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Higher-dimensional fractional time-independent Schrödinger equation via fractional derivative with generalised pseudoharmonic potential

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Abstract. In this paper, we obtain approximate bound-state solutions of N-dimensional time-independent fractional Schrödinger equation for the generalised pseudoharmonic potential which has the form $V(r^\alpha) = a_1 r^{2\alpha} + (a_2/r^{2\alpha}) + a_3$. Here α $(0 < \alpha < 1)$ acts like a fractional parameter for the space variable r. The entire study consists of the Jumarie-type fractional derivative and the elegance of Laplace transform. As a result, we can successfully express the approximate bound-state solution in terms of Mittag–Leffler function and fractionally defined confluent hypergeometric function. Our study may be treated as a generalisation of all previous works carried out on this topic when $\alpha = 1$ and N arbitrary. We provide numerical result of energy eigenvalues and eigenfunctions for a typical diatomic molecule for different α close to unity. Finally, we try to correlate our work with a Cornell potential model which corresponds to $\alpha = 1/2$ with $a_3 = 0$ and predicts the approximate mass spectra of quarkonia.

Keywords. Fractional radial Schrödinger equation; generalised pseudoharmonic potential; bound-state solutions; Mittag–Leffler function.

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1. Introduction

Although the history of fractional calculus is almost as old as that of integer-order calculus, for many years fractional calculus was not used in physics and applied sciences. The reason for such unpopularity could be that there exist several other definitions of fractional derivatives [1] and also there is a lack of geometrical interpretation [2]. The situation changed after 1970 when Mandelbrot [3] proposed fractional dimension and a close interconnection between the Brownian motion and the Riemann–Liouville fractional calculus. From then on, fractional calculus started to attract physicists to explore the complex phenomena originated from different dissipative forces present in nature. To that aim, Riewe [4] introduced the fractional Hamiltonian and Lagrangian equations of motion for non-conservative systems. This prompted the physicists to develop fractional Schrödinger equation, because Hamiltonian canonical equations are the basic starting theory of non-relativistic quantum mechanics. Using the fractional canonical equation of motion, Muslih *et al* [5] derived the fractional Schrödinger equation containing partial left and right Riemann–Liouville fractional derivatives. Later, Laskin [6,7] was able to generalise the Feynman path integral to Lévy one and developed the space-fractional Schrödinger equation. Soon after, Guo and Xu [8], Dong and Xu [9] studied the space-fractional Schrödinger equation with a few specific potential models.

Despite all these studies, still there is a dilemma to use the Riemann–Liouville derivative rules in quantum mechanics as these rules are not quite parallel to the well-known classical calculus [10]. The embarrassing fact of classical Riemann–Liouville derivative is that fractional derivative of a constant is not zero. Although the Caputo [11] derivative solved the problems, but it has its own disadvantage as it cannot work on

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non-differentiable functions. To get rid of this problem, Jumarie [12] modified the Riemann–Liouville derivative that can run parallel with the classical calculus rules. In quantum mechanics, the Jumarie-type derivative rules are always welcome because the quantum mechanics, in general, deals with predetermined boundary values of the wave function or its derivative on the boundary as zero. During the last few years, the use of Jumarie-type derivative and its application in various fields, including quantum mechanics, has been studied a lot [13–18].

Recently, the subject of higher-dimensional Schrödinger equation has attracted lots of interest in the field of quantum mechanics [19]. Motivated by this new subject, we developed a higher-dimensional fractional Schrödinger equation [20] and studied the fractional Mie-type spherical symmetric potential composed of Jumarie-type fractional derivative. The study was mainly devoted to realise the fractional Laplacian operator in hyperspherical coordinate system in N spatial dimensions and tried to find the solution of N-dimensional Schrödinger equation for bound state of generalised Mie-type potential $V(r^\alpha) = (A/r^{2\alpha}) + (B/r^\alpha) + C$ characterised by the fractional parameter α. To achieve the goal, we used the rules of Laplace transformation of fractional differ-integrals. Finally, going through the rigorous mathematics of fractional differential equation with Jumarie-type differential coefficient, under well-behaved boundary conditions, we succeeded in obtaining approximate bound-state solution of the potential for $\alpha \approx 1.00$. The present paper heavily follows our previous work [20].

In this paper, we have just replaced the generalised Mie-type potential with the generalised pseudoharmonic potential which has the same popularity as Mie-type potential in molecular and chemical physics. The main reason behind the study of the generalised pseudoharmonic potential separately is: under the substitution of the potential into the fractional Schrödinger equation, the resulting fractional differential equation does not offer an easy way to solve it. Even the Laplace transform becomes quite tedious if the aforementioned fractional differential equation is not manipulated properly. Keeping all these odds, in this paper, we have approached the generalised pseudoharmonic potential which can be written as

$$V(r^\alpha) = a_1 r^{2\alpha} + \frac{a_2}{r^{2\alpha}} + a_3, \tag{1}$$

where $a_i\,(i = 1, 2, 3)$ are some suitable constants. When $\alpha = 1$ this potential converts into the original form of pseudoharmonic potential [21–25]

$$V(r) = D_0\left(\frac{r}{r_\mathrm{e}} - \frac{r_\mathrm{e}}{r}\right)^2. \tag{2}$$

D_0 stands for the dissociation energy and it is given by $D_0 = (1/8)K_\mathrm{e} r_\mathrm{e}^2$, where K_e is the force constant due to the bonding of the diatomic molecule and r_e is the equilibrium constant for the same.

The present paper is organised as follows: The next section gives a detailed calculation of the bound-state spectrum for the pseudoharmonic potential, where we have recalled the N-dimensional fractional-order hyperradial or in short 'radial' equation from our previous work mentioned earlier. The discussion appears in §3 where theoretical as well as numerical results are discussed by plotting a few eigenfunctions. We have also furnished the approximate mass spectra of quarkonia via the Cornell potential model, which is equivalent to our potential model that corresponds to $\alpha = 0.5$. Finally, the conclusion of the work is given in §4.

