2x_2) = 2U_0. \tag{12}$$

From dispersion equation (12) we can find dependence of the nonlinear surface wave frequency on medium and interface parameters.

The wave described by eq. (9) can have a maximum at the interface of the media, or its maximum can be located in a nonlinear medium (to the right or left of the interface). Its position is determined by the parameters of the system. Therefore, having the ability to control such parameters, we can adjust the location of the maximum field intensity.

We can choose one of the possible control (free) parameters on the dependence of the setting of specific experiments. We propose the position x_1 (or x_2) or the value of the wave amplitude at the interface u_0:

$$u_0 = \frac{q_j}{\gamma_j^{1/2}\cosh(q_j x_j)}. \qquad (13)$$

If we choose as the free parameter the position x_1 (or x_2), then from the dispersion equation (12) we can determine the dependence of nonlinear surface wave frequency on the medium and the interface parameters as a function $\omega = \omega(U_0, \Omega_{1,2}, \gamma_{1,2}, x_1)$.

If we choose as the free parameter the wave amplitude at the interface u_0, then from the dispersion equation (12) we can determine the dependence of nonlinear surface wave frequency on the medium and the interface parameters as a function $\omega = \omega(U_0, \Omega_{1,2}, \gamma_{1,2}, u_0)$.

3.1.1 *Analytical results.*

Case A: Symmetrical surface waves. In the case of $x_1 = x_2 = x_0$, $\gamma_1 = \gamma_2 = \gamma$ and $\Omega_1 = \Omega_2 = \Omega$ (then $q_1 = q_2 = q$), from eq. (12) we obtain the dispersion equation derived in ref. [34]:

$$q \tanh(q x_0) = U_0. \qquad (14)$$

Explicit solution of dispersion equation (14) in the limit case of a long-wave approximation when $q x_0 \ll 1$ (it means $|\Omega - \omega| \ll 1/2x_0^2$) has the form

$$q^2 = U_0/x_0.$$

The frequency of the symmetrical surface wave has the form

$$\omega = \Omega - U_0/2x_0.$$

If we choose the amplitude of interface oscillations u_0 defined by eq. (13) as a free parameter, we can find the exact solution of dispersion equation (14):

$$q^2 = U_0^2 + \gamma u_0^2.$$

Then the frequency has the form

$$\omega = \Omega - (U_0^2 + \gamma u_0^2)/2.$$

Case B: Non-symmetrical surface waves. Non-symmetrical surface waves can exist near the interface between media having different characteristics: $\gamma_1 \neq \gamma_2$ and $\Omega_1 \neq \Omega_2$.

Case B.1: At first we consider the special case of $x_1 = 0$ and $x_2 \neq 0$. In this case we find the exact solution of dispersion equation (12) in the form

$$q_1^2 = \frac{2(2U_0^2 - \Delta\Omega)}{1 - \eta^2}, \qquad (15)$$

where

$$\eta = (\gamma_2/\gamma_1)^{1/2}.$$

The frequency of the non-symmetrical surface wave has the form

$$\omega = \Omega_1 - \frac{2U_0^2 - \Delta\Omega}{1 - \eta^2}. \qquad (16)$$

From eqs (11) and (15), we find

$$x_2 = \sqrt{\frac{1-\eta^2}{2(2U_0^2 - \eta^2\Delta\Omega)}}\cosh^{-1}\left(\frac{1}{\eta}\sqrt{\frac{2U_0^2 - \eta^2\Delta\Omega}{2U_0^2 - \Delta\Omega}}\right). \qquad (17)$$

The non-symmetrical surface wave of such a special kind exists under one of the following cases: (i) $\gamma_1 > \gamma_2$ and $U_0^2 > \Delta\Omega/2$ and (ii) $\gamma_1 < \gamma_2$ and $U_0^2 < \Delta\Omega/2$.

Case B.2: Now we consider the case of $x_1 \neq x_2 \neq 0$. Explicit solution of dispersion equation (12) in the limit case of a long-wave approximation when $q_j x_j \ll 1$ (it means $|\Omega_j - \omega| \ll 1/2x_j^2$) has the form

$$q_1^2 = \frac{2\Delta\Omega}{\eta^2 - 1}. \qquad (18)$$

The frequency of the non-symmetrical surface wave has the form

$$\omega = \Omega_1 - \frac{\Delta\Omega}{\eta^2 - 1}. \qquad (19)$$

From eqs (11) and (18), we find

$$x_1 = \frac{(\eta^2 - 1)U_0}{\Delta\Omega} - \eta x_2. \qquad (20)$$

We choose x_2 as a free parameter here.

The non-symmetrical surface wave of such a kind exists under one of the following cases: (i) $\gamma_1 > \gamma_2$ and $\Omega_1 > \Omega_2$ and (ii) $\gamma_1 < \gamma_2$ and $\Omega_1 < \Omega_2$.

If we choose now the amplitude of interface oscillations defined by eq. (13) as a free parameter we can find the exact solution of dispersion equation (12):

$$q_1^2 = \gamma_1 u_0^2 + Q_c^2. \qquad (21)$$

Here

$$Q_c = \frac{2\Delta\Omega + (\gamma_1 - \gamma_2)u_0^2 - 4U_0^2}{4U_0}.$$

The frequency of the non-symmetrical surface wave has the form

$$\omega = \Omega_1 - (\gamma_1 u_0^2 + Q_c^2)/2. \qquad (22)$$

From eq. (13), we find

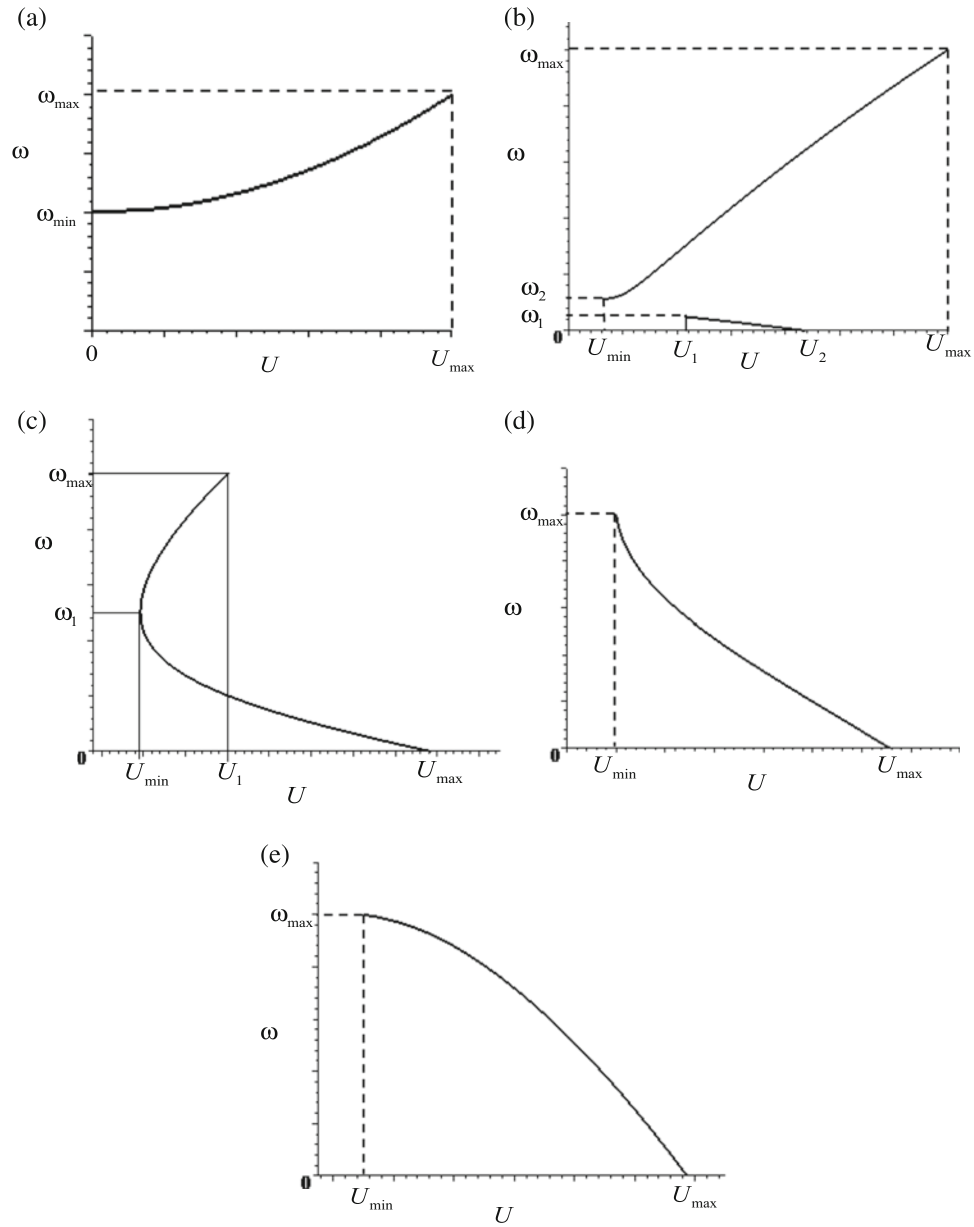

Figure 1. The dependence of surface wave frequency (type-one) on the power of the interface for fixed values of parameters in model conventional units: $\Omega_1 = 1$, $\Omega_2 = 1.5$, $\eta = 1.5$, (**a**) $x_1 = 0$, (**b**) $x_1 = 0.4668$, (**c**) $x_1 = 0.49$, (**d**) $x_1 = 0.7$ and (**e**) $x_1 = 5$.

$$x_1 = \frac{1}{\sqrt{\gamma_1 u_0^2 + Q_c^2}} \cosh^{-1}\left(\sqrt{\frac{Q_c^2}{\gamma_1 u_0^2} + 1}\right),$$

$$x_2 = \frac{1}{\sqrt{2\Delta\Omega + \gamma_1 u_0^2 + Q_c^2}} \cosh^{-1}\left(\sqrt{\frac{2\Delta\Omega + Q_c^2}{\gamma_1 u_0^2} + 1}\right).$$

3.1.2 *Numerical results.* To numerically analyse the dependence $\omega = \omega(U_0, \Omega_{1,2}, \zeta, x_1)$ we exclude x_2 from eq. (12) using eq. (11) and obtain the dispersion equation in the form

$$q_1 \tanh(q_1 x_1) + q_2 \left\{1 - \left(\frac{\eta q_1}{q_2 \cosh(q_1 x_1)}\right)^2\right\}^{1/2} = 2U_0.$$

We analyse the dependence of frequency ω on defect power U_0. We find five different regimes depending on the control parameter (x_1) values (see figure 1):

(1) In the region $0 \le x_1 < x_{c1}$ the wave exists in the frequency range $\omega_{\min} < \omega < \omega_{\max}$ for a limited interface power from the interval $U_{\min} < U_0 < U_{\max}$ (figure 1a).