2. Bound-state spectrum

In [20], we developed the N-dimensional fractional-order $(0 < \alpha < 1)$ radial Schrödinger equation in natural units as

$$\mathscr{O}_\alpha(N) R(r^\alpha) = 0, \tag{3}$$

where the operator $\mathscr{O}_\alpha(N)$ is given by

$$\begin{aligned}\mathscr{O}_\alpha(N) = {} & \frac{\mathrm{d}^{2\alpha}}{\mathrm{d}r^{2\alpha}} + \frac{\Gamma(1+\alpha(N-1))}{\Gamma(1+\alpha(N-2))}\frac{1}{r^\alpha}\frac{\mathrm{d}^\alpha}{\mathrm{d}r^\alpha} \\ & - \frac{\ell(\ell+N-2)[\Gamma(1+\alpha)]^2}{r^{2\alpha}} \\ & + 2M[\Gamma(1+\alpha)]^2[\mathscr{E}_\alpha - V(r^\alpha)].\end{aligned} \tag{4}$$

The operator $\mathscr{O}_\alpha(N)$ contains the highest-order fractional derivative operator $\mathrm{d}^{2\alpha}/\mathrm{d}r^{2\alpha}$ followed by $\mathrm{d}^\alpha/\mathrm{d}r^\alpha$. That means the range $0 < \alpha < 1$ helps to debunk the actual form of radial Schrödinger equation at $\alpha = 1$ and $N = 3$. Any other range of α, i.e. $\alpha > 1$ is not accepted here and our model is not compatible with that. The dynamic system, like quantum mechanics, follows the coarse-grained phenomena which eventually deals with the fractal space and time within the limit $0 < \alpha < 1$ [26,27]. The various symbols of eq. (4) are as follows: $\mathscr{E}_\alpha$ is the total energy, $V(r^\alpha)$ is the fractional potential energy and M is the fractional mass. They all have unit GeV^α. The symbol ℓ stands for orbital angular momentum quantum number which can take quantised values 0, 1, 2, 3, ... only. All the fractional derivatives, used in eq. (4), are taken with Jumarie sense [28,29], i.e.

$$ {}_0^J D_x^\alpha [f(x)] = \frac{\mathrm{d}^\alpha f(x)}{\mathrm{d}x^\alpha} = \begin{cases} \dfrac{1}{\Gamma(-\alpha)} \displaystyle\int_0^x (x-\xi)^{-\alpha-1} f(\xi)\,\mathrm{d}\xi, & \alpha < 0, \\ \dfrac{1}{\Gamma(1-\alpha)} \dfrac{\mathrm{d}}{\mathrm{d}x} \displaystyle\int_0^x (x-\xi)^{-\alpha} (f(\xi) - f(0))\,\mathrm{d}\xi, & 0 < \alpha < 1, \\ (f^{(\alpha-n)}(x))^{(n)}, & n \le \alpha < n+1. \end{cases} \tag{5} $$

It is also called modified left-Riemann–Liouville (RL) fractional derivative. The function $f(x)$ is continuous but not necessarily differentiable in the interval 0 to x and also $f(x)|_{x<0} = 0$. Now in potential (1), we restrict ourselves to accept the same α $(0 < \alpha < 1)$ for making different indices of r. Obviously it is to lower the mathematical stress. The range of α, i.e. $0 < \alpha < 1$ makes it possible to study the bound state. Any other value such as $\alpha > 1$ will make the potential more anharmonic and this will drastically shift the aim of the present study. Inserting potential (1) into eq. (3) we have

$$ {}_0^J D_r^{2\alpha}[R(r^\alpha)] + \frac{\Gamma(1+\alpha(N-1))}{\Gamma(1+\alpha(N-2))} \frac{1}{r^\alpha} {}_0^J D_r^\alpha [R(r^\alpha)] + \left[\epsilon_\alpha^2 - \mu_\alpha^2 r^{2\alpha} - \frac{\nu_\alpha(\nu_\alpha+1)}{r^{2\alpha}}\right] R(r^\alpha) = 0, \tag{6} $$

where

$$ \nu_\alpha(\nu_\alpha+1) = \ell(\ell+N-2)[\Gamma(1+\alpha)]^2 + 2M[\Gamma(1+\alpha)]^2 a_2, \tag{7a} $$

$$ \epsilon_\alpha^2 = 2M[\Gamma(1+\alpha)]^2(\mathscr{E}_\alpha - a_3), \tag{7b} $$

$$ \mu_\alpha^2 = 2M[\Gamma(1+\alpha)]^2 a_1. \tag{7c} $$

Quantum mechanical bound-state eigenfunctions are generally well behaved in nature, which means that they are bound or $R(r^\alpha)$ approaches zero for $r \to 0$ and $r \to \infty$. We predict the solution of eq. (6) as

$$ R(r^\alpha) = (r^\alpha)^{-k} f(r^\alpha)|_{k>0}. \tag{8} $$

Here the term $(r^\alpha)^{-k}$ ensures the fact that $R(r \to \infty) = 0$. The unknown function $f(r^\alpha)$ is expected to behave like $f(r \to 0) = 0$. After deriving ${}_0^J D_r^{2\alpha}[f(r^\alpha)]$, ${}_0^J D_r^\alpha [f(r^\alpha)]$ and inserting them into eq. (6) we have

$$ {}_0^J D_r^{2\alpha} f(r^\alpha) + \frac{Q_1(\alpha,k,N)}{r^\alpha} {}_0^J D_r^\alpha f(r^\alpha) + \left[\frac{Q_2(\alpha,k,N,\nu_\alpha)}{r^{2\alpha}} - \mu_\alpha^2 r^{2\alpha} + \epsilon_\alpha^2\right] f(r^\alpha) = 0, \tag{9} $$

where

$$ Q_1(\alpha,k,N) = \frac{2\Gamma(1-\alpha k)}{\Gamma(1-\alpha k-\alpha)} + \frac{\Gamma(1+\alpha(N-1))}{\Gamma(1+\alpha(N-2))}, \tag{10a} $$

$$ Q_2(\alpha,k,N,\nu_\alpha) = \frac{\Gamma(1-\alpha k)}{\Gamma(1-\alpha k-\alpha)} \times \left[\frac{\Gamma(1-\alpha k-\alpha)}{\Gamma(1-\alpha k-2\alpha)} + \frac{\Gamma(1+\alpha(N-1))}{\Gamma(1+\alpha(N-2))}\right] - \nu_\alpha(\nu_\alpha+1). \tag{10b} $$

Here we have a plan to solve eq. (9) using the Laplace transform method. The strong singular term $Q_2(\alpha,k,N,\nu_\alpha)/r^{2\alpha}$ is the main obstacle to that path. Let us choose

$$ Q_2(\alpha,k,N,\nu_\alpha) = 0. \tag{11} $$

This restriction (not mandatory though) only helps us to ease out the mathematical calculations. Denoting the solution of the above equation for k (>0) as k_α^*, we can rewrite eq. (9) as

$$ {}_0^J D_r^{2\alpha} g(r) + \frac{Q_1(\alpha,k_\alpha^*,N)}{r^\alpha} {}_0^J D_r^\alpha g(r) + (\epsilon_\alpha^2 - \mu_\alpha^2 r^{2\alpha}) g(r) = 0, \tag{12} $$

where $f(r^\alpha)$ is replaced with $g(r)$. In spite of the condition $Q_2 = 0$ again the present equation is not suitable for Laplace transform because the term containing $r^{2\alpha}$ will generate higher-order fractional differential equation in the transformed space, which will be difficult to tackle. There is an alternative way. If we adopt a change in the variable, then the situation becomes much more easy. To that aim, the rule of Ghosh *et al.* [14], i.e. ${}_0^J D_x^\alpha (f[u(x)]) = f_u^{(\alpha)}(u)(\mathrm{d}u/\mathrm{d}x)^\alpha$ is appropriate here. Taking $y = r^2$, the rule facilitates ${}_0^J D_r^\alpha y = {}_0^J D_y^\alpha y(\mathrm{d}y/\mathrm{d}r)^\alpha$ and we have the following operators:

$$ {}_0^J D_r^\alpha = 2^\alpha y^{\alpha/2} {}_0^J D_y^\alpha, \tag{13a} $$

$$ {}_0^J D_r^{2\alpha} = 4^\alpha y^\alpha {}_0^J D_y^{2\alpha} + 4^\alpha \frac{\Gamma(1+(\alpha/2))}{\Gamma(1-(\alpha/2))} {}_0^J D_y^\alpha. \tag{13b} $$

These operators help to rewrite eq. (12) in a new form with $g(r) \Leftrightarrow \chi(y)$ as

$$ y^\alpha {}_0^J D_y^{2\alpha} \chi(y) + \left(\frac{\Gamma(1+(\alpha/2))}{\Gamma(1-(\alpha/2))} + \frac{Q_1}{2^\alpha}\right) {}_0^J D_y^\alpha \chi(y) + \frac{1}{4^\alpha}(\epsilon_\alpha^2 - \mu_\alpha^2 y^\alpha)\chi(y) = 0. \tag{14} $$

This is a familiar type of fractional differential equation which we have solved in [20]. Realising $\mathscr{L}\{\chi(y)\} = \varphi(s)$ with $\chi(0) = 0$, it is easy to obtain the following fractional differential equation in the transformed space:

$$ {}_0^J D_s^\alpha \varphi(s) + \eta(s^\alpha)\varphi(s) = 0, \tag{15} $$

where

$$ \eta(s^\alpha) = \frac{\lambda_1}{s^\alpha + (\mu_\alpha/2^\alpha)} + \frac{\lambda_2}{s^\alpha - (\mu_\alpha/2^\alpha)}, \tag{16a} $$

$$ \lambda_1 = \frac{\gamma_\alpha}{2} + \frac{\epsilon_\alpha^2}{2^{\alpha+1}\tau^2\mu_\alpha}, \tag{16b} $$

$$ \lambda_2 = \frac{\gamma_\alpha}{2} - \frac{\epsilon_\alpha^2}{2^{\alpha+1}\tau^2\mu_\alpha}, \tag{16c} $$

$$ \gamma_\alpha = \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} - \frac{1}{\tau^2}\left(\frac{\Gamma(1+(\alpha/2))}{\Gamma(1-(\alpha/2))} + \frac{Q_1}{2^\alpha}\right). \tag{16d} $$

$\tau = -[\mathrm{cosec}((\alpha-\delta)\pi)/\mathrm{cosec}(-\delta\pi)]$ with $-1 < \delta < 0$ comes from the Laplace transform relation $\mathscr{L}\{x^\alpha f(x)\} = -\tau(\mathrm{d}^\alpha F(s)/\mathrm{d}s^\alpha)$ which has been established in ref. [20], where $\mathscr{L}\{f(x)\} = F(s)$. The exact solution of eq. (15) is very complicated in fractional domain. Following ref. [20], the approximated solution for $\alpha \approx 1.00$ in the transformed space comes out as

$$ \varphi(s) = C_1\left(s^\alpha + \frac{\mu_\alpha}{2^\alpha}\right)^{-\lambda_1}\left(s^\alpha - \frac{\mu_\alpha}{2^\alpha}\right)^{-\lambda_2}, \tag{17} $$

where the 'α-logarithmic' function [30] has been used in the Jumarie sense, i.e. $\int(\mathrm{d}^\alpha t/t) = \mathrm{Ln}_\alpha(t/C), t = E_\alpha(\mathrm{Ln}_\alpha t)$, where C denotes a constant such that $(t/C) > 0$. C_1 is the integration constant. The second factor of eq. (17) is a multivalued function when the power $-\lambda_2$ is a non-integer. The quantum mechanical eigenfunction must be single valued in nature. So we must take

$$ -\lambda_2 = \frac{\epsilon_\alpha^2}{2^{\alpha+1}\tau^2\mu_\alpha} - \frac{\gamma_\alpha}{2} = n, \quad n = 0, 1, 2, 3, \ldots. \tag{18} $$