(2) In the region $x_{c1} < x_1 < x_{c2}$ we find bifurcation. There exist two branches of surface waves (figure 1b). The branch with $\omega_2 < \omega$ exists for $U_{\min} < U_0 < U_1$. Two branches with $0 < \omega < \omega_1$ and $\omega_2 < \omega$ exist for $U_1 < U_0 < U_2$. Then one branch with $\omega_2 < \omega$ exists for $U_2 < U_0 < U_{\max}$. There is a band gap with frequencies in the interval $\omega_1 < \omega < \omega_2$.

(3) In the region $x_{c2} < x_1 < x_{c3}$ there exist two branches of surface waves for the limited interface power from the interval $U_{\min} < U_0 < U_1$ (figure 1c). The branch with $\omega < \omega_1$ exists for $U_1 < U_0 < U_{\max}$. This is the gapless bifurcation regime.

(4) In the region $x_{c3} < x_1 < x_{c4}$ the frequency monotonically decreases downwards for the limited interface power from the interval $U_{\min} < U_0 < U_{\max}$ (figure 1d).

(5) In the region $x_1 > x_{c4}$ the frequency monotonically decreases upwards for the limited interface power from the interval $U_{\min} < U_0 < U_{\max}$ (figure 1e).

The values of the interval limits $U_{\min} < U_0 < U_{\max}$ are almost independent of the control parameter x_1.

We note that wave with $x_1 = 0$ can exist only when $\Omega_1 \neq \Omega_2$. Also the surface waves cannot exist for any values of interface power.

3.2 *Nonlinear surface waves near the interface of the defocussing media*

In the case of the nonlinear defocussing media contact with $\gamma_{1,2} < 0$ NSE (5) and (6) have the solution in the range $\omega < \omega_{\max}$:

$$u(x) = \begin{cases} \dfrac{q_1}{g_1^{1/2}\sinh(q_1(x+x_1))}, & x < 0, \\ \dfrac{q_2}{g_2^{1/2}\sinh(q_2(x-x_2))}, & x > 0. \end{cases} \tag{23}$$

Here $g_{1,2} = -\gamma_{1,2} > 0$. The spatial damping factors are determined by eq. (10).

Substituting solution (23) into the boundary condition (7) leads to the relation

$$q_1 g_2^{1/2} \sinh(q_2 x_2) = -q_2 g_1^{1/2} \sinh(q_1 x_1). \tag{24}$$

Substituting solution (9) into the boundary condition (8) leads to the dispersion equation

$$q_1 \coth(q_1 x_1) + q_2 \coth(q_2 x_2) = 2U_0. \tag{25}$$

The amplitude of interface oscillations is determined by the equation

$$u_0 = \mp \frac{q_j}{g_j^{1/2} \sinh(q_j x_j)}. \tag{26}$$

3.2.1 *Numerical results.* To numerically analyse the dependence $\omega = \omega(U_0, \Omega_{1,2}, \eta, x_1)$ of the nonlinear surface waves near the interface of defocussing media we exclude x_2 from eq. (25) using eq. (24) and obtain the dispersion equation in the form

$$q_1 \coth(q_1 x_1) + q_2 \left\{ 1 + \left(\frac{\eta q_1}{q_2 \sinh(q_1 x_1)} \right)^2 \right\}^{1/2} = 2U_0.$$

We find four different regimes depending on control parameter (x_1) values (see figure 2):

(1) In the region $0 \leq x_1 < x_{c1}$ the surface wave frequency almost linearly increases in the range $0 < \omega < \omega_{\max}$ for the limited interface power from the interval $U_{\min} < U_0 < U_{\max}$ (figure 2a).

(2) In the region $x_{c1} < x_1 < x_{c2}$ the surface wave frequency monotonically increases in the same ranges (figure 2b).

(3) In the region $x_{c2} < x_1 < x_{c3}$ there exist two branches of surface waves for the limited interface power from the interval $U_{\min} < U_0 < U_1$ (figure 2c). One branch with $\omega > \omega_1$ exists for $U_1 < U_0 < U_{\max}$. This is the gapless bifurcation regime.

(4) In the region $x_1 > x_{c3}$ the frequency monotonically decreases upwards for the limited interface power from the interval $U_{\min} < U_0 < U_{\max}$ (figure 2d).

In the difference from the results of §3.1.2 here we note that the values of the interface power interval $U_{\min} < U_0 < U_{\max}$ depend on the control parameter x_1. The values of the boundaries of the interval $U_{\min} < U_0 < U_{\max}$ shift to the right as the value of x_1 increases.