The inverse transform of eq. (17) will provide the solution of the problem in actual space. To that aim, we expand eq. (17) with the help of eq. (18) as

$$ \begin{aligned} \varphi(s) &= C_1\left(s^\alpha + \frac{\mu_\alpha}{2^\alpha}\right)^{-n-\gamma_\alpha}\left(s^\alpha - \frac{\mu_\alpha}{2^\alpha}\right)^n \\ &= C_1\left(s^\alpha + \frac{\mu_\alpha}{2^\alpha}\right)^{-\gamma_\alpha}\left(\frac{s^\alpha - (\mu_\alpha/2^\alpha)}{s^\alpha + (\mu_\alpha/2^\alpha)}\right)^n \\ &= C_1\left(s^\alpha + \frac{\mu_\alpha}{2^\alpha}\right)^{-\gamma_\alpha}\left[1 - \frac{\mu_\alpha}{s^\alpha + (\mu_\alpha/2^\alpha)}\right]^n \\ &= C_1\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(-1)^j(\mu_\alpha)^j\left(s^\alpha + \frac{\mu_\alpha}{2^\alpha}\right)^{-(\gamma_\alpha+j)} \\ &= C_1\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(-1)^j(\mu_\alpha)^j \\ &\quad\times\frac{1}{(s^\alpha + (\mu_\alpha/2^\alpha))^{m_j+1}}, \\ &\quad\text{where } m_j = (\gamma_\alpha + j - 1). \end{aligned} \tag{19} $$

Using the formula [31,32] $\mathscr{L}\{x^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm a x^\alpha)\} = k!s^{\alpha-\beta}/(s^\alpha \mp a)^{k+1}$ for $\alpha = \beta$, we can find the inverse of eq. (19) quite easily:

$$ \begin{aligned} \chi(y) &= C_1\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(-1)^j(\mu_\alpha)^j \\ &\quad\times\frac{1}{\Gamma(\gamma_\alpha+j)}y^{\alpha(\gamma_\alpha+j)-1}E_\alpha^{(m_j)}\left(-\frac{\mu_\alpha}{2^\alpha}y^\alpha\right) \\ &= \frac{C_1}{\Gamma(\gamma_\alpha)}y^{\alpha\gamma_\alpha-1}\sum_{j=0}^{n}\frac{n!}{j!(n-j)!}(-1)^j \\ &\quad\times\frac{\Gamma(\gamma_\alpha)}{\Gamma(\gamma_\alpha+j)}(\mu_\alpha y^\alpha)^j E_\alpha^{(m_j)}\left(-\frac{\mu_\alpha}{2^\alpha}y^\alpha\right) \\ &= \mathscr{N}_c y^{\alpha\gamma_\alpha-1}E_\alpha^{(m_j)}\left(-\frac{\mu_\alpha}{2^\alpha}y^\alpha\right){}_1F_1(-n, \gamma_\alpha, \mu_\alpha y^\alpha), \end{aligned} \tag{20} $$

where

$$ \begin{aligned} E_\alpha^{(m_j)}\left(-\frac{\mu_\alpha}{2^\alpha}y^\alpha\right) &= \frac{\mathrm{d}^{m_j}}{\mathrm{d}y^{m_j}}E_\alpha\left(-\frac{\mu_\alpha}{2^\alpha}y^\alpha\right) \\ &= \sum_{p=0}^{\infty}\frac{(p+m_j)!}{p!}\left[\frac{(-(\mu_\alpha/2^\alpha)y^\alpha)^p}{\Gamma(\alpha p + \alpha m_j + \alpha)}\right]. \end{aligned} $$

The symbol $E_\alpha(-(\mu_\alpha/2^\alpha)y^\alpha)$ represents the Mittag–Leffler function, which is generally defined as [33]

$$ E_\alpha(z) = \sum_{\kappa=0}^{\infty}\frac{z^\kappa}{\Gamma(\alpha\kappa+1)} \quad (\alpha > 0). $$

Hence

$$ \begin{aligned} g(r) = f(r^\alpha) &= \mathscr{N}_c r^{2\alpha\gamma_\alpha-2}E_\alpha^{(m_j)}\left(-\frac{\mu_\alpha}{2^\alpha}r^{2\alpha}\right) \\ &\quad\times{}_1F_1(-n, \gamma_\alpha, \mu_\alpha r^{2\alpha}). \end{aligned} \tag{21} $$

This yields the complete radial eigenfunction

$$ \begin{aligned} R_{n\alpha N\ell}(r^\alpha) &= r^{-\alpha k_\alpha^*}f(r^\alpha) = \mathscr{N}_c r^{[\alpha(2\gamma_\alpha - k_\alpha^*)-2]} \\ &\quad\times E_\alpha^{(m_j)}\left(-\frac{\mu_\alpha}{2^\alpha}r^{2\alpha}\right){}_1F_1(-n, \gamma_\alpha, \mu_\alpha r^{2\alpha}), \end{aligned} \tag{22} $$

where $\mathscr{N}_c = C_1/\Gamma(\gamma_\alpha)$ acts like a normalisation constant and ${}_1F_1$ is the fractionally defined confluent hypergeometric function, i.e.

$$ {}_1F_1(-n, \gamma_\alpha, \mu_\alpha y^\alpha) = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!}(-1)^j \times \frac{\Gamma(\gamma_\alpha)}{\Gamma(\gamma_\alpha + j)}(\mu_\alpha y^\alpha)^j. $$

The energy eigenvalue equation of the potential model comes out from eq. (18) as

$$ \mathscr{E}_{n\alpha N\ell} = a_3 + \frac{\tau^2}{\Gamma(\alpha+1)}\left(n + \frac{\gamma_\alpha}{2}\right)\sqrt{\frac{2^{2\alpha+1}a_1}{M}}. \tag{23} $$

3. Results and discussion

In this section, at first we have shown theoretically that the results obtained in §2 are compatible with several special cases, both for lower and higher dimensions, especially when $\alpha = 1$. Secondly, we have furnished numerical results of our work for specified potential parameters in different dimensions. At last, we shall try to review the famous Cornell potential for $\alpha = 0.5$.