3.2.2 *Analytical results.*

Case A: Symmetrical surface waves. When $x_1 = x_2 = x_0$, $g_1 = g_2 = g$ and $\Omega_1 = \Omega_2 = \Omega$ (then $q_1 = q_2 = q$) from eq. (25) we obtain the dispersion equation derived in ref. [34]:

$$q \coth(q x_0) = U_0. \tag{27}$$

If we choose the amplitude of interface oscillations u_0 defined by eq. (13) as a free parameter we can find exact solution of dispersion equation (27):

$$q^2 = U_0^2 - g u_0^2.$$

Then the frequency has the form

$$\omega = \Omega - (U_0^2 - g u_0^2)/2.$$

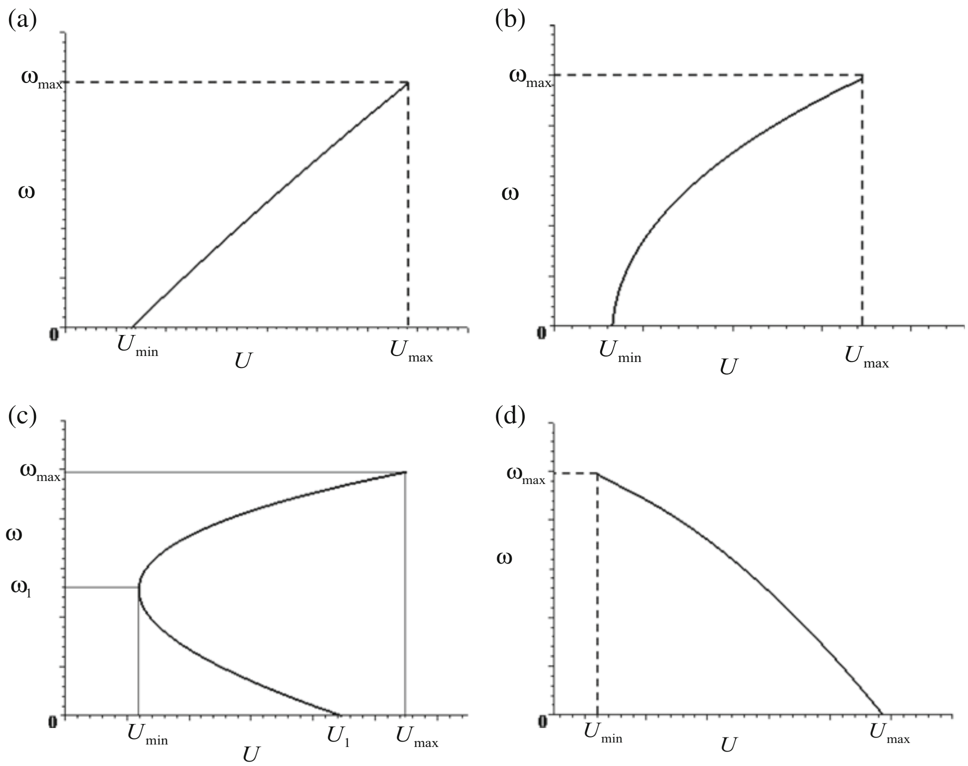

Figure 2. The dependence of surface wave frequency (type-two) on the power of the interface for fixed values of parameters in model conventional units: $\Omega_1 = 1$, $\Omega_2 = 1.5$, $\eta = 4.5$, (**a**) $x_1 = 0.4$, (**b**) $x_1 = 0.9$, (**c**) $x_1 = 1.5$ and (**d**) $x_1 = 8$.

The symmetrical surface waves exist under the condition $u_0^2 < U_0^2/g$.

Case B: Non-symmetrical surface waves ($g_1 \neq g_2$ and $\Omega_1 \neq \Omega_2$). If we choose now the amplitude of interface oscillations defined by eq. (13) as a free parameter we can find the exact solution of dispersion equation (25):

$$q_1^2 = Q_s^2 - g_1 u_0^2. \tag{28}$$

Here

$$Q_s = \frac{4U_0^2 - 2\Delta\Omega + (g_1 - g_2)u_0^2}{4U_0}.$$

The frequency of the non-symmetrical surface wave has the form

$$\omega = \Omega_1 - (Q_s^2 - g_1 u_0^2)/2. \tag{29}$$

From eq. (26) we find

$$x_1 = \frac{1}{\sqrt{Q_s^2 - g_1 u_0^2}} \sinh^{-1}\left(\sqrt{\frac{Q_s^2}{g_1 u_0^2} - 1}\right),$$

$$x_2 = -\frac{1}{\sqrt{2\Delta\Omega + Q_s^2 - g_1 u_0^2}} \sinh^{-1}\left(\sqrt{\frac{2\Delta\Omega + Q_s^2}{g_1 u_0^2} - 1}\right).$$

3.3 *Nonlinear surface waves near the interface between the self-focussing and defocussing media*

In the case of contact of self-focussing and defocussing media with $\gamma_1 < 0$ and $\gamma_2 > 0$ NSE (5) and (6) have the solution in the range $\omega < \min\{\Omega_{1,2}\}$:

$$u(x) = \begin{cases} \dfrac{q_1}{g_1^{1/2} \sinh(q_1(x + x_1))}, & x < 0, \\[2ex] \dfrac{q_2}{\gamma_2^{1/2} \cosh(q_2(x - x_2))}, & x > 0. \end{cases} \tag{30}$$

Solution (30) describes the nonlinear surface wave of the non-symmetrical shape only.

Substituting solution (30) into the boundary condition (7) leads to the relation

$$q_1 \gamma_2^{1/2} \cosh(q_2 x_2) = q_2 g_1^{1/2} \sinh(q_1 x_1). \tag{31}$$

Substituting solution (30) into the boundary condition (8) leads to the dispersion equation

$$q_1 \coth(q_1 x_1) + q_2 \tanh(q_2 x_2) = 2U_0. \tag{32}$$

The amplitude of interface oscillations is determined by the equation

$$u_0 = \frac{q_1}{g_1^{1/2} \sinh(q_1 x_1)} = \frac{q_2}{\gamma_2^{1/2} \cosh(q_2 x_2)}. \tag{33}$$

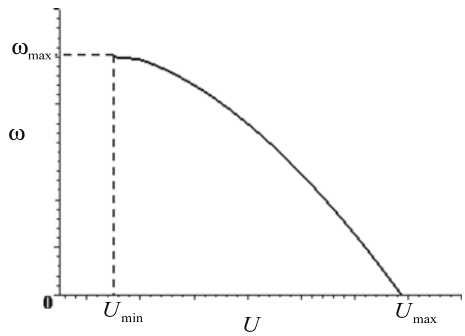

Figure 3. The dependence of surface wave frequency (type-three) on the power of the interface for fixed values of parameters in model conventional units: $\Omega_1 = 1$, $\Omega_2 = 1.5$, $\eta = 1.5$ and $x_1 = 5$.

3.3.1 *Numerical results.* To numerically analyse the dependence $\omega = \omega(U_0, \Omega_{1,2}, \zeta, x_1)$ of the nonlinear surface waves near the interface of the defocusing media, we exclude x_2 from eq. (32) using eq. (31) and obtain the dispersion equation in the form

$$q_1 \coth(q_1x_1)+q_2\left\{1-\left(\frac{\zeta q_1}{q_2\sinh(q_1x_1)}\right)^2\right\}^{1/2} = 2U_0,$$

where

$$\zeta = (\gamma_2/g_1)^{1/2}.$$

We find only one regime in dependence on control parameter x_1 values. The surface wave frequency monotonically decreases upwards for any value of x_1 for the interface power from the interval $U_{\min} < U_0 < U_{\max}$ (figure 3).

3.3.2 *Analytical results.*
Case A: At first we consider the special case of $x_2 = 0$ and $x_1 \neq 0$.

In this case we find the exact solution of dispersion equation (32) in the form

$$q_1^2 = \frac{2(2U_0^2\zeta^2 - \Delta\Omega)}{1+\zeta^2}. \tag{34}$$

The frequency of the surface wave has the form

$$\omega = \Omega_1 - \frac{2U_0^2\zeta^2 - \Delta\Omega}{1+\zeta^2}. \tag{35}$$

From eqs (31) and (34) we find

$$x_1 = \sqrt{\frac{1+\zeta^2}{2(2U_0^2\zeta^2-\Delta\Omega)}}\sinh^{-1}\left(\sqrt{\frac{2U_0^2\zeta^2-\Delta\Omega}{2U_0^2+\Delta\Omega}}\right). \tag{36}$$

The surface wave (30) of such a special kind exists under the condition: $-2U_0^2 < \Delta\Omega < 2U_0^2\zeta^2$.

Case B: Now we consider the case of $x_1 \neq x_2 \neq 0$. If we choose now the amplitude of interface oscillations defined by eq. (13) as a free parameter we can find the exact solution of dispersion equation (32):

$$q_1^2 = Q^2 - g_1u_0^2. \tag{37}$$

Here

$$Q = \frac{4U_0^2 - 2\Delta\Omega + (g_1+\gamma_2)u_0^2}{4U_0}.$$

The frequency of the surface wave has the form

$$\omega = \Omega_1 - (Q^2 - g_1u_0^2)/2. \tag{38}$$

From eq. (33) we find

$$x_1 = \frac{1}{\sqrt{Q^2-g_1u_0^2}}\sinh^{-1}\left(\sqrt{\frac{Q^2}{g_1u_0^2}-1}\right),$$

$$x_2 = \frac{1}{\sqrt{2\Delta\Omega+Q^2-g_1u_0^2}}\sinh^{-1}\left(\sqrt{\frac{2\Delta\Omega+Q^2}{g_1u_0^2}-1}\right).$$

4. The energy flux of nonlinear surface waves

4.1 *The energy flux near the interface of the self-focussing media*

We substitute solution (9) into eq. (4) and obtain the energy flux in the form

$$N = \frac{q_1}{\gamma_1}\{1+\tanh(q_1x_1)\} + \frac{q_2}{\gamma_2}\{1+\tanh(q_2x_2)\}. \tag{39}$$

Using eq. (12) we can exclude x_2 from eq. (39) and derive the energy flux in the form

$$N = \frac{\gamma_1-\gamma_2}{\gamma_1\gamma_2}q_1\tanh(q_1x_1) + \frac{q_1}{\gamma_1} + \frac{q_2+2U_0}{\gamma_2}. \tag{40}$$

Equation (40) defines the flux as a function $N = N(\omega, x_1)$ where we choose x_1 as a free parameter. The typical curves of dependence $N(\omega, x_1)$ defined by eq. (40) are shown in figure 4 for different values of x_1.