3.1 *Isotropic harmonic oscillator potential*

In this case $a_2 = a_3 = 0$ and $a_1 = (1/2)M\omega^2$, where ω is the circular frequency of the oscillator. Hence eq. (11) provides $k(k+1) - k(N-1) - \ell(\ell+N-2) = 0$ which yields $k_1^* = k_{\ell N} = \ell + N - 2$ as $k > 0$. So we have

$$ Q_1 = -2k_{\ell N} + N - 1, \tag{24a} $$

$$ \gamma_1 = 2 + k_{\ell N} - \frac{N}{2}. \tag{24b} $$

The energy eigenvalues of the oscillator become

$$ \mathscr{E}_{n1N\ell} = \left(2n + \ell + \frac{N}{2}\right)\omega. \tag{25} $$

To find the eigenfunctions, we take eq. (20) with $\alpha = 1$

$$ \chi(y) = \mathscr{N}' y^{\gamma_1 - 1} \mathrm{e}^{-(\mu_1/2)y} {}_1F_1(-n, \gamma_1, \mu_1 y), \tag{26} $$

where $\mathscr{N}' = \mathscr{N}_c(-\mu_1/2)^{m_j}$. Hence

$$ g(r) = \chi(y)|_{y=r^2} = \mathscr{N}' r^{2\gamma_1 - 2}\mathrm{e}^{-(\mu_1/2)r^2} \times {}_1F_1(-n, \gamma_1, \mu_1 r^2). \tag{27} $$

The radial eigenfunctions in this case are

$$ R_{n1\ell N}(r) = r^{-k_{\ell N}} g(r) = \mathscr{N}' r^\ell \mathrm{e}^{-(\mu_1/2)r^2} \times {}_1F_1\left(-n, \ell + \frac{N}{2}, \mu_1 r^2\right), \tag{28} $$

where we have used eq. (24b) with $k_{\ell N} = \ell + N - 2$. All these results have already been achieved in ref. [34]. Inserting $N = 3$ one can get the required results for isotropic harmonic oscillator in ordinary space.

3.2 *Pseudoharmonic potential*

In this case a_i $(i = 1, 2, 3) \neq 0$. So eq. (11) provides

$$ k_{\ell N} = \frac{N}{2} - 1 + \frac{1}{2}\sqrt{(N + 2\ell - 2)^2 + 8Ma_2}. \tag{29} $$

The expressions of Q_1, γ_1 are same as eqs (24a) and (24b). The energy eigenvalues for pseudoharmonic potential are

$$ \mathscr{E}_{n1N\ell} = a_3 + \sqrt{\frac{8a_2}{M}}\left[n + \frac{1}{2} + \frac{1}{4}\sqrt{(N + 2\ell - 2)^2 + 8Ma_2}\right]. \tag{30} $$

To extract the eigenfunctions we again use eq. (20) with $\alpha = 1$ as earlier. The expression of $g(r)$ is the same as what we have derived in eq. (27) except $\gamma_1 = 2 + k_{\ell N} - (N/2)$, where $k_{\ell N}$ is given by eq. (29). Finally, the radial eigenfunctions are

$$ R_{n1\ell N}(r) = r^{-k_{\ell N}} g(r) = \mathscr{N}' r^{2 + k_{\ell N} - N} \mathrm{e}^{-(\mu_1/2)r^2} \times {}_1F_1\left(-n, 2 + k_{\ell N} - \frac{N}{2}, \mu_1 r^2\right), \tag{31} $$

where $\mathscr{N}' = \mathscr{N}_c(-\mu_1/2)^{m_j}$ acts like a normalisation constant. All these results match with the work cited in ref. [25].

Apart from verifying the earlier works for $\alpha = 1$, we also provide numerical results of our entire model. The potential parameters $a_{i=1,2,3}$ are assigned different values close to the relevant experimental situation. Assuming the diatomic molecular mass $M = 1\ \mathrm{GeV}^\alpha$, we have taken $a_1 = 10^{-3}\ \mathrm{GeV}^{3\alpha}$, $a_2 = 0.1\ \mathrm{GeV}^{-\alpha}$ and $a_3 = 0$ for constructing table 1. The value of τ against α has been enlisted in table 2.

All the figures, viz. figures 1–6 for eigenfunctions are a direct consequence of table 1. It is clear that, for lower α far from unity, the graphs are loosing its periodicity specially for ordinary three-dimensional space. There is a critical value of α, below which the nature of the eigenfunction becomes physically unsuitable. For three-dimensional space, numerical results predict the value just near about 0.8. The critical value depends on the dimension and the potential function (figure 7). This can be seen in figures 3–6 where the critical value lies more near to $\alpha = 1$.

3.3 *Review of mass spectra of quarkonium via Cornell potential*

Although our entire model is approximated for $\alpha \approx 1.00$, we cannot resist ourselves to check the situation for $\alpha = 0.5$. Taking the potential parameters $a_1 = a$ and $a_2 = b$ with $a_3 = 0$ the potential given by eq. (1)

Table 1. $\ell = 1$ state energy spectrum of the molecule (in GeV$^{\alpha}$).

N	α	k^*_α	Q_1	γ_α	$\mathscr{E}(n=1)$	$\mathscr{E}(n=2)$
3	0.70	1.4028884	− 25.3206	6.5298	0.9870	1.2184
	0.75	1.2795312	− 9.6668	4.0106	0.4919	0.6555
	0.80	2.2839501	− 4.0078	2.6515	0.2971	0.4248
	0.85	2.1074364	− 3.0707	2.3910	0.2796	0.4070
	0.90	1.9666677	− 2.5078	2.4630	0.2141	0.3100
	0.95	2.5295886	− 2.4127	2.5719	0.2066	0.2970
	1.00	2.0652475	− 2.1305	2.5652	0.2041	0.2936
4	0.70	8.2964030	− 10.0649	3.2853	0.6116	0.8430
	0.75	1.3248506	− 64.0555	20.1808	1.8151	1.9788
	0.80	4.6602700	− 5.3314	3.1491	0.3289	0.4566
	0.85	7.5867908	− 5.6188	3.3372	0.3399	0.4672
	0.90	3.0443684	− 3.5661	2.9759	0.2387	0.3346
	0.95	2.8882250	− 3.1616	2.9501	0.2237	0.3141
	1.00	3.0439015	− 3.0988	3.0494	0.2258	0.3153
5	0.70	8.3549641	− 12.4554	3.7937	0.6704	0.9018
	0.75	7.6900754	− 8.6112	3.6967	0.4662	0.6299
	0.80	7.0975390	− 6.7123	3.6681	0.3620	0.4898
	0.85	4.4198401	− 5.3874	3.2513	0.3344	0.4618
	0.90	7.0863034	− 5.4862	3.9065	0.2833	0.3793
	0.95	3.9098650	− 4.1146	3.4313	0.2455	0.3358
	1.00	4.0396850	− 4.0794	3.5397	0.2477	0.3372

Table 2. The variation of τ against α when $\delta = -0.5$.

α	τ
0.70	1.7013
0.75	1.4142
0.80	1.2361
0.85	1.2223
0.90	1.0515
0.95	1.0125
1.00	1.00

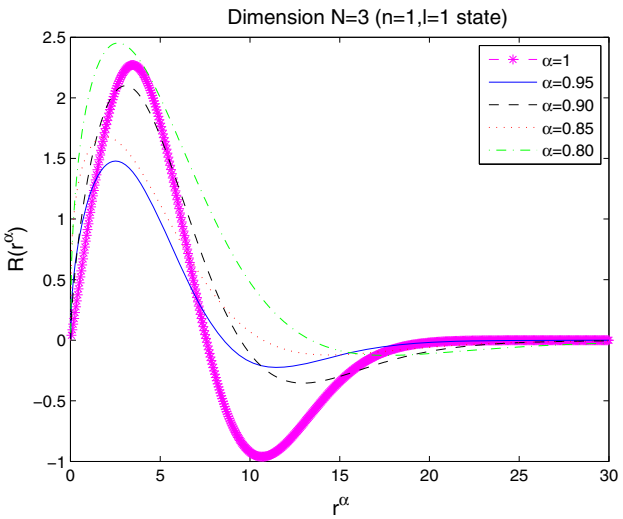

Figure 1. $n = 1$ state eigenfunctions in $N = 3$ dimensions for the generalised fractional pseudoharmonic potential when $\alpha = 1.0, 0.95, 0.90, 0.85$ and 0.80.

becomes $V(r^{1/2}) = V_c(r) = ar + (b/r)$. This is the renowned Cornell potential [35], generally taken in non-relativistic quantum chromodynamics (NRQCD) for realising the quarkonium states. Apart from the light quarks, *bottom* and *charm* are much heavier. The speed of the charm quark is $0.3c$ and that of the bottom quark is $0.1c$. Thus, relativistic effects on charm and bottom are small. That is why NRQCD is enough for computing the states of quarkonium. In addition to that, it is also seen that NRQCD fits much better for bottom quark due to its larger mass m_b than for charm quark with mass m_c.

Quarkonium is flavourless meson with a combination of two quarks, which is symbolised as $q\bar{q}$, where q is a quark and $\bar{q}$ is its antiquark. For heavy quarks, bottom (b) and charm (c), the quarkonia are written as $b\bar{b}$ and $c\bar{c}$. The first one is bottomonium and the second is charmonium. In NRQCD, the study of the quarkonium states is done effectively via a static potential. The most popular potential in the list is the Cornell potential $V_c(r)$. The first part, i.e. the linear part ar is responsible for the confinement of the quarks and the second part b/r is the usual Coulomb part which defines the e.m. force between the constituent quarks in certain quarkonium.

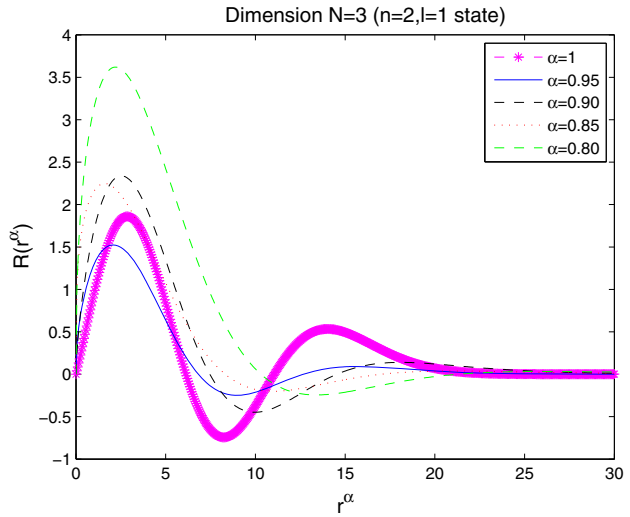

Figure 2. $n = 2$ state eigenfunctions in $N = 3$ dimensions for the generalised fractional pseudoharmonic potential when $\alpha = 1.0, 0.95, 0.90, 0.85$ and 0.80.

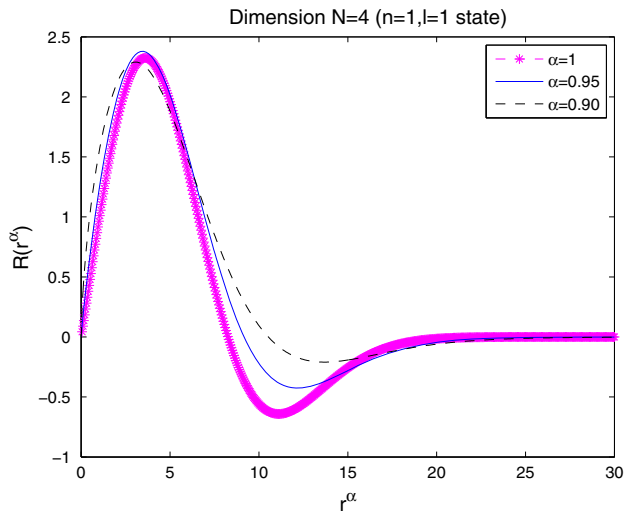

Figure 3. $n = 1$ state eigenfunctions in $N = 4$ dimensions for the generalised fractional pseudoharmonic potential when $\alpha = 1.0, 0.95$ and 0.90.