Case A: Symmetrical surface waves

From eq. (39) with eq. (14) in the case of $x_1 = x_2 = x_0$, $\gamma_1 = \gamma_2 = \gamma$ and $\Omega_1 = \Omega_2 = \Omega$ we obtain the flux derived in ref. [34]:

$$N = \frac{2}{\gamma}(q+U_0). \tag{41}$$

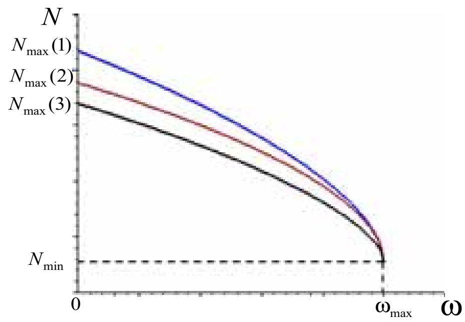

Figure 4. The dependence of surface wave flux (40) on the frequency for fixed values of parameters in model conventional units: $\Omega_1 = 1$, $\Omega_2 = 1.5$, $\gamma_1 = 1$, $\gamma_2 = 1.5$, $U_0 = 0.3$. Line (1): $x_1 = 0$, line (2): $x_1 = 0.5$ and line (3): $x_1 = 10$.

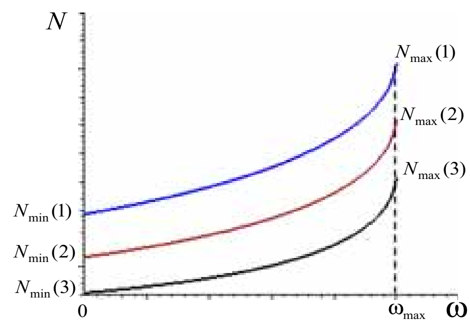

Figure 5. The dependence of surface wave flux (47) on the frequency for fixed values of parameters in model conventional units: $\Omega_1 = 1$, $\Omega_2 = 1.5$, $g_1 = 1$, $g_2 = 4.5$, $U_0 = 0.3$. Line (1): $x_1 = 0.6$, line (2): $x_1 = 0.75$ and line (3): $x_1 = 1$.

Case B: Non-symmetrical surface waves
Case B.1: At first we consider the special case of $x_1 = 0$ and $x_2 \neq 0$. In this case, from eq. (40), we obtain

$$N = \frac{q_1}{\gamma_1} + \frac{q_2 + 2U_0}{\gamma_2}. \tag{42}$$

Using eq. (15) we can find from eq. (42) the energy flux in the form

$$N = \frac{2U_0}{\gamma_2} + \sqrt{\frac{2}{1-\eta^2}}\left\{\frac{1}{\gamma_1}\sqrt{2U_0^2 - \Delta\Omega} + \frac{1}{\gamma_2}\sqrt{2U_0^2 - \eta^2\Delta\Omega}\right\}. \tag{43}$$

In the case of a 'powerful' interface when $U_0^2 \gg |\Delta\Omega|$, from eq. (43) we get an approximate estimate

$$N = 2U_0\Gamma(\gamma_1, \gamma_2), \tag{44}$$

where

$$\Gamma(\gamma_1, \gamma_2) = \frac{1}{\gamma_2} + \left(\frac{1}{\gamma_1} + \frac{1}{\gamma_2}\right)\sqrt{\frac{\gamma_1}{\gamma_1 - \gamma_2}}.$$

Case B.2: Now we consider the case of $x_1 \neq x_2 \neq 0$. If we choose now the amplitude of interface oscillations defined by eq. (13) as a free parameter we can find the exact expression of flux. Using eqs (11), (12) and (21) from (39) we obtain

$$N = \frac{1}{\gamma_1}\left(\sqrt{\gamma_1 u_0^2 + Q_c^2} - Q_c\right) + \frac{1}{\gamma_2}\left(\sqrt{\gamma_1 u_0^2 + Q_c^2 + 2\Delta\Omega} + \sqrt{(\gamma_1 - \gamma_2)u_0^2 + Q_c^2 + 2\Delta\Omega}\right). \tag{45}$$

Equation (45) defines the flux as the function $N = N(u_0)$, where we choose u_0 as a free parameter.

4.2 *The energy flux near the interface of the defocussing media*

We substitute solution (23) into eq. (4) and obtain the energy flux in the form

$$N = -\frac{q_1}{g_1}\{1 + \coth(q_1 x_1)\} - \frac{q_2}{g_2}\{1 + \coth(q_2 x_2)\}. \tag{46}$$

Using eq. (12) we can exclude x_2 from eq. (39) and derive the energy flux in the form

$$N = \frac{g_1 + g_2}{g_1 g_2} q_1 \coth(q_1 x_1) - \frac{q_1}{g_1} - \frac{q_2 + 2U_0}{g_2}. \tag{47}$$

Equation (47) defines the flux as the function $N = N(x_1)$ where we choose x_1 as a free parameter. The typical curves of dependence $N(\omega, x_1)$ defined by eq. (44) are shown in figure 5 for different values of x_1.