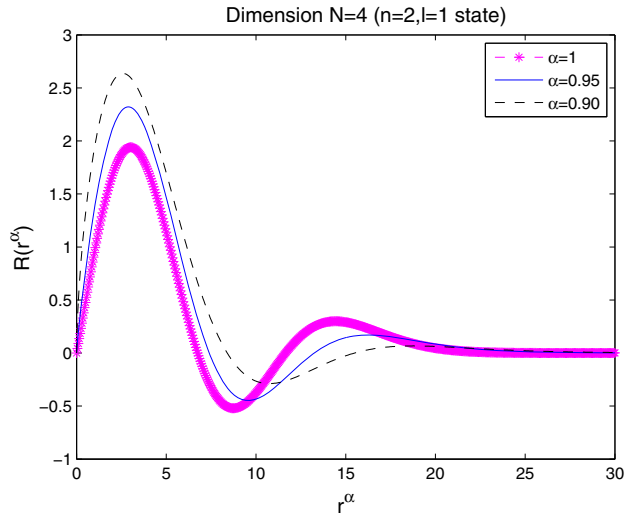

Figure 4. $n = 2$ state eigenfunctions in $N = 4$ dimensions for the generalised fractional pseudoharmonic potential when $\alpha = 1.0, 0.95$ and 0.90.

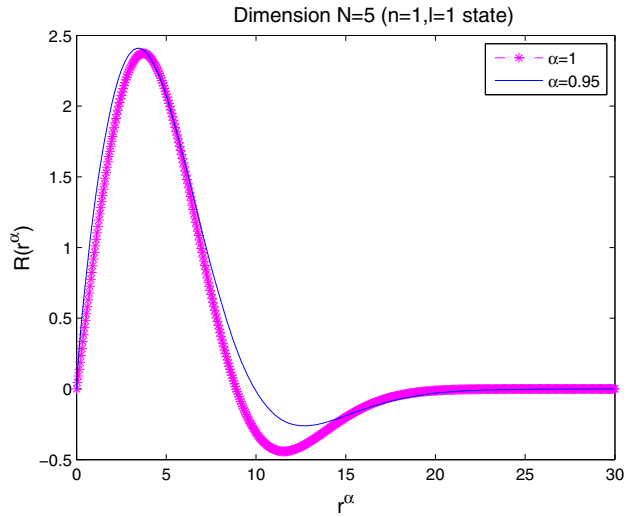

Figure 5. $n = 1$ state eigenfunctions in $N = 5$ dimensions for the generalised fractional pseudoharmonic potential when $\alpha = 1.0$ and 0.95.

Now coming back to our model, the Cornell potential is the obvious consequence of $\alpha = 0.5$ and $a_3 = 0$. As our model is approximated for $\alpha \approx 1.00$ it will not be possible to determine the exact bound-state eigenfunctions as well as energy eigenvalue equation directly. We may use the present energy value equation with slight modification to investigate the situation for the mass spectra of quarkonium states. The modification is to correct the dimension or unit of the energy eigenvalue equation to incorporate the situation for $\alpha = 0.5$. Once again we shall use the natural unit scheme here. Cornell potential with $a_1 = a$ and $a_2 = b$ will make the unit of a as $\text{GeV}^{2\alpha}$ because the unit of r (in fractional sense) is $\text{GeV}^{-\alpha}$ and b will be unit free to make the unit of the potential GeV^{α}. This will turn the unit of $[\mathscr{E}_{n\ell}]_{a_3=0}$ as $\text{GeV}^{\alpha/2}$ because the unit of the mass is GeV^{α}. We propose a dimensional term ζ with the unit $\text{GeV}^{\alpha/2}$ such that the unit of the energy value equation emerges as GeV^{α}. The expression of ζ is unknown to us. Furthermore, we assigned the value of ζ very close to unity to check the validity of the energy value equation numerically. Accepting all these we write $[\mathscr{E}_{n\ell}]_{a_3=0} \rightarrow \zeta[\mathscr{E}_{n\ell}]_{a_3=0}$ for $\alpha = 0.5$.

The mass spectra of quarkonium state in three dimensions is $M_q = 2m_q + \mathscr{E}_{n\ell}$ [36]. So, along with the modification proposed, we have from eq. (23)

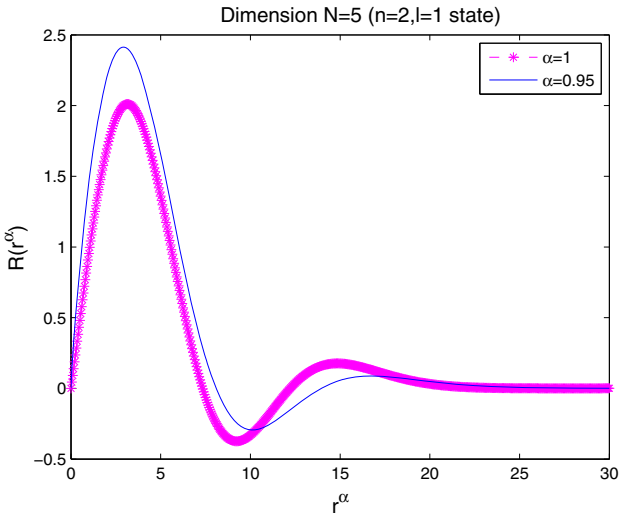

Figure 6. $n = 2$ state eigenfunctions in $N = 5$ dimensions for the generalised fractional pseudoharmonic potential when $\alpha = 1.0$ and 0.95.