Case A: Symmetrical surface waves. From eq. (46) with eq. (27) in the case of $x_1 = x_2 = x_0$, $\gamma_1 = \gamma_2 = \gamma$ and $\Omega_1 = \Omega_2 = \Omega$, we obtain the flux derived in ref. [34]:

$$N = -\frac{2}{g}(q + U_0). \tag{48}$$

Case B: Non-symmetrical surface waves
Case B.1: At first we consider the case of long wave states when $q_j x_j \ll 1$. In this case, from eqs (24) and (25) we find

$$x_1 = \eta x_2 \quad \text{and} \quad x_2 = \frac{2U_0\eta}{\eta - 1}. \tag{49}$$

Using eq. (49) from eq. (47) we obtain the flux in the form

$$N = \frac{g_1 + g_2}{g_2^2}\frac{\eta - 1}{2U_0} - \frac{q_1}{g_1} - \frac{q_2 + 2U_0}{g_2}. \tag{50}$$

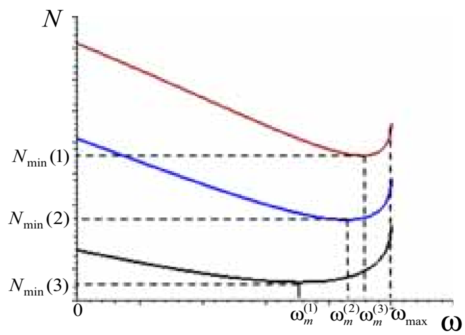

Figure 6. The dependence of surface wave flux (53) on the frequency for fixed values of parameters in model conventional units: $\Omega_1 = 1$, $\Omega_2 = 1.5$, $\gamma_1 = 1$, $g_2 = 1.5$, $U_0 = 0.3$. Line (1): $x_2 = 0.75$, line (2): $x_2 = 0.45$ and line (3): $x_2 = 0.25$.

Case B.2: Flux as a function of amplitude. Using eqs (14), (25) and (28), from (46) we obtain

$$N = -\frac{1}{g_1}\left(\sqrt{Q_s^2 - g_1u_0^2} - Q_s\right) -\frac{1}{g_2}\left(\sqrt{Q_s^2 - g_1u_0^2 + 2\Delta\Omega} -\sqrt{(g_2 - g_1)u_0^2 + Q_s^2 + 2\Delta\Omega}\right). \tag{51}$$

Equation (51) defines the flux as the function $N = N(u_0)$ where we choose u_0 as a free parameter.

4.3 *The energy flux near the interface between the self-focussing and defocussing media*

We substitute solution (30) into eq. (4) and obtain the energy flux in the form

$$N = -\frac{q_1}{g_1}\{1 + \coth(q_1x_1)\} + \frac{q_2}{\gamma_2}\{1 + \tanh(q_2x_2)\}. \tag{52}$$

Using eq. (32) we can exclude x_1 from eq. (52) and derive the energy flux in the form

$$N = \frac{g_1 + \gamma_2}{g_1\gamma_2}q_2\tanh(q_2x_2) + \frac{q_2}{\gamma_2} - \frac{q_1 + 2U_0}{g_1}. \tag{53}$$

Equation (53) defines the flux as a function $N = N(x_2)$ where we choose x_2 as a free parameter. The typical curves of dependence $N(\omega, x_2)$ defined by eq. (53) are shown in figure 6 for different values of x_2.

Case A: At first we consider the special case of $x_2 = 0$ and $x_1 \neq 0$. In this case, from eq. (53) we obtain

$$N = \frac{q_2}{\gamma_2} - \frac{q_1 + 2U_0}{g_1}. \tag{54}$$

Using eq. (34) we can find from eq. (54) the energy flux in the form

$$N = \sqrt{\frac{2}{1+\zeta^2}}\left\{\frac{\zeta}{\gamma_2}\sqrt{2U_0^2 + \Delta\Omega} -\frac{1}{g_1}\sqrt{2U_0^2\zeta^2 - \Delta\Omega}\right\} - \frac{2U_0}{g_1}. \tag{55}$$

In this case flux (55) is defined by the interface 'power', nonlinearity coefficients and the difference between the characteristics of media $\Delta\Omega$.

In the case of 'powerful' interface when $U_0^2 \gg |\Delta\Omega|$ from eq. (55) we get an approximate estimate

$$N = 2U_0\Gamma(g_1, \gamma_2), \tag{56}$$

where

$$\Gamma(g_1, \gamma_2) = \left(\frac{1}{\gamma_2} - \frac{1}{g_1}\right)\sqrt{\frac{\gamma_2}{g_1 + \gamma_2}} - \frac{1}{g_1}.$$

Case B.2: Now we consider the case of $x_1 \neq x_2 \neq 0$ If we now choose the amplitude of interface oscillations defined by eq. (33) as a free parameter we can find the exact expression of flux. Using eqs (31), (32) and (37), from (52), we obtain

$$N = \frac{1}{\gamma_2}\left(\sqrt{Q^2 - g_1u_0^2 + 2\Delta\Omega} +\sqrt{Q^2 + 2\Delta\Omega - (g_1 + \gamma_2)u_0^2}\right) -\frac{1}{g_1}\left(\sqrt{Q^2 - g_1u_0^2} - Q\right). \tag{57}$$

Equation (57) defines the flux as a function of $N = N(u_0)$ where we choose u_0 as a free parameter.

5. Discussion

In this paper, we propose two approaches to determine the flux depending on the formulation of specific experiments when we can choose one of the possible free parameters x_j or u_0. As a result, we obtained the flow in explicit form as the functions $N = N(\mathbf{p}, u_0)$ and $N = N(\mathbf{p}, x_j)$. Here $\mathbf{p} = \{\gamma_{1,2}, \Omega_{1,2}, U_0\}$ is the vector of parameters characterising the media and the interface.

We can give alternative to the interpretation of these dependencies. Also we can consider them as normalisation conditions. In this case, the flux is fixed. Then we consider N as a control parameter defining the soliton 'centre' positions as the functions $x_j = x_j(\mathbf{p}, N)$ or the interface amplitude oscillations as the function $u_0 = u_0(\mathbf{p}, N)$.

For example, in the simplest case of contact of two nonlinear media with positive nonlinearity from eq. (41), we can find the damping space factor

$$q(N) = N\gamma/2 - U_0 \tag{58}$$

and frequency ω as a function of N in the form

$$\omega(N) = \Omega - (N\gamma/2 - U_0)^2/2. \tag{59}$$

Here the condition for the flux $N > 2U_0/\gamma$ must be fulfilled. This condition makes it possible to control the stationary state localisation by the relation between the interface 'power' and the value of N.

The value $q = -U_0$ corresponds to the linear mode with the frequency $\omega_L = \Omega - U_0^2/2$ and $N = 0$ exists near the repelled interface only with $U_0 < 0$.

The critical value $N = 2U_0/\gamma$ corresponds to the frequency Ω. The local state is divided into two solitons, which are infinitely distant from the interface, in each of which two elemental excitations are connected.

As we have noted, the first NSE integral (4) can be interpreted as the total number of particles in the system in the Bose–Einstein condensation theory. Also, the first NSE integral (4) can be interpreted as the total number of elementary excitations bounded in soliton for quasiclassical quantisation [34]. The value N is conserved in both the cases. Localised state is treated as the bound state of a large number of elementary excitations.

The difference between the characteristics of contacting media leads to new features of surface wave propagation. These features do not appear in the case of contact of media with the same characteristics on both sides of the interface. Surface waves with a non-symmetrical amplitude profile can exist. Also the frequency bifurcation is possible in the case of differing media characteristics (see figures 1b, 1c and 2c). The stability analysis of the two frequency branches is beyond the scope of this paper.

Also in such a case the surface waves cannot exist for any value of the interface power. Valid interface power values are bounded below and above in the interval $U_{\min} < U_0 < U_{\max}$. In the case of contact of media with the same characteristics, the frequency decreases monotonically on interface power. Valid interface power values are bounded above $U_0 < U_{\max}$. For example, for symmetrical surface waves, we find critical value from eq. (14) $U_{\max} = 2\sqrt{\Omega}\tanh(2x_0\sqrt{\Omega})$ or from eq. (27) $U_{\max} = 2\sqrt{\Omega}\coth(2x_0\sqrt{\Omega})$.

The flux N acts differently for focussing media contact and defocusing media contact. The flux N defined by eq. (40) depends monotonically on the frequency near the focussing media contact (see figure 4). The flux N defined by eq. (47) depends monotonically on the frequency near the defocussing media contact (see figure 5). The flux N defined by eq. (53) depends non-monotonically on the frequency near the contact of focussing and defocussing media (see figure 6). The specific frequency ω_m corresponding to the flux minimum $N_{\min} = N(\omega_m)$ exists. For example, for the simplest case $x_2 = 0$, the specific frequency value is derived from eq. (54):

$$\omega_m = \frac{g_1^2\Omega_1 - \gamma_2^2\Omega_2}{g_1^2 - \gamma_2^2}. \tag{60}$$

The flux minimum cannot exist for $g_1 = \gamma_2$.

6. Conclusions

We analysed the peculiarities of excitation localisation near the interface between two different nonlinear media within the framework of a simple model based on the NSE with the Dirac delta function potential. The NSE nonlinearity we consider has the cubic form (Kerr-type nonlinearity). We analytically described the localised states of three types of dependence on the sign of nonlinearity coefficients and frequency range.

We obtained NSE solutions and dispersion equations in cases of the contact of two self-focussing (attractive) media corresponding to the positive nonlinearity, contact of two defocussing (repulsive) media corresponding to the negative nonlinearity and the contact of self-focussing and defocussing media corresponding to the opposite signs of nonlinearity from the different sides of interface.

We calculated the exact solutions of dispersion equations in cases of special kinds of localised states. Also we derived solutions of dispersion equations in explicit analytical form for a long-wave approximation. We found the frequencies of non-symmetrical surface waves as exact solutions of dispersion equations in the form of functions of amplitude of interface oscillations. We analysed the conditions for the existence of such waves.

The main difference between the results of this work lies in the fact that the characteristics of the media on different sides from the interface between them are different. As a result, it becomes possible to analyse the influence of the differences of such characteristics on the existence of localised states.

We calculated and analysed the first NSE integral interpreted as an energy flux conserved along the interface between the media for all the cases considered. The fluxes of non-symmetrical surface waves are defined by the interface 'power' U_0, nonlinearity coefficients γ_j and the difference between the characteristics of media $\Delta\Omega$. In the case of the 'powerful' interface, the fluxes of non-symmetrical surface waves are proportional to the interface 'power' U_0.

The obtained estimates of the energy fluxes carried away by nonlinear surface waves are important for the design of optical waveguide systems with given characteristics of optical conductivity, as well as optical control devices based on layered media [8,64]. The results obtained can be used to improve various optical switches in waveguides and optical power limiters, capable of transmitting light pulses only above/below a fixed value of the energy flux [65–67]. The combination of manageable threshold and limiting actions can be used to determine the optimal modes of transmission of energy fluxes that carry nonlinear surface waves.

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