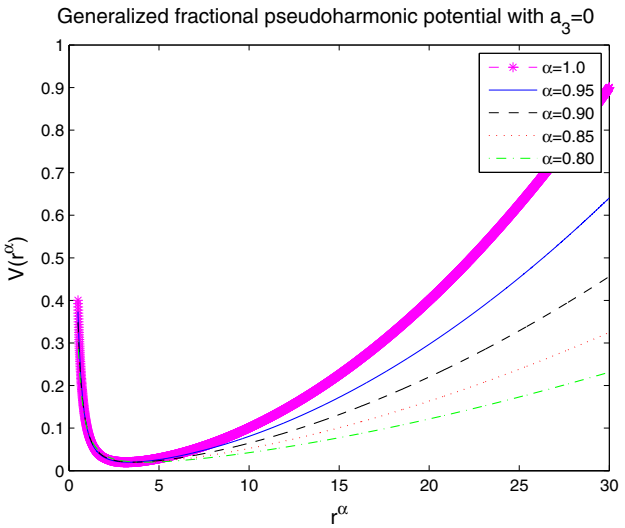

Figure 7. Variation of the generalised fractional pseudoharmonic potential with α.

$$M_q = 2m_q + \zeta \frac{2\tau^2}{\Gamma(3/2)}\left(n + \frac{\gamma_{0.5}}{2}\right)\sqrt{\frac{a}{M}}, \tag{32}$$

where $M = m_b/2$ or $m_c/2$ depending on whether q is b or c. The major factor in the above equation is $\tau = -[\text{cosec}((\alpha - \delta)\pi)/\text{cosec}(-\delta\pi)]$ with $-1 < \delta < 0$. At $\alpha = 0.5$ it offers infinite or large value when $\delta = -0.5$. In this situation we choose $\delta = -1/3$ to evaluate τ for $\alpha = 0.5$. Any other value of δ may give somehow similar results, but for the best results we have chosen $\delta = -1/3$. This can be seen in figure 8, where the variation of τ with δ has been shown. $\delta = -1/3$ provides much more symmetric variation under the range of $0 < \alpha < 1$.

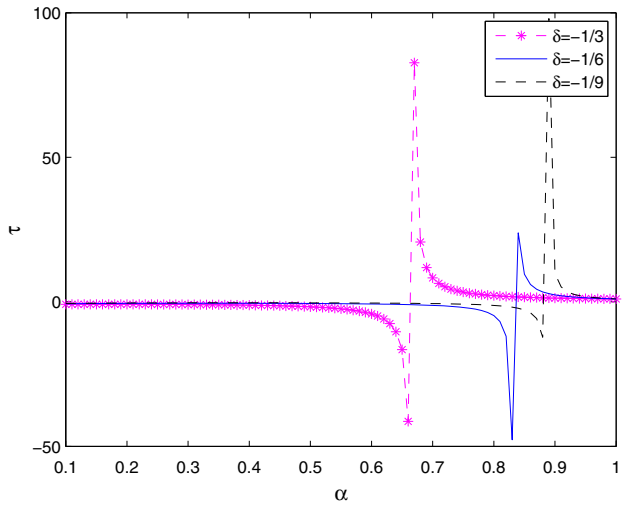

Figure 8. Variation of τ with α when δ acts as a parameter.

Table 3. Quarkonium mass spectra in GeV^α unit: bottomonium ($b\bar{b}$) ($m_b = 4.803\,\text{GeV}^\alpha, a = 0.095\,\text{GeV}^{2\alpha}, b = -1.0$) and charmonium ($c\bar{c}$) ($m_c = 1.480\,\text{GeV}^\alpha, a = 0.010\,\text{GeV}^{2\alpha}$, $b = -2.0$).

Quarkonium	State (n, ℓ)	From eq. (32)	Exp. [38,39]
$b\bar{b}$	1S (1,0)	9.5700	9.460
	1P (1,1)	9.1360	9.900
	2S (2,0)	10.9166	10.023
	2P (2,1)	10.4826	10.260
$c\bar{c}$	1S (1,0)	2.7053	3.068
	1P (1,1)	2.4289	3.525
	2S (2,0)	3.4923	3.663
	2P (2,1)	3.2160	3.773

The mass spectra of bottomonium and charmonium are calculated from (32) by taking standard parameter values [37]. The results are displayed in table 3. The mass spectra confirm that our model is very close to the earlier experimental results in this field. Moreover, the obtained values of mass spectra also indicate that bottomonium obeys NRQCD better than the charmonium.

4. Conclusion

This present study is a sequel of our previous work which was on fractional Mie-type potential and cited in ref. [20]. In this paper, we have studied the approximate bound-state solutions of N-dimensional fractional Schrödinger equation for the generalised pseudoharmonic potential, namely, $V(r^\alpha) = a_1 r^{2\alpha} + (a_2/r^{2\alpha}) + a_3$, where α $(0 < \alpha < 1)$ acts like a fractional parameter for the space variable r. We have composed the entire

study by Jumarie-type derivative rules with the desirability of Laplace transform. The obtained results are verified for harmonic and pseudoharmonic potentials in lower as well as in higher dimension with $\alpha = 1$. We have also furnished the numerical results and a few eigenfunction plots for different α close to unity. In addition to that we have tried to obtain the mass spectra of quarkonia through the model of Cornell potential which is a special case of our potential model that corresponds to $\alpha = 0.5$.

The generalised pseudoharmonic potential for different α is shown in figure 7. It is clear that, as α goes to the lower value of unity the potential graph tends toward the r^{α} axis, which means that the effect of potential is gradually fading. The eigenfunctions, especially for higher dimensions with lower α, are becoming wider than the same for $\alpha = 1$. This means that the particle under the potential experiences less resistance to its motion when α goes by the value less than unity. Since the motion of the particle is less affected by the potential, its position will become more uncertain and the graph will be more wider. This is what we have achieved in all the figures starting from figures 1–7. We need further study to explore more on this topic.

